Notre Dame Journal of Formal Logic
Volume XI, Number 3, July 1970

THE COMPLETENESS OF COPI'S SYSTEM OF NATURAL DEDUCTION

JOHN A. WINNIE

I. *Introduction.* This note will outline a proof of the completeness of the system of sentential logic developed by Copi in [2] which also provides an effective proof-method for this system. Although the completeness of the Copi system is well known, the method to be used here does not involve a detour through an auxiliary axiomatic system (as in [1], where the completeness of the system presented in [3] is established). Since the method is of some interest in itself, the general procedure is sketched first.

Let $P_1, P_2, \ldots, P_n, Q$ be any sequence of sentential schemata. Then a sentential system of natural deduction is here said to be *complete* if and only if there is a derivation in the system of Q from $P_1, P_2, \ldots, P_n$ whenever the schema $(P_1 \cdot P_2 \cdot, \ldots, \cdot P_n) \supset Q$ is a (standard) truth-table tautology. The notion of a derivation used here will, of course, depend on the particular rules of inference or rules of replacement which are peculiar to the system under study. In the method used below, completeness is demonstrated as follows. First, we show that any tautology is derivable in the system from any non-empty sequence of sentences whatsoever. It now follows as a corollary that $(P_1 \cdot P_2 \cdot, \ldots, \cdot P_n) \supset Q$ is derivable from $P_1, P_2, \ldots, P_n$ whenever $(P_1 \cdot P_2 \cdot, \ldots, \cdot P_n) \supset Q$ is a tautology. Repeated use of the rule of conjunction (or an equivalent device) will now yield $(P_1 \cdot P_2, \ldots, \cdot P_n)$. A single application of *modus ponens* (i.e., the rule of detachment) then gives us Q, the desired result. In what follows, it is assumed that the reader is familiar with the inference and replacement rules of [2], here called **CND** (Copi's system of natural deduction), along with their abbreviations.[1]

1. The system of natural deduction developed in [3] is called **CMD** by Canty in [1]. The method of proving the completeness of **CND** developed here is not immediately applicable to **CMD**, however, due to the fact that the rule of Absorption (*Abs.*) is dropped in that system and replaced by rules of Conditional Proof (**C.P.**) and Indirect Proof (**I.P.**). These last rules are so formulated as to prevent the *continuation* of the proof once they are applied, and this is crucial to the method presented here.

Received January 10, 1969

II. *The Completeness of* **CND**. Since the proof to be given here leans heavily on the notion of a conjunctive normal form of a schema, and the main features of this construction are well-known (cf. [4], pp. 11-15; and [3] pp. 239-240), the following results will be stated without proofs.

R.1. Every sentential schema S which is a tautology has a conjunctive normal form. Every conjunctive normal form of S, CNF(*S*), *is such that each conjunct is a disjunction which contains some sentential variable,* p, *together with its negation,* $\sim p$. *Furthermore, any* CNF(*S*) *may be effectively obtained from S (and conversely) by a finite number of applications of the following rules of replacement: De M., Com., Assoc., Dist., D.N., Impl., and Equiv.*

Next, we show that any statement of the form $p \vee \sim p$ is derivable in **CND** from any non-empty set of premisses whatsoever.

R.2. Let $P_1, P_2, \ldots, P_n$ $(n > 0)$ *be any sequence of sentential schemata in* **CND**. *Then* $p \vee \sim p$ *is derivable in* **CND** *from* $P_1, P_2, \ldots, P_n$.
Proof: The derivation-schema is as follows.

(1)	P_1	
(2)	P_2	
$\vdots$		
(n)	P_n	$\therefore / p \vee \sim p$
$(n+1)$	$P_1 \vee \sim p$	1, *Add.*
$(n+2)$	$\sim p \vee P_1$	$(n+1)$, *Com.*
$(n+3)$	$p \supset P_1$	$(n+2)$, *Impl.*
$(n+4)$	$p \supset (p \cdot P_1)$	$(n+3)$, *Abs.*
$(n+5)$	$\sim p \vee (p \cdot P_1)$	$(n+4)$, *Impl.*
$(n+6)$	$(\sim p \vee p) \cdot (\sim p \vee P_1)$	$(n+5)$, *Dist.*
$(n+7)$	$\sim p \vee p$	$(n+6)$, *Simp.*
$(n+8)$	$p \vee \sim p$	$(n+7)$, *Com.*

The following result shows that any disjunction containing some sentential variable p and its negation $\sim p$ as disjuncts is also derivable in **CND** from any non-empty set of sentential schemata.

R.3. Let $Q = (Q_1 \vee Q_2 \vee p \vee, \ldots, \vee \sim p \vee, \ldots, \vee Q_m)$ $(m \geq 0)$ *be any sentential schemata, and* $P_1, P_2, \ldots, P_n$ $(n > 0)$ *be any sequence of sentential schemata. Then Q is derivable in* **CND** *from* $P_1, P_2, \ldots, P_n$.

