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A LOGICAL CALCULUS OF ANALOGY INVOLVING FUNCTIONS OF ORDER 2

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1. INTRODUCTION The logic of analogy is hardly an uninvestigated subject. At another place,[1] we consider the syntax and pragmatics of analogical arguments. Bocheński,[2] under the inspiration of Aquinas and Cajetan,[3] has given some thought to the semantics of analogical statements, and recently Hesse[4] has proposed a theory of analogy that both involves and presupposes some of the results of 1. However, it would seem that little, if any, effort has been expended on a detailed formal description of the logic of similarity or likeness. In this respect, the logic that follows should be thought of as a methodological proposal. Whatever its difficulties, it at least purports to show that

(1) analogy (in the sense of similarity or likeness) of individuals can be expressed within the framework of either a standard predicate or a standard set calculus, but

(2) it cannot be so expressed without resort to semantical considerations.[5]

Further, it can be demonstrated that, like that of identity,

(3) this logic of similarity is a model for the logic of the Universal Relation $\dot{V}$ of *Principia Mathematica.*[6]

Finally,

(4) the definition of analogy for sets is not the usual one, in terms of isomorphism. However, under suitable restrictions, the two definitions may be reduced to one another.[7]

2. NOTATION Our logic will be called $\mathbf{A.S.}_1$, i.e., proposal of an analogical system for individuals. In stating and developing $\mathbf{A.S.}_1$, we make the following assumptions: (1) there exists a standard elementary logic of the sort developed by Quine[8] and Copi[9] included in which is a consistent and deductively complete propositional calculus[10]; (2) the rules governing the quantification and substitution of predicate variables are formally identical to those for individual variables[11]; and (3) Post's criterion of consistency is

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formally satisfied by **A.S.$_1$**. We also assume familiarity with the notation and techniques of *Principia Mathematica.*

A.S.$_1$ contains a denumerably infinite number of primitive symbols of the following kinds:

1. English capital letters:

P_1, Q_1, R_1, . . . from the middle of the alphabet and called propositional variables

A, B, C, . . . from the first part of the alphabet with or without left-hand superscripts and called predicate constants

P, Q, R, . . . from the middle of the alphabet with or without left-hand superscripts but without subscripts and called predicate variables

2. English and Greek lower case letters:

a, b, c, . . . from the first part of the English alphabet with or without left-hand superscripts and called individual constants

α, β, γ, . . . from the first part of the Greek alphabet with or without left-hand superscripts and called set constants

v, w, x, . . . from the latter part of the English alphabet with or without left-hand superscripts and called individual variables

ϕ, X, Ψ, . . . from the latter part of the Greek alphabet with or without left-hand superscripts and called set variables

3. Operators:), (, $\sim$, ., $\Diamond$

The latter two subsets of 1 will be collectively referred to as predicate symbols. Since both sets and particulars may be treated as individuals,[12] all of the symbols included in 2 will be regarded as individual symbols.

At this point we define a formula for **A.S.$_1$** to be any finite sequence of symbols. A well-formed formula (wff) is defined recursively thus:

RD-I If G is a propositional variable, G is well formed

RD-II If G is an n-adic predicate symbol, $G(x_1, x_2, \ldots, x_n)$ is well formed

RD-III If G is an individual symbol, G is well formed

RD-IV If G and H are well formed then $(G) \cdot (H)$ is also well formed

RD-V If G is well formed in the sense of RD-I or RD-II and if x is an individual variable, then $(x)G$ is well formed

RD-VI If G is well formed in the sense of RD-I or RD-V, then $\Diamond G$ is also well formed

RD-VII If G is well formed in the sense of RD-I, II, V, VI, then $\sim(G)$ is also well formed

We next define what we mean by the word 'proof' in the following way: A proof is any finite sequence of wffs S_1, S_2, . . ., S_t such that for any S_j,[13] it is either an axiom or a premise[14] or the result of applying one of the usual rules of inference[15] or the result of using an associated propositional formula (apf). But the latter result demands a definition, which we presently give. An apf is a wff, $\overline{G}$, of the propositional calculus which results from replacing every well formed part of the formula, G, of the predicate calculus having the form $P^n(x_1, x_2, \ldots, x_n)$ by a propositional symbol, the differences in replacement being contingent solely upon different P^n's and not upon the differences among individual symbols that succeed the P^n's.

Now, it can be demonstrated for a standard elementary logic like that of Copi that if G is a provable proposition then $\overline{G}$ is a theorem and if $\overline{G}$ is a provable proposition then G is a theorem.[16] But since all of the wffs of **A.S.**$_1$ are either G's or reducible by definition to G's,[17] we shall occasionally use the theoremhood of a $\overline{G}$ to warrant the theoremhood of a G and this process will be referred to as "proof by apf."

An interpreted metalanguage composed of ordinary English, a few well-known mathematical symbols, and the special symbols '$\Diamond$', '$\vdash$', and '$=_{df}$' are to be used in developing **A.S.**$_1$. The expression '$\Diamond$' is to be read "it is possible that ..." and denotes its counterpart in **A.S.**$_1$. The expression '$\vdash P_1$' may be read "a provable proposition, 'P_1'" where 'provable' is understood in the sense defined above for the word 'proof.' The expression '$=_{df}$' is directly translatable by the English expressions 'is defined as' or 'is understood to mean.' Any symbol or formula within single quotes is always metalinguistic and denotes its counterpart in the object language. Finally, we introduce the operator '$\exists$' and define it in the following way:

$$\exists G =_{df} \sim(x) \sim G \tag{D-I}$$

We are guided in setting up our system by the interpretation we intend to give it. Thus, the individual symbols are to be the symbolic translations of such English expressions as nouns (both common and proper respectively) and definite descriptions. Our predicate symbols are to be the symbolic translations of those expressions in English which typically and legitimately occupy the predicate position in an indicative sentence.

3. THE DEFINITION OF ANALOGY Before proposing a definition of analogy we shall discuss, *in a somewhat cursory fashion*, the rationale for such a definition. This step seems appropriate since the all-pervasive character of the notion of analogy makes a complete conventionalism with respect to a definition inadvisable. Moreover, as will be noted below, a definite formal problem could arise with an unwise choice of definition. Among some of the informal considerations to which we could first turn our attention, the most fruitful would seem to be some *prima facie* examples of analogy. Thus: (A) 'In this respect, John is similar to Joe'; (B) 'With respect to these considerations, the two situations are analogous'; (C) 'Chinese bears greater similarity to Japanese than to Korean'; (D) 'The two men are similarly related to their respective governments'; (E) 'Mary is like her mother in that she always retires early'; (F) 'The

Corvair bears some resemblance to the Volkswagen'; (G) 'Mongoloids are similar to Negroids with respect to the amount and disposition of body hair.' Some of the above examples might be rejected as those of analogy on the grounds that subtle distinctions exist between, on the one hand, uses of such expressions as 'like,' 'resembles,' 'similar (to),' and on the other hand the use of 'analogous (to).' Thus, the dictionary indicates that the former group are more often used with respect to properties whereas the latter suggest a parallelism with respect to relations. While such a distinction will not be entirely ignored in **A.S.**$_1$, the symbol '$\approx$', as yet undefined, is here introduced and may be read indiscriminately as 'is analogous to' or 'is similar to' or 'is like,' etc. Although we do want to pinpoint some of the characteristics of '$\approx$' which are propaedeutic to a definition, no attempt will be made here to carry out an exhaustive analysis (if there be such a thing) of ordinary language with respect to the notion of analogy.

First of all, we should like to call attention to a distinction between two types of logical connectives: One, which we shall call the intrasentential type, is exemplified by 'is greater than,' 'plus', and 'is equal to' in the propositions, (1) '2 is greater than 1'; or (2) '5 plus 6 is equal to 11'; the other, which we call the intersentential type, is exemplified by 'and' in (3) '2 is greater than 1 and 5 is greater than 4'.[18] In (1) and (2) the relational expressions 'greater than' and 'equal to' serve to join two nonsentential expressions whereas in (3) the connective 'and' joins two sentential expressions.[19] Now, it would seem obvious that the analogy connective '$\approx$' is of the intrasentential variety and resembles in this respect such other intrasentential connectives as '=', '<', '>', etc. This of course suggests that any formal treatment of analogy be within the framework of predicate or set calculus rather than a propositional calculus. Further, if we analyze what is being asserted by our examples above, it would seem that in each case two or more things[20] are said to be analogous or similar on the basis of some property or relation which they have in common. Although the property or relation in question may not be specifically indicated in the analogy statement itself (e.g., (C) and (F), above), all such statements can be rendered in such a way as to at least indicate that the analogy asserted is on the basis of some property or other, some relation or other. Thus, (F) could be translated, without residue,[21] into 'There is at least one property in terms of which the Corviar resembles the Volkswagen.' (C) could be translated into 'There are more characteristics (properties or relations) in terms of which Chinese and Japanese are similar than there are in terms of which Chinese and Korean are similar. Finally, in view of the foregoing, it would seem that while the analogy relation holds among individuals it is clear that the relation is not confined to particulars.[22] Thus, as illustrated by (G) above, classes as well as physical objects may be related analogically. Higher order properties and classes also may be so related but their logic is not discussed here.

Now, all of our considerations thus far would make it curious indeed to define analogy in terms of, say, either alternation or simple implication. It would seem more appropriate to define such a relation either in terms of complication or in terms of conjunction. Hence, for the present we shall entertain the two following definitions:

$$a \approx b =_{df} (\exists P)\ (Pa \equiv Pb)^{23}, \qquad \text{(D-II)}$$
$$a \approx b =_{df} (\exists P)\ (Pa \cdot Pb). \qquad \text{(D-III)}$$

A highly suggestive modus operandi for choosing from among one of the above definitions exists in the formalization of Leibniz's definition of '=', which can be stated for individuals thus:

$$a = b =_{df} (P)\ (Pa \equiv Pb)^{24} \qquad \text{(D-IV)}$$

Because of D-IV, it would seem sensible to select D-II above as a more acceptable definition of the analogy relation than D-III. Substantiation of such a selection can be had by consulting *Principia Mathematica*,[25] where the twin notions of 'universal class' (V) and 'universal relation' ($\dot{V}$) are discussed.[26] Definitions of these are respectively:

$$V =_{df} \hat{x}\ (x = x), \qquad \text{(D-V)}$$

where "$\hat{x}\ (x = x)$" is to be read "the class 'x' determined by the propositional function '$x = x$,'" and

$$\dot{V} =_{df} \hat{x}\hat{y}\ (x = x) \cdot (y = y), \qquad \text{(D-VI)}$$

where "$\hat{x}\hat{y}\ (x = y) \cdot (y = y)$" is to be read "the relation between x and y determined by the propositional function '$(x = x) \cdot (y = y)$.'" Now, it is obvious that D-II above corresponds formally to the definition D-VI of the universal relation ($\dot{V}$) of *Principia Mathematica* and should for this reason be eschewed as a definition of the analogy relation. Moreover, it can be demonstrated that defining analogy in terms of complication, i.e., in terms of D-II, allows one to derive both the universal property, universal relation, and *pari passu*, D-III above.[27] Therefore, in terms of the foregoing considerations, our choice of a definition of analogy for individuals will be that of D-II. It should be noted, however, that this definition will undergo a few slight modifications as **A.S.**$_1$ is developed.[28] The first of these modifications is presently discussed.

Our proposed definition of analogy has one very obvious drawback. For, according to it, the following is also acceptable:

$$a \approx b =_{df} (\exists P)\ ({\sim}Pa \equiv {\sim}Pb)$$

But the defining portion of such a formula reduces to,

(1) $$(\exists P) \sim({\sim}Pa \equiv Pb),$$

and hence to,

(2) $$\sim(P)\ ({\sim}Pa \equiv Pb),$$

which makes a triviality of analogy so defined. Hence, we introduce the following semantical stipulation:

(3) $$a \approx b \equiv (\exists P)\ (Pa \equiv Pb) \cdot \sim \Diamond(\exists P)\ ({\sim}Pa \equiv {\sim}Pb)$$

This precludes the predicate variables of our definition taking negative values. Such a qualification fits fairly well with "common sense," since the similarity between two individuals is seldom predicated upon the basis of properties which neither of them have but rather upon the basis of those which they both possess.

4. A PREDICATE CALCULUS FOR ANALOGY We state our first theorem thus:

$$\vdash (x)\ (y)\ ((x \approx y) \equiv (\exists P)\ (Px \equiv Py)), \tag{T-I}$$

the proof of which via Universal Generalization (U.G.) from D-II is obvious.

Lemma I: $\vdash (x)\ (x = x)$

Proof: The proof of this lemma, within restrictions of type and order, has been given by Quine.[29]

$$\vdash (x)\ (y)\ (\exists P)\ (Px \equiv Py) \tag{T-II}$$

Proof: (1) $\mathrm{a} = \mathrm{a}$, from Lemma I by Universal Instantiation (U.I.); (2) $\mathrm{b} = \mathrm{b}$, from Lemma I by U.I.; (3) $V\mathrm{a}$, from 1 by D-V and the Principle of Extensionality (E); (4) $V\mathrm{b}$, from 2 by D-V; (5) $V\mathrm{a} \equiv V\mathrm{b}$ from 3 and 4 by truth functional inference; (6) $(\exists P)\ (P\mathrm{a} \equiv P\mathrm{b})$ from 5 by Existential Generalization (E.G.); (7) $(x)\ (y)\ (\exists P)\ (Px \equiv Py)$, by U.G.

$$\vdash (x)\ (x \approx x) \tag{T-III}$$

Proof: (1) $\mathrm{a} \approx \mathrm{a} \equiv (\exists P)(P\mathrm{a} \equiv P\mathrm{a})$ from T-I by U.I.; (2) $\mathrm{a} \approx \mathrm{a} \supset (\exists P)(P\mathrm{a} \equiv P\mathrm{a})$. $(\exists P)\ (P\mathrm{a} \equiv P\mathrm{a}) \supset \mathrm{a} \approx \mathrm{a}$ from 1 by definition; (3) $(\exists P)\ (P\mathrm{a} \equiv P\mathrm{a}) \supset \mathrm{a} \approx \mathrm{a}$ from 2 by simplification (S); (4) $(\exists P)\ (P\mathrm{a} \equiv P\mathrm{a})$ from T-II by U.I.; (5) $\mathrm{a} \approx \mathrm{a}$ from 3 and 4 by Modus Ponens (M.P.); (6) $(x)\ (x \approx x)$ from 5 by U.G.

$$\vdash (\exists x)\ (\exists y)\ (x \approx y). \tag{T-IV}$$

Proof: (1) $(\exists P)\ (Px \equiv Py)$ from T-II by U.I. (2) $x \approx y$ from 1 by D-II and E; (3) $(\exists x)\ (\exists y)\ (x \approx y)$ from 2 by E.G.

Where 'A' is the constant for the analogy relation and in the notation of *Principia Mathematica*, the above theorem can be translated into:

$$\vdash \exists!\, \dot{\mathrm{A}}. \tag{T-IV.I}$$

Presently, we augment our notation so as to introduce a minor revision in our definition of analogy. Thus, the symbol '$\approx_{\mathrm{p}}$' is defined in almost identically the same way as '$\approx$':

$$\mathrm{a} \approx_{\mathrm{p}} \mathrm{b} =_{df} (\exists P)\ (P\mathrm{a} \equiv P\mathrm{b}) \tag{D-II.I}$$

The subscript, 'p', is merely for keeping track of analogy connectives. Such tagging of connectives has not been important up till the present but is necessary now for (1) the adequate statement of an important axiom of analogy and for (2) the adequate statement and proof of a transitivity theorem for analogy. The former can be stated thus:

$$(x)\ (y)\ (x \approx_{\mathrm{p}} y) \equiv \hat{x}\hat{y}\ (\exists P)\ (Pxyx \ldots \supset Pxyy \ldots), \tag{A-I}$$

where 1) x and y are distinct, i.e., take on different values; and 2) the right-hand side of the equivalency allows y to be substituted for x at any and all of the latter's occurrences in any and only schemata bearing the generalized form '--- $\approx_p \ldots$'.

The foregoing axiom seems necessary to rigorize a kind of extensionality principle for analogy. Loosely stated, such a principle would be: 'Under certain conditions, similars may be substituted for one another.' As an example, if we assert the existence of the property denoted by 'green' then it should be possible to substitute for the expression 'the grass' in the statement 'the grass is green' those expressions which refer to individuals that exemplify the property in question, e.g., 'the tree,' 'the Tokay grape,' 'the grasshopper,' etc. Hence, in the analogical statement 'the grass is related to its color as the tree is to its color', the expression, 'the Tokay grape' and 'the grasshopper' could be substituted for 'the grass' and 'the tree' respectively without changing the relation entailed by the original analogical statement.

Turning now to the derivation of more theorems, it would seem propaedeutic to the deduction of a transitivity relation for analogy to assert the existence of at least one property (or relation) in terms of which individuals are interchanged:[30]

$$\vdash (\exists P)\ (P\mathrm{aba} \ldots \supset P\mathrm{abb} \ldots). \qquad \text{T-V}$$

Proof: (1) $\vdash (x)\ (y)\ ((x \approx_p y) \equiv (\exists P)\ (Pxyx \ldots \supset Pxyy \ldots))$ from A-I. (2) $(\exists x)\ (\exists y)\ (x \approx_p y)$ from T-IV. (3) $\mathrm{a} \approx_p \mathrm{b}$ from 2, by Existential Instantiation (E.I.). (4) $(\mathrm{a} \approx_p \mathrm{b}) \equiv (\exists P)\ (P\mathrm{aba} \ldots \supset P\mathrm{abb} \ldots)$ from 1, by U.I. (5) $(\mathrm{a} \approx_p \mathrm{b}) \supset (\exists P)\ (P\mathrm{aba} \ldots \supset P\mathrm{abb} \ldots) \cdot (\exists P)\ (P\mathrm{aba} \ldots \supset P\mathrm{abb} \ldots) \supset (\mathrm{a} \approx_p \mathrm{b})$ by apf,[31] from 4. (6) $(\mathrm{a} \approx_p \mathrm{b}) \supset (\exists P)\ (P\mathrm{aba} \ldots \supset P\mathrm{abb} \ldots)$ from 5, by S. (7) $(\exists P)\ (P\mathrm{aba} \ldots \supset P\mathrm{abb} \ldots)$ from 6 and 3 by M.P.

The next two derivations are of the properties of commutativity and transitivity.

$$\vdash (x)\ (y)\ ((x \approx_p y) \supset (y \approx_p x)). \qquad \text{T-VI}$$

Proof: (1) $\vdash (x)\ (y)\ ((x \approx_p y) \equiv (x \approx_p y))$ which follows by D-II.I and E. from T-I. From 1 we derive (2) $(x \approx_p y) \equiv (x \approx_p y)$ by U.I. and this in turn yields (3) $(x \approx_p y) \supset (x \approx_p y) \cdot (x \approx_p y) \supset (x \approx_p y)$ by apf. (4) $(x \approx_p y) \supset (x \approx_p y)$ from 3 by S. This in turn yields (5) $(x \approx_p y) \supset (y \approx_p x)$ by T-V. Finally, (6) $\vdash (x)\ (y)\ ((x \approx_p y) \supset (y \approx_p x))$ from 5, by U.G.

$$\vdash (x)\ (y)\ (z)\ (((x \approx_p y) \cdot (y \approx_p z)) \supset (x \approx_p z)). \qquad \text{T-VII}$$

Proof: Derivation is via Conditional Proof (C.P.). Hence, the statement to be derived is $(x)\ (z)\ (x \approx_p z)$. The assumed condition is (1) $\vdash (x)\ (y)\ (z)\ (x \approx_p y) \cdot (y \approx_p z)$, from which is derived (2) $(x \approx_p y) \cdot (y \approx_p z)$ by U.I. (3) $(z \approx_p x) \cdot (x \approx_p z)$ from 2 by T-V. (4) $z \approx_p x$ follows from 3 by S. T-VI converts to (5) $(x \approx_p y) \supset (x \approx_p z)$ by U.I. 5 transforms to (6) $(z \approx_p x) \supset (x \approx_p z)$ by T-V. 4 and 6 yield (7) $x \approx_p z$ by M.P. 7 becomes (8) $(x)\ (z)\ (x \approx_p z)$ by U.G.

At this point "Euclid's Law" for the analogy relation may be asserted and demonstrated:

$$\vdash (x)\,(y)\,(z)\,(((x \approx_{\mathrm{P}} y) \cdot (z \approx_{\mathrm{P}} y)) \supset (x \approx z)). \qquad \text{T-VIII}$$

Proof: The proof of this is almost obvious. By instantianting T-VII and then exchanging '$z \approx_{\mathrm{P}} y$' for '$y \approx_{\mathrm{P}} z$' a la T-V and finally universally generalizing the result, we obtain the theorem to be proved.

We wish now to consider and eventually demonstrate the legitimate use of the connective '·' between two individual symbols of the same type. The reason for such considerations will become apparent with respect to some theorems which we shall subsequently derive. Our considerations arise in the following example: 'Lyndon B. Johnson is similar to Charles de Gaulle.' Such a statement can be translated into: 'The thirty-sixth President of the United States is similar to the first President of the fifth Republic of France. In other words, using the usual notation for the definite description,[32] '$\mathrm{a} \approx_{\mathrm{P}} \mathrm{b}$' can be translated into '$(\iota x)\mathrm{F}x \approx_{\mathrm{P}} (\iota y)\mathrm{G}y$.' Moreover, in principle at least,[33] it is always possible to make such a translation, i.e., to translate proper names into definite descriptions. And the converse of such a translation is also possible. More than this, it is always possible to derive '$(\iota x)\mathrm{F}x$' from either '$(x)\mathrm{F}x$' or '$(\exists x)\mathrm{F}x$' by the 'natural deduction" techniques.[34] By the same techniques, it is also possible to derive '$(\exists x)\mathrm{F}x$' from '$(\iota x)\mathrm{F}x$.' With these considerations in mind it becomes possible to assert our first metatheorem and its corollary:

Metatheorem I: *Given any two individual symbols*, 'a' *and* 'b', *and any two propositional symbols*, P_1 *and* Q_1, *then* $\mathrm{a} \cdot \mathrm{b} = P_1 \cdot Q_1$.[35]

Corollary: $\mathrm{a} \vee \mathrm{b} = P_1 \vee Q_1$

Proof:[36] Evidence for this is to be found in the definition of the iota operator. Where such a definition is $(\iota x)\mathrm{F}x =_{df} (\exists x)\,((\mathrm{F}x \cdot (y)\,(\mathrm{F}y \supset y = x))$, it becomes evident that all uses of the iota operator are eliminable, by this definition, in favor of a proposition.

It is now possible to state both the associative and distributive principles of the analogy relation. However, before proving the theorems asserting both of these principles, we shall assert and prove the following theorem:

$$(x)\,(y)\,(z)\,((x \cdot y) \approx_{\mathrm{P}} z). \qquad \text{T-IX}$$

Proof: By T-II and E, (1) $\vdash (x)\,(y)\,(x \approx_{\mathrm{P}} y)$. From T-VII, step 8, (2) $\vdash (x)\,(y)\,(x \approx_{\mathrm{P}} z)$. From 1 and 2 by T-VIII (3) $\vdash (z)\,(y)\,(z \approx_{\mathrm{P}} y)$. From 2 and 3 respectively (4) $x \approx_{\mathrm{P}} z$ and (5) $z \approx_{\mathrm{P}} y$ by U.I. 4 and 5 yield (6) $(x \approx_{\mathrm{P}} z) \cdot (z \approx_{\mathrm{P}} y)$ by Adjunction (A).[37] 6 yields (7) $\vdash (x)\,(y)\,(z)\,((z \approx_{\mathrm{P}} x) \cdot (z \approx_{\mathrm{P}} y))$ which is asserting that z is analogous to both x and y. Hence, by metatheorem I, 7 yields (8) $\vdash (x)\,(y)\,(z)\,(z \approx_{\mathrm{P}} (x \cdot y))$ which in turn yields (9) $(x)\,(y)\,(z)\,((x \cdot y) \approx_{\mathrm{P}} z))$ by T-VI.

$$\vdash (x)\,(y)\,(z)\,(((x \cdot y) \approx_{\mathrm{P}} z) \equiv ((x \cdot z) \approx_{\mathrm{P}} y)). \qquad \text{T-X}$$

Proof: The proof of this theorem follows steps 1-3 of T-IX, above. Step 4 would be to notice that in virtue of 1-3 and by means of E., T-X is a case of A-I, where two or more distinct variables that are connected by the same analogy connective are interchangeable.

$$\vdash (x)\ (y)\ (z)\ (((x \vee y) \approx_{\mathrm{P}} z) \equiv ((x \vee z) \approx_{\mathrm{P}} y)). \qquad \text{T-XI}$$

Proof: (Same as T-IX and T-X).

The distributive property of analogy can be stated thus:

$$\vdash (x)\ (y)\ (z)\ ((x \approx_{\mathrm{P}} (y \cdot z)) \equiv (x \approx_{\mathrm{P}} y) \cdot (x \approx_{\mathrm{P}} z)). \qquad \text{T-XII}$$

Proof: We proceed via C.P. where (1) $\vdash (x)\ (y)\ (z)\ ((x \approx_{\mathrm{P}} y) \supset ((y \approx_{\mathrm{P}} z) \cdot (x \approx_{\mathrm{P}} z)))$ is derived from T-VII.[38] 1 yields (2) $(x \approx_{\mathrm{P}} y) \supset (y \approx_{\mathrm{P}} z) \cdot (x \approx_{\mathrm{P}} z)$ by U.I. (3) $x \approx_{\mathrm{P}} y$ from T-II. 2 and 3 yield (4) $(y \approx_{\mathrm{P}} z) \cdot (x \approx_{\mathrm{P}} z)$ by M.P. But 4 yields (5) $x \approx_{\mathrm{P}} z$ by S. 3 and 5 together yield $(x \approx_{\mathrm{P}} y) \cdot (x \approx_{\mathrm{P}} z)$ by A. At this point the proof proceeds as in steps 6-9 in T-IX to demonstrate that the adjunction of '$x \approx_{\mathrm{P}} y$' and '$x \approx_{\mathrm{P}} z$' yield '$x \approx_{\mathrm{P}} (y \cdot z)$.'

$$\vdash (x)\ (y)\ (z)\ ((x \approx_{\mathrm{P}} (y \vee z)) \equiv ((x \approx_{\mathrm{P}} y) \vee (x \approx_{\mathrm{P}} z))). \qquad \text{T-XIII}$$

Proof: (Same as T-XII).

A principle of addition for the analogy relation can be asserted:

$$\vdash (x)\ (y)\ (z)\ ((x \approx_{\mathrm{P}} y) \supset x \approx_{\mathrm{P}} (y \vee z)). \qquad \text{T-XIV}$$

Proof: This theorem is demonstrated via metatheorem I from the apf '$P_1 \supset (P_1 \vee Q_1)$' in the following way: T-XIV yields (1) $(x)\ (y)\ (z)\ ((x \approx_{\mathrm{P}} y) \supset ((x \approx_{\mathrm{P}} y) \vee (x \approx_{\mathrm{P}} z)))$ by T-XIII and E. But 1 bears the form of the apf cited above.

We shall next consider the notion of negation with respect to analogy. In this connection, we employ the symbol '$\not\approx$' in 'a $\not\approx$ b' to indicate indifferently that 'a is not analogous to b,' or 'it is not the case that a is analogous to b.' At first it might seem that defining analogy negtaion is a quite straightforward process, i.e., such negation is defined simply thus:

$$\mathrm{a} \not\approx \mathrm{b} =_{df} \sim (\exists P)\ (P\mathrm{a} \equiv P\mathrm{b}). \qquad \text{D-II.II}$$

However, this approach to defining analogy negation has two objections: (1) The definition to which it gives rise leads to a contradiction within the system and (2) the consequent definition also offers considerable violation to common sense.[39] The former can be demonstrated thus:

1. $\sim (\exists P)\ (P\mathrm{a} \equiv P\mathrm{b})$ assumption (C.P.)
2. $(P) \sim (P\mathrm{a} \equiv P\mathrm{b})$ from 1 by D-I
3. $(P)\ (P\mathrm{a} \cdot \sim P\mathrm{b}) \vee (P\mathrm{b} \cdot \sim P\mathrm{a})$ from 2 by apf
4. $(\exists x)\ (\exists y)\ (P)\ (Px \cdot \sim Py) \vee (Py \cdot \sim Px)$ from 3 by E.G.
5. $(x)\ (y)\ (\exists P)\ (\sim Px \vee Py) \cdot (\sim Py \vee Px)$ from T-II by apf

It is evident that 4 and 5 are logical duals and are therefore contradictories. On the intended, "common sense," interpretation of **A.S.**$_1$, D-II.II is incongruous. For, when one says of two individuals, 'a' and 'b' that (1) "a and b are not analogous," one never means that (2) "it is not the case that for any property whatever, a possesses it if and only if b does." Rather, one usually intends as a translation of 1 the statement that (3) "with respect to a particular property (or relation), it is not the case that a has it if and only if b has it." Hence, we define analogy negation in the following way:

$$\mathrm{a} \not\approx_{\mathrm{P}} \mathrm{b} =_{df} (\exists P) \sim (P\mathrm{a} \equiv P\mathrm{b}), \qquad \text{D-II.III}$$

which will yield the theorem,

$$\vdash (x)\,(y)\,((x \not\approx_{\mathrm{P}} y) \equiv (\exists P) \sim (Px \equiv Py)). \qquad \text{T-XV}$$

Proof: T-XV follows from D-II.III by U.G.

Our next problem is to note and prove the synonymity of use between '$\not\approx$,' defined above, and '$\sim$,' of ordinary propositional negation.

Metatheorem II: *If* $\vdash \mathrm{a} \not\approx \mathrm{b}$ *then* $\vdash \sim (\mathrm{a} \approx \mathrm{b})$.

Proof: (1) $\mathrm{a} \not\approx \mathrm{b} \equiv (\exists P) \sim (P\mathrm{a} \equiv P\mathrm{b})$ from T-XV, by U.I. (2) $\mathrm{a} \not\approx \mathrm{b} \equiv \sim (P)(P\mathrm{a} \equiv P\mathrm{b})$ from 1 by D-I. But 2 involves, by D-II.III, the possibility of substituting '$\sim(\mathrm{a} \approx \mathrm{b})$' for '$\mathrm{a} \not\approx \mathrm{b}$,' since the expression on the right-hand side of the equivalency in 1 is always translatable into a clear example of propositional negation.

We can now assert some relations between identity[40] and analogy. Thus:

$$\vdash (x)\,(y)\,((x = y) \supset (x \approx y)). \qquad \text{T-XVI}$$

Proof: Using Lemma I, we assert (1) $(x)\,(y)\,(x = y)$ with $(x)\,(y)\,(x \approx y)$ to be proved. By D-IV and E., 1 yields (2) $(x)\,(y)\,(P)\,(Px \equiv Py)$. 2 yields (3) $A\mathrm{a} \equiv A\mathrm{b}$ by U.I. This in turn yields (4) $(\exists P)\,(P\mathrm{a} \equiv P\mathrm{b})$ by E.G. By definition and E., 4 yields (5) $\mathrm{a} \approx \mathrm{b}$, which in turn yields (6) $(x)\,(y)\,(x \approx y)$ by U.G.

$$\vdash (x)\,(y)\,(x \not\approx y) \equiv (x \neq y). \qquad \text{T-XVII}$$

Proof: First T-XVII translates, by definition and E., into (1) $(x)\,(y)\,((\exists P) \sim (Px \equiv Py) \equiv \sim (P)(Px \equiv Py))$.[41] 1 yields (2) $(\exists P) \sim (P\mathrm{a} \equiv P\mathrm{a}) \equiv \sim (P)(P\mathrm{a} \equiv P\mathrm{b})$ by U.I. and this is patently valid.[42]

Because we wish to accurately express comparisons among different analogy connectives and in view of our definition of analogy, it now becomes important to be able to indicate precisely, but economically, the range of values permissible to a predicate variable such that the domain of the function in which the variable occurs is thereby clearly specified.[43] In order to accomplish the foregoing, we shall adopt the following convention:

$$(\exists P),\ (\exists^2 P),\ (\exists^3 P), \ldots,\ (\exists^h P),$$

where the right-hand superscript to any '$\exists$' may be any integer and where '$(\exists^2 P)P\mathrm{a}$' may be read "there are exactly two values of 'P' which take 'a' for an argument." At this point we introduce one of our symbols for comparison among analogical connectives, ':', by the following definition:

$$\mathrm{a} \approx_{\mathrm{P}} \mathrm{b} : \mathrm{c} \approx_{\mathrm{P}} \mathrm{b} =_{df} (\exists^i P)\,(P\mathrm{a} \equiv P\mathrm{b}) \cdot (\exists^h P)\,(P\mathrm{c} \equiv P\mathrm{b}) \cdot i > h,^{[44]} \qquad \text{D-VII}$$

which will yield by U.G. the theorem:

$$\vdash (x)\,(y)\,(z)\,((x \approx_{\mathrm{p}} y : z \approx_{\mathrm{p}} y) \equiv (\exists^{i} P)\,(Px \equiv Py) \cdot (\exists^{h} P)\,(Pz \equiv Py) \cdot i > h). \qquad \text{T-XVIII}$$

The transitivity of ':' may be stated thus:

$$\vdash (x)\,(y)\,(z)\,(w)\,(((x \approx_{\mathrm{p}} y : z \approx_{\mathrm{p}} y) \cdot (z \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y)) \supset (x \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y)). \qquad \text{T-XIX}$$

Proof: We proceed via C.P. by assuming (1) $(x)\,(y)\,(z)\,(x \approx_{\mathrm{p}} y : z \approx_{\mathrm{p}} y)$ and attempting to prove $(x)\,(y)\,(w)\,((x \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y) \cdot (x \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y))$. (2) $x \approx_{\mathrm{p}} y : z \approx_{\mathrm{p}} y$ is derived from 1 by U.I. (3) a) $z \approx_{\mathrm{p}} y$, b) $w \approx_{\mathrm{p}} y$, and c) $x \approx_{\mathrm{p}} y$ can all be derived from T-II by U.I. But, by T-VIII and A-I, 3 allows (4) $z \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y$ and (5) $x \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y$ to be derived from 2. Hence, (6) $(z \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y) \cdot (x \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y)$ by A, and this in turn yields 7) $(z)\,(y)\,(w)\,((x \approx_{\mathrm{p}} y : z \approx_{\mathrm{p}} y) \cdot (x \approx_{\mathrm{p}} y : w \approx_{\mathrm{p}} y))$.

The symbol '$.\dot{}.$' can be similarly introduced by definition:

$$\mathrm{a} \approx_{\mathrm{p}} \mathrm{b}\,.\dot{}.\,\mathrm{c} \approx_{\mathrm{p}} \mathrm{b} =_{df} (\exists^{i} P)\,(P\mathrm{a} \equiv P\mathrm{b}) \cdot (\exists^{h} P)\,(P\mathrm{b} \equiv P\mathrm{c}) \cdot i = h, \qquad \text{D-VIII}$$

which also yields, by U.G., the theorem:

$$\vdash (x)\,(y)\,(z)\,((x \approx_{\mathrm{p}} y\,.\dot{}.\,z \approx_{\mathrm{p}} y) \equiv (\exists^{i} P)\,(Px \equiv Py) \cdot (\exists^{h} P)\,(Py \equiv Pz) \cdot i = h). \qquad \text{T-XX}$$

A transitivity property can also be stated for '$.\dot{}.$' in the following way:

$$\vdash (x)\,(y)\,(z)\,(w)\,(((x \approx_{\mathrm{p}} y\,.\dot{}.\,z \approx_{\mathrm{n}} y) \cdot (z \approx_{\mathrm{p}} y\,.\dot{}.\,w \approx_{\mathrm{p}} y))) \supset (x \approx_{\mathrm{p}} y\,.\dot{}.\,w \approx_{\mathrm{p}} y), \qquad \text{T-XXI}$$

the proof of which is essentially the same as the one for T-XIX.

5. INTRODUCTION TO AN ANALOGICAL SET CALCULUS Up to this point we have described what seem to be the rudiments of a calculus of analogy for individuals. We wish now to extrapolate to the rudiments of a similar calculus for sets. In our list of primitive symbols we indicate that lower case Greek letters α, β, ... or ϕ, Ψ, ... denote sets. A la *Principia Mathematica*,[45] each such letter is translatable into $\hat{x}(\mathrm{A}x)$. But this allows us to define the connective 'ϵ'. Thus, where $\mathrm{b}\epsilon\mathrm{a}$ then

$$\mathrm{b}\,\epsilon\,\hat{x}(\mathrm{A}x) =_{df} \mathrm{Ab}. \qquad \text{D-IX}$$

Hence, any two sets 'α' and 'β' may be said to be analogous according to the following definition:

$$\alpha \approx \beta =_{df} (\exists x)\,(\exists P)\,(\exists Q)\,((x \epsilon \alpha \equiv Px) \cdot (x \epsilon \beta \equiv Qx) \cdot P = Q). \qquad \text{D-IX}$$

This is a somewhat restrictive definition in that (1) by virtue of its quantifiers, null sets are never analogous and (2) it permits the analogy relation to hold only among those sets in which the set properties[46] are complex such that P and Q are elements thereof. Provided that these semantical restrictions obtain, the definition of analogous sets may be abbreviated to:

$$\alpha \approx \beta =_{df} (\exists x)\,(x \epsilon \alpha \equiv x \epsilon \beta). \qquad \text{D-IX.I}$$

Such a definition with its restrictions may induce a number of negative reactions from philosophers of logic. However, since we are attempting neither to dissolve nor to resolve problems in the philosophy of logic we shall succumb to giving only a brief rationale for restriction 2. This restriction can be grasped from the consideration that the set of (1) 'green things' and the set of (2) 'red things' are not, *as they stand*, analogous sets. But the set of (3) 'blue and red things' and the set of (4) 'green and red things' are, *as they stand*, analogous because an element, i.e., 'red,' of each of their set properties is identical. Moreover, two sets (5) and (6) whose determining properties[47] are in both cases 'red,' 'green,' and 'blue' may be regarded as identical. Now, the expression "as they stand" used above is to be understood in the sense of "without reference to set inclusion." In other words, someone *might* regard 1 and 2 as analogous in terms of both being subsets of the set of (7) 'colored things,' whereas the analogy between 3 and 4 requires no such reference. Further, we note that our logic, which is obviously cast within the framework of type-theory, is prevented as much as possible from engendering paradoxes by having only properly stratified formulae[48] within it. This can be demonstrated by translating our definitions and theorems into primitive notation[49] and then submitting them to Quine's test.[50] Within the extensions of the foregoing provisos, we assert the theorem:

$$(\phi)\ (\Psi)\ (\phi \approx \Psi), \qquad \text{T-XXII}$$

which is the assertion of the universality of the analogy relation for sets.[51] We now assert the theorems:

$$\vdash (\phi)\ (\Psi)\ ((\phi \approx_{\mathrm{p}} \Psi) \equiv (\exists x)\ (x \epsilon \phi \equiv x \epsilon \Psi)) \qquad \text{T-XXIII}$$

$$\vdash (\exists \phi)\ (\exists \Psi)\ (\phi \approx_{\mathrm{p}} \Psi) \qquad \text{T-XXIV}$$

$$\vdash (\phi)\ (\Psi)\ ((\phi \approx_{\mathrm{p}} \Psi) \supset (\Psi \approx_{\mathrm{p}} \phi)) \qquad \text{T-XXV}$$

$$\vdash (\phi)\ (\Psi)\ (\mu)\ (((\phi \approx_{\mathrm{p}} \Psi) \cdot (\Psi \approx_{\mathrm{p}} \mu)) \supset (\phi \approx_{\mathrm{p}} \mu)) \qquad \text{T-XXVI}$$

the proofs of which proceed as those of respectively T-I, T-IV, T-VI, and T-VII. At this point analogy negation for sets is introduced by the following definition:

$$\alpha \not\approx \beta =_{df} (\exists x) \sim (x \epsilon \alpha \equiv x \epsilon \beta). \qquad \text{D-X}$$

The rationale for the choice of this definition parallels closely that for the choice of D-II.III, where conditions of consistency and conformity to "common sense" are the dicta.

We now turn our attention to the status of Boolean products relative to the analogy relation. Thus, where in classical set theory $\alpha \cap \beta =_{df} (\exists x)$ $(x \epsilon \alpha \cdot x \epsilon \beta)$ then

$$\vdash (\phi)\ (\Psi)\ ((\phi \approx_{\mathrm{p}} \Psi) \supset (\phi \cap \Psi)). \qquad \text{T-XXVII}$$

Proof: Proceeding via C.P., from T-XXII we derive (1) $\alpha \approx_{\mathrm{p}} \beta$ by U.I. 1 yields (2) $(\exists x)\ (x \epsilon \alpha \equiv x \epsilon \beta)$ by D-IX.I. Next, we derive (3) $a \epsilon \alpha \equiv a \epsilon \beta$ from

2 by E.I. We then derive (4) $(a\epsilon\alpha \supset a\epsilon\beta) \cdot (a\epsilon\beta \supset a\epsilon\alpha)$ from 3 by apf. We assume (5) $a\epsilon\alpha$. 4 yields (6) $a\epsilon\alpha \supset a\epsilon\beta$ by S. 6 and 5 yield (7) $a\epsilon\beta$ by M.P. 5 and 7 yield (8) $a\epsilon\alpha \cdot a\epsilon\beta$ by A. 8 yields (9) $(\exists x)(x\epsilon\alpha \cdot x\epsilon\beta)$ by E.G. This latter yields (10) $\alpha \cap \beta$ by definition (above), which in turn yields $(\phi)(\Psi)(\phi \cap \Psi)$ by U.G.

$$\vdash (\phi)(\Psi)(\sim(\phi \cap \Psi) \supset (\phi \not\approx \Psi)). \qquad \text{T-XXVIII}$$

Proof: We first instantiate T-XXVIII, above, to yield (1) $(\alpha \approx_p \beta) \supset (\alpha \cap \beta)$ by U.I. This in turn yields (2) $\sim(\alpha \cap \beta) \supset \sim(\alpha \approx_p \beta)$ by apf. But 2 yields (3) $\sim(\alpha \cap \beta) \supset (\alpha \not\approx_p \beta)$ by metatheorem II. 3 yields $(\phi)(\Psi)(\sim(\phi \cap \Psi) \supset (\phi \not\approx \Psi))$ by U.G.

In classical set theory, a distinction is usually made between an improper and a proper subset of any given set. Where the former is indicated by the symbol '$\subseteq$' and defined in the following way: A) $\alpha \subseteq \beta =_{df} (x)(x\epsilon\alpha \equiv x\epsilon\beta)$ then the theorem:

$$\vdash (\phi)(\Psi)((\phi \subseteq \Psi) \supset (\phi \approx \Psi)). \qquad \text{T-XXIX}$$

Proof: Using C.P. we universally instantiate the definiens of A above to obtain (1) $x\epsilon\alpha \equiv x\epsilon\beta$. Then, by applying E.G. to 1, we obtain $(\exists x)(x\epsilon\alpha \equiv x\epsilon\beta)$, which is the definiens of $\alpha \approx \beta$. But since the definiens may be substituted by E. for its corresponding definiendum, we have proven our theorem.

At this point, we assert the analogy relation between a set and its proper subset, which in classical set theory is usually indicated by the symbol '$\subset$' and defined in the following way:

$$\alpha \subset \beta =_{df} (x)(x\epsilon\alpha \supset x\epsilon\beta). \qquad \text{B}$$

$$\vdash (\phi)(\Psi)((\phi \subset \Psi) \supset (\phi \approx \Psi)). \qquad \text{T-XXX}$$

Proof: We first assert the definitional translation of T-XXIX, above: (1) $(x)(x\epsilon\beta \equiv x\epsilon\alpha) \supset (\exists x)(x\epsilon\beta \equiv x\epsilon\alpha)$. This yields (2) $(a\epsilon\beta \equiv a\epsilon\alpha) \supset (\exists x)(x\epsilon\beta \equiv x\epsilon\alpha)$ by U.I. From 2 we derive (3) $(a\epsilon\beta \supset a\epsilon\alpha) \cdot (a\epsilon\alpha \supset x\epsilon\beta) \supset (\exists x)(x\epsilon\beta \equiv x\epsilon\alpha)$ by apf. 3 yields (4) $(a\epsilon\alpha \supset a\epsilon\beta) \supset (\exists x)(a\epsilon\beta \equiv x\epsilon\alpha)$ by S. 4 yields (5) $(x)(x\epsilon\alpha \supset x\epsilon\beta) \supset (\exists x)(x\epsilon\beta \supset x\epsilon\alpha)$ which in turn yields, by definition B and E., (6) $(\alpha \subset \beta) \supset (\alpha \approx \beta)$. From 6 we derive (7) $(\phi)(\Psi)((\phi \subset \Psi) \supset (\Phi \approx \Psi))$.

It now becomes possible to assert

$$\vdash (\phi)(\mathrm{X})(\Psi)(((\phi \subset \mathrm{X}) \cdot (\mathrm{X} \subset \Psi)) \supset (\phi \approx \Psi)). \qquad \text{T-XXXI}$$

Proof: A la T-XXX, we translate each of the expressions in T-XXXI containing a '$\subset$' symbol into one containing an '$\approx$' symbol. Thus, universally instantiating and translating gives us the following result $((\alpha \approx \beta) \cdot (\beta \approx \gamma)) \supset (\alpha \approx \gamma)$, which reveals T-XXXI to be only a case of theorem T-XXVI. It should be noted here, however, that the statement $((\alpha \subset \beta) \cdot (\gamma \subset \beta)) \supset (\alpha \approx \gamma)$ does not hold, simply because there is no guarantee that the set properties by virtue of which α and β are subsets of γ overlap. With the assertion and proof of T-XXX and where $\alpha \cup \beta =_{df} (\exists x)(x\epsilon\alpha \vee x\epsilon\beta)$, it becomes possible to assert and prove:

$$\vdash (\phi)(\Psi)((\phi \cap \Psi) \approx (\phi \cup \Psi)). \qquad \text{T-XXXII}$$

Proof: Since, in classical set theory, for any intersect, '$\alpha \cap \beta$,' and its corresponding join, '$\alpha \cup \beta$,' the statement (1) $(\alpha \cap \beta) \subset (\alpha \cup \beta)$ holds then, by T-XXX, the statement (2) $(\alpha \cap \beta) \approx (\alpha \cup \beta)$ must also hold. This in turn yields (3) $(\phi)\ (\Psi)\ ((\phi \cap \Psi) \approx (\phi \cup \Psi))$ by U.G.

NOTES AND REFERENCES

1. D. C. Dorrough, *A Prolegomenon to Some Future Considerations of the Notion of Analogy.* University Microfilms, Inc., #65-9004, Ann Arbor, Michigan.

2. I. M. Bocheński, *On Analogy*, The Thomist, XI, No. 4 (1948).

3. Thomas Aquinas, *Questiones Disputate*, Vol. I: *De Veritate*, Rome (1949). Also, Thomas de Vio Caietanus, *De Nominum Analogia*, Scarzzini Edition, n.d.

4. Maria B. Hesse, *Models and Analogies in Science*, University of Notre Dame Press, Notre Dame, Indiana (1966). See especially "Material Analogy" and "The Logic of Analogy."

5. Where any qualification with respect to the values of variables in a proposition affects what that proposition is about rather than the syntax of the sentence making the proposition.

6. A. N. Whitehead and B. Russell, *Principia Mathematica*, Vol. I, University Press, Cambridge, England (1950). (2nd Edition),* **$\bar{2}5$**.

7. Such a reduction is not carried out here.

8. W. V. O. Quine, *Mathematical Logic*, Harvard University Press, Cambridge, Massachusetts (1958).

9. I. M. Copi, *Symbolic Logic*, The Macmillan Co., New York (1953). Cf. Second Edition.

10. *Ibid.*, pp. 226-236.

11. An informal demonstration of which can be found in D. Hilbert and W. Ackermann, *Principles of Mathematical Logic*, Chelsea Publishing Co., New York (1950).

12. See Reference 22.

13. Where $1 \leq j \leq t$.

14. Especially in a conditional proof.

15. E.g., *modus ponens* or any of the "natural deduction" rules.

16. For proof of this see Copi, *op. cit.*, pp. 274-276.

17. Evidence for this can be found in subsequent discussions.

18. As will be noted later, these are not to be thought of as mutually exclusive categories.

19. Here, the word 'sentential' is a predicate attached to any collocation of symbols which under usual conditions is used to assert a complete proposition.

20. Where 'things' may be applied to classes or particulars.

21. I.e., without change in meaning or truth value.

22. Where a particular is anything of which there can be no instances whereas an individual is anything whose name is substitutable for variables of the lowest type. Hence, the class of individuals has a greater extension than that of the class of particulars (cf. P. F. Strawson, "Logical Subjects and