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BINARY CLOSURE-ALGEBRAIC OPERATIONS THAT ARE FUNCTIONALLY COMPLETE

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1. *Preliminaries*.* It is well known that the modal system S4 is related to closure algebras in the same way that the classical propositional calculus is related to Boolean algebras, namely: a wff is a theorem of S4 if and only if its algebraic transliteration is valid in every closure algebra ([3], p. 130). Consequently, many results about closure algebras carry over to S4, and conversely. In this paper we exploit the aforementioned relationship to introduce binary closure-algebraic operations that are functionally complete in closure algebras in the same sense that the operations of nonunion and nonintersection are functionally complete in Boolean algebras. By a *closure-algebraic operation of* a closure algebra $\langle K, -, \cap, *\rangle$ we shall understand an operation on K that is generable by finite composition from the operations $*$ (closure), $\cap$ (intersection), and - (complementation). A set $\triangle$ of closure-algebraic operations of a closure algebra $\langle K, -, \cap, *\rangle$ shall be called *functionally complete in* $\langle K, -, \cap, *\rangle$ if every closure-algebraic operation of $\langle K, -, \cap, *\rangle$ can be generated by finite composition from the members of $\triangle$. We can now state precisely the theorem that will be proved:

If $\langle K, -, \cap, *\rangle$ *is a closure algebra, then (the unit set of) the binary closure-algebraic operation* $\ast$ *of* $\langle K, -, \cap, *\rangle$ *is functionally complete in* $\langle K, -, \cap, *\rangle$, *where*

$$A \ast B =_{Df} [-(-A \cap *A \cap -*B) \cup A] \cap [(-A \cap *A \cap -*B) \cup -(A \cap B)].$$

The same is also true of the closure-algebraic operation dual to $\ast$.

2. *Proof of Theorem*. In view of the aforementioned relationship between S4 and closure algebras, it is sufficient proof of the theorem to show that the binary connective '$\ast$' serves by itself to define the S4 connectives '$\sim$', '$\cdot$', and '$\Diamond$' (or '$\Box$'), where

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$$A * B =_{Df} [\sim A \cdot \Diamond A \cdot \sim \Diamond B \supset A] \cdot [\sim(\sim A \cdot \Diamond A \cdot \sim \Diamond B) \supset \sim (A \cdot B)].$$

To establish the definability of '$\sim$', '$\cdot$', and '$\Box$' in terms of '$*$', we will make liberal use of Kripke's semantics for S4 as explained in [2]. Concerning the semantical evaluation of '$*$', notice that $\ulcorner A * B \urcorner$ has the same value in a world W as $\ulcorner \sim (A \cdot B) \urcorner$ has in W, unless W satisfies the following three conditions:

(1) A is false in W;
(2) A is true in at least one world accessible to W;
(3) B is false in every world accessible to W (hence, by the reflexivity of accessibility, B is false in W).

If W satisfies all three conditions, then $\ulcorner A * B \urcorner$ has the same value in W that A has in W, namely falsehood. So, clearly,

$$\sim A =_{Df} A * A \qquad \mathbf{T} =_{Df} A * \sim A \qquad \mathbf{F} =_{Df} \sim \mathbf{T}$$

We define an auxiliary connective '$\odot$' as follows.

$$A \odot B =_{Df} \sim (A * B)$$

Notice that $\ulcorner A \odot B \urcorner$ has the same value in a world W that $\ulcorner A \cdot B \urcorner$ has in W, unless W satisfies the three conditions mentioned above; in the latter event, $\ulcorner A \odot B \urcorner$ is true in W. Conjunction may now be defined as follows.

$$A \cdot B =_{Df} (A \odot B) \odot (A \odot \mathbf{T})$$

To see that our definition of conjunction is correct, observe that the *definiens* behaves semantically like conjunction so long as no special case (i.e. a world satisfying the three conditions listed above) arises in the semantical evaluation of any of the occurrences of '$\odot$' in the *definiens*. Therefore, we need consider only what happens when such special cases arise. The special case cannot arise in evaluating the third occurrence of '$\odot$' in a world W, since its right-hand component $\mathbf{T}$ will be true in every world accessible to W. Moreover, the special case cannot arise in evaluating the second occurrence of '$\odot$' in a world W for the following reasons. Suppose W were the special case for the second occurrence of '$\odot$' in the *definiens*. Then $\ulcorner A \odot \mathbf{T} \urcorner$ would be false in every world accessible to W. Hence, by the semantics of '$\odot$', A must also be false in every world accessible to W. By the definition of the special case for the second occurrence of '$\odot$', $\ulcorner A \odot B \urcorner$ must be true in some world W_1 accessible to W. Since A is false in W_1, W_1 must be a special case for $\ulcorner A \odot B \urcorner$; otherwise, $\ulcorner A \odot B \urcorner$ would be false in W_1. So there must be a world W_2 accessible to W_1 in which A is true. But, by the transitivity of accessibility, W_2 is accessible to W, and we have already established that A is false in every world accessible to W. This contradiction shows that the special case cannot arise in evaluating the second occurrence of '$\odot$' in the *definiens*. It is easily verified that the *definiens* has the same value as $\ulcorner A \cdot B \urcorner$ in any world W which is the special case relative to the first occurrence of '$\odot$'. Therefore, our definition of conjunction is correct.

We define necessity as follows:

$$\Box A =_{Df} A \cdot (\sim A * \mathsf{F})$$

To verify the correctness of this definition, observe that both $\ulcorner \Box A \urcorner$ and the *definiens* are false in any world in which A is false. If A is true in a world W and in every world accessible to W, then $\ulcorner \Box A \urcorner$ is true in W. But $\ulcorner \sim A * \mathsf{F} \urcorner$ is also true in W, since W is not the special case relative to it. So, the *definiens* has the same value in W that $\ulcorner \Box A \urcorner$ has. But suppose that A is true in a world W and that A is false in at least one world accessible to W. Then, because W is the special case for $\ulcorner \sim A * \mathsf{F} \urcorner$, both $\ulcorner \Box A \urcorner$ and the *definiens* are false in W. Thus our definition of necessity is correct. This completes our proof that '$*$' serves to define the S4 connectives '$\sim$', '$\cdot$', and '$\Box$', i.e. that '$*$' is a Sheffer connective for S4 and containing systems. (See [1] concerning the notion of a Sheffer connective for modal systems.) The dual of '$*$' is also a Sheffer connective for S4. To obtain a definition of the dual of '$*$', substitute '$\Box$', 'v', and '$\not\subset$' (converse nonimplication symbol) for '$\Diamond$', '$\cdot$', and '$\supset$', respectively, throughout the *definiens* of '$*$'.

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