Proof: By *R.2.*, $p \vee \sim p$ is derivable in **CND** from $P_1, P_2, \ldots, P_n$. Hence, by *Add.*, $(p \vee \sim p) \vee (Q_1 \vee Q_2 \vee, \ldots, \vee Q_m)$ is now derivable. Repeated uses of *Assoc.* and *Com.* now yield Q.

It now follows that the conjunctive normal form of any tautology is derivable in **CND** from an arbitrary non-empty sequence of schemata.

R.4. Let $P_1, P_2, \ldots, P_n$ $(n > 0)$ *be any sequence of sentential schemata, and S be any tautology. Then if* CNF(*S*) *is a conjunctive normal form of S, then* CNF(*S*) *is derivable in* **CND** *from* $P_1, P_2, \ldots, P_n$.

Proof: By *R.1.*, each conjunct of CNF(S) is a disjunction containing a sentential variable and its negation. Hence, by *R.3.*, each conjunct of CNF(S) is derivable in **CND** from $P_1, P_2, \ldots, P_n$. Repeated use of the rule *Conj.* now yields CNF(S).

The above result, together with *R.1.*, shows that any tautology is derivable in **CND** from an arbitrary sequence of schemata.

R.5. Let $P_1, P_2, \ldots, P_n$ $(n > 0)$ *be any sequence of sentential schemata. Let* S *be any schema which is a tautology. Then* S *is derivable in* **CND** *from* $P_1, P_2, \ldots, P_n$.

Proof: Let CNF(S) be a conjunctive normal form of S. Then, by *R.4.*, CNF(S) is derivable from $P_1, P_2, \ldots, P_n$. Hence, by *R.1.* (since **CND** contains all of the necessary replacement rules), S is now derivable in **CND** from CNF(S). Thus S is derivable from $P_1, P_2, \ldots, P_n$.

The following result now establishes the completeness of **CND**.

Completeness. Let $P_1, P_2, \ldots, P_n, Q (n > 0)$ *be a sequence of sentential schemata. Then if* $(P_1 \cdot P_2 \cdot, \ldots, \cdot P_n) \supset Q$ *is a tautology,* Q *is derivable in* **CND** *from* $P_1, P_2, \ldots, P_n$.

Proof: Assume $(P_1 \cdot P_2 \cdot, \ldots, \cdot P_n) \supset Q$ is a tautology. Then by *R.5.*, $(P_1 \cdot P_2 \cdot, \ldots, \cdot P_n) \supset Q$ is derivable in **CND** from $P_1, P_2, \ldots, P_n$. By repeated uses of *Conj.* (n - 1 times), $P_1, P_2, \ldots, P_n$ yields $(P_1 \cdot P_2 \cdot, \ldots \cdot P_n)$ as well. Hence, by *M.P.*, we may now derive Q.

Since the procedures used in the above proofs are all effective (i.e., constructive), and the only rules of **CND** used were those *omitted* below, we also have the following corollary.

Cor. 1. There is a complete and effective proof-method for **CND** *when the rules M.T., H.S., D.S., C.D., Trans., Exp., and Taut. are dropped.*

The following derivation-schema also shows that the rule *Conj.* may be dropped without loss of the completeness of **CND**.

1)	P_1	
2)	P_2	$\therefore / (P_1 \cdot P_2)$
3)	$P_2 \vee \sim P_1$	2, *Add.*
4)	$\sim P_1 \vee P_2$	3, *Com.*
5)	$P_1 \supset P_2$	4, *Impl.*
6)	$P_1 \supset (P_1 \cdot P_2)$	5, *Abs.*
7)	$P_1 \cdot P_2$	1, 5, *M.P.*

The system which now results when the unnecessary rules are dropped contains only the rules *M.P.*, *Abs.*, *Simp.*, *Add.*, *De M.*, *Com.*, *Assoc.*, *Dist.*, *D.N.*, *Impl.*, and *Equiv.* I leave as an open problem the question as to whether or not one or more of the above rules may be omitted while retaining completeness.

REFERENCES

[1] Canty, J. T., "Completeness of Copi's method of deduction," *Notre Dame Journal of Formal Logic*, vol. IV (1963), pp. 142-144.

[2] Copi, Irving, *Introduction to Logic*, 3rd ed., Macmillan Co., New York (1968).

[3] Copi, Irving, *Symbolic Logic*, 2nd ed., Macmillan Co., New York (1965).

[4] Hilbert and Ackermann, *Principles of Mathematical Logic*, Chelsea Publishing Co., New York (1950).

University of Hawaii
Honolulu, Hawaii