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NOTE ON G. J. MASSEY'S CLOSURE—ALGEBRAIC OPERATION

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1. In [1] it was shown by Massey that a binary operation defined as follows:

Df1 $A * B =_{Df} [-(-A \cap *A \cap -*B) \cup A] \cap [(-A \cap *A \cap -*B) \cup -(A \cap B)]$

where $*$, $\cap$, $\cup$ and $-$ are the symbols of closure, intersection, union and complementation operations respectively is functionally complete in closure algebras in the same sense that an operation of nonunion $(-(A \cup B))$ is functionally complete in Boolean algebras. In order to prove this Massey used a well known fact that closure algebras are in some sense strictly related to system S4 of Lewis, and, therefore, Df1 corresponds to the following definition in the latter system:

Df2 $A * B =_{Df} (\sim A . \Diamond A . \sim\Diamond B \supset A) . [\sim(\sim A . \Diamond A . \sim\Diamond B) \supset \sim(A . B)]$

Subsequently, using Kripke's semantics for S4 he has proved that the functor defined in Df2 can be adopted as a single primitive term of the modal system S4. Hence, operation $*$ defined in Df1 also possesses the required properties.

2. Below, using elementary algebraic calculations I shall show that in the field of closure algebras Df1 is inferentially equivalent to a much shorter formula, and, subsequently, starting from this new formula I shall prove algebraically the results which in [1] are obtained semantically. Moreover, it will be shown that a formula due to Massey in which the definability of intersection by operation $*$ is established can be substituted by a shorter one. An acquaintance with Boolean and closure algebras, as also with paper [1], is presupposed. Instead of $*$ the more common C will be used as a symbol of closure operation, and 0 and 1 will mean algebraic zero and unit elements. In the proof lines the calculations obtained by Boolean algebra will be indicated simply by **BA**. From closure algebras only the following theses will be used:

C1 $[a b] : a \epsilon A . b \epsilon A . \supset . \mathrm{C}(a \cup b) = \mathrm{C}a \cup \mathrm{C}b$
C2 $[a] : a \epsilon A . \supset . a \leqslant \mathrm{C}a$
C3 $[a] : a \epsilon A . \supset . \mathrm{CC}a = \mathrm{C}a$
C4 $\mathrm{C}0 = 0$

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DC $[a]: a \epsilon A . \supset . -\mathrm{C}-a = \mathrm{I}\, a$

C5 $\mathrm{C}1 = 1$

C6 $[a\, b]: a \epsilon A . b \epsilon A . \supset . \mathrm{C}(a \cap b) \leqslant \mathrm{C}\, a$

C7 $[a\, b\, c]: a \epsilon A . b \epsilon A . c \epsilon A . \supset . \mathrm{C}(a \cap b \cap c) \leqslant \mathrm{C}\, b$

C8 $[a]: a \epsilon A . \supset . \mathrm{I}\, a \leqslant a$

Clearly, C1 - C4 are proper axioms of closure algebras, DC is a definition of the interior operation, and C5 - C8 are elementary theorems of these systems.

3. Let us assume a closure algebra $\mathfrak{A} = \langle A, \cap, -, \mathrm{C} \rangle$ and add to it Massey's definition:

D1 $[a\, b]: a \epsilon A . b \epsilon A . \supset . a * b = [-(-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, b) \cup a] \cap [(-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, b) \cup -(a \cap b)]$

Then, in the field of $\mathfrak{A}$ D1 is inferentially equivalent to:

T1 $[a\, b]: a \epsilon A . b \epsilon A . \supset . a * b = (a \cup -\mathrm{C}\, a \cup \mathrm{C}\, b) \cap (-a \cup -b)$

Proof: $[a\, b]: a \epsilon A . b \epsilon A . \supset .$

1. $a * b = [-(-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, b) \cup a] \cap [(-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, b) \cup -(a \cap b)]$ [D1]
2. $= (a \cup -\mathrm{C}\, a \cup \mathrm{C}\, b) \cap [(-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, b) \cup (-a \cup -b)]$ [1; **BA**]
3. $= (a \cup -\mathrm{C}\, a \cup \mathrm{C}\, b) \cap [(-a \cup -b) \cap (\mathrm{C}\, a \cup -a \cup -b) \cap (-\mathrm{C} b \cup -a \cup -b)]$ [2; **BA**]

$= (a \cup -\mathrm{C}\, a \cup \mathrm{C}\, b) \cap (-a \cup -b)$ [3; **BA**]

Thus, D1 is inferentially equivalent to T1.

4. Assume now algebra $\mathfrak{A}$ together with T1 as a definition of operation $*$. Then:

T2 $[a]: a \epsilon A . \supset . a * a = -a$

Proof: $[a]: a \epsilon A . \supset .$

1. $a * a = (a \cup -\mathrm{C}\, a \cup \mathrm{C}\, a) \cap (-a \cup -a) = 1 \cap -a = -a$ [T1, b/a; **BA**]

T3 $[a]: a \epsilon A . \supset . a * -a = 1$

Proof: $[a]: a \epsilon A . \supset .$

1. $a * -a = (a \cup -\mathrm{C}\, a \cup \mathrm{C}\, -a) \cap (-a \cup --a)$ [T1, $b/-a$]

$= (a \cup -\mathrm{C}\, a \cup -a \cup \mathrm{C}\, -a) \cap 1 = 1$ [1; C2, $a/-a$, **BA**]

T4 $0 = -1$ [**BA**]

T5 $[a\, b]: a \epsilon A . b \epsilon A . \supset . -(a * b) = (-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, b) \cup (a \cap b)$ [T1; **BA**]

T6 $[a]: a \epsilon A . \supset . -(a * 1) = a$

Proof: $[a]: a \epsilon A . \supset .$

1. $-(a * 1) = (-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, 1) \cup (a \cap 1) = (-a \cap \mathrm{C}\, a \cap -1) \cup a$ [T5, $b/1$; C5; **BA**]

$= (-a \cap \mathrm{C}\, a \cap 0) \cup a = 0 \cup a = a$ [1; T4; **BA**]

T7 $[a\, b]: a \epsilon A . b \epsilon A . \supset . -(a * b) \cap a = a \cap b$

Proof: $[a\, b]: a \epsilon A . b \epsilon A . \supset .$

1. $-(a * b) \cap a = [(-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, b) \cup (a \cap b)] \cap a$ [T5]

$= (-a \cap \mathrm{C}\, a \cap -\mathrm{C}\, b \cap a) \cup (a \cap b) = 0 \cup (a \cap b) = a \cap b$ [1; **BA**]

T8 $[a\, b]: a \epsilon A .\, b \epsilon A . \supset . \mathrm{C}(a \cap b) \cap -\mathrm{C}a = 0$

Proof: $[a\, b]: a \epsilon A .\, b \epsilon A . \supset .$

1. $\mathrm{C}(a \cap b) \cap -\mathrm{C}a = \mathrm{C}(a \cap b) \cap \mathrm{C}a \cap -\mathrm{C}a = 0$ [C6; **BA**]

T9 $[a\, b]: a \epsilon A .\, b \epsilon A . \supset . \mathrm{C}(-a \cap \mathrm{C}a \cap -\mathrm{C}b) \cap -\mathrm{C}a = 0$

Proof: $[a\, b]: a \epsilon A .\, b \epsilon A . \supset .$

1. $\mathrm{C}(-a \cap \mathrm{C}a \cap -\mathrm{C}b) \cap -\mathrm{C}a = \mathrm{C}(-a \cap \mathrm{C}a \cap -\mathrm{C}b) \cap \mathrm{C}\mathrm{C}a \cap -\mathrm{C}a$ [C7, $a/-a$, $b/\mathrm{C}a$, $c/-\mathrm{C}a$; **BA**]
$= \mathrm{C}(-a \cap \mathrm{C}a \cap -\mathrm{C}b) \cap \mathrm{C}a \cap -\mathrm{C}a = 0$ [1; C3; **BA**]

T10 $[a\, b]: a \epsilon A .\, b \epsilon A . \supset . \mathrm{C} -(a * b) \cap -\mathrm{C}a = 0$

Proof: $[a\, b]: a \epsilon A .\, b \epsilon A . \supset .$

1. $\mathrm{C} -(a * b) \cap -\mathrm{C}a = \mathrm{C}[(-a \cap \mathrm{C}a \cap -\mathrm{C}b) \cup (a \cap b)] \cap -\mathrm{C}a$ [T5]
2. $= [\mathrm{C}(-a \cap \mathrm{C}a \cap -\mathrm{C}b) \cup \mathrm{C}(a \cap b)] \cap -\mathrm{C}a$ [1; C1, $a/-a \cap \mathrm{C}a \cap -\mathrm{C}b, b/a \cap b$]
3. $= [\mathrm{C}(-a \cap \mathrm{C}a \cap -\mathrm{C}b) \cap -\mathrm{C}a] \cup [\mathrm{C}(a \cap b) \cap -\mathrm{C}a]$ [2; **BA**]
$= 0 \cup 0 = 0$ [3; T9; T8; **BA**]

T11 $[a\, b]: a \epsilon A .\, b \epsilon A . \supset . -(-(a * b) * a) = a \cap b$

Proof: $[a\, b]: a \epsilon A .\, b \epsilon A . \supset .$

1. $-(-(a * b) * a) = [--(a * b) \cap \mathrm{C} -(a * b) \cap -\mathrm{C}a] \cup [-(a * b) \cap a]$ [T5, $a/-(a * b)$, b/a]
2. $= [(a * b) \cap 0] \cup (a \cap b) = 0 \cup (a \cap b) = a \cap b$ [1; T10; T7; **BA**]

T11 shows that there is a shorter definition of intersection by operation $*$, than that which is given in [1]. Namely, there Massey introduced an auxiliary definition:

D2 $[a\, b]: a \epsilon A .\, b \epsilon A . \supset . a \odot b = -(a * b)$

and later defined intersection as:

T12 $[a\, b]: a \epsilon A .\, b \epsilon A . \supset . (a \odot b) \odot (a \odot 1) = a \cap b$ [T11; T6; D2]

T13 $[a]: a \epsilon A . \supset . a \cup -(a * 0) = \mathrm{C}a$

Proof: $[a]: a \epsilon A . \supset .$

1. $a \cup -(a * 0) = a \cup [(-a \cap \mathrm{C}a \cap -\mathrm{C}0) \cup (a \cap 0)]$ [T5, $b/0$]
2. $= a \cup [(-a \cap \mathrm{C}a \cap -0) \cup 0] = a \cup (-a \cap \mathrm{C}a \cap 1)$ [1; C4; **BA**]
$= a \cup (-a \cap \mathrm{C}a) = (a \cup -a) \cap (a \cup \mathrm{C}a) = 1 \cap \mathrm{C}a = \mathrm{C}a$ [2; C2; **BA**]

T14 $[a]: a \epsilon A . \supset . a \cap (-a * 0) = \mathrm{I}a$

Proof: $[a]: a \epsilon A . \supset .$

1. $a \cap (-a * 0) = a \cap [(-a \cup -\mathrm{C} -a \cup \mathrm{C}0) \cap (--a \cup -0)$ [T1; $a/-a$, $b/0$]
2. $= a \cap [(-a \cup \mathrm{I}a \cup 0) \cap (a \cup 1)] = a \cap (-a \cup \mathrm{I}a)$ [1; DC; C4; **BA**]
$= (a \cap -a) \cup (a \cap \mathrm{I}a) = \mathrm{I}a$ [2; C8; **BA**]

Thus, since theses T2, T3, T11 and T13 are consequences of $\mathfrak{A}$ on the base of definition T1 (or D1), it is proved that operation $*$ can be adopted as a single operation in closure algebras in the sense given in [1].

5. In an entirely analogous way it can be shown that in the field of modal system S4 constant $*$ defined by Df2 can serve as the single primitive term of this theory. In Łukasiewicz symbolism with symbol *W* taken for $*$ Massey's definition Df2 has the following form:

DI *EWpqKCKNpKMpNMqpCNKNpKMpNMqNKpq*

And, in an elementary way it can be proved that DI is inferentially equivalent to:

W1 *EWpqKCNpCMpNMqCpNq*

since in any system of modal logic the following formula:

W2 *EKCKNpKMpNMqpCNKNpKMpNMqNKpqKCNpCMpNMqCpNq*

is provable. Then, using deductions analogous to those which were given above we can prove that in the field of S4 W1 implies the following theses:

W3 *EWppNp*
W4 *EWpNpT*
W5 *ENWNWpqpKpq*
W6 *EApNWpFMp*
W7 *EKpWNpFLp*

i.e. theorems corresponding to those proven above in the field of $\mathfrak{A}$: theses T2, T3, T11, T13 and T14.

6. It should be remarked that in both cases, i.e. in $\mathfrak{A}$ or in S4, the desired formulas are proved only with the use of all proper axioms of closure algebras or of S4. This suggests that the above deductions do not hold in systems weaker than closure algebras or S4. I did not investigate this problem.

REFERENCES

[1] Massey, Gerald J., "Binary closure—algebraic operations that are functionally complete," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 340-342.

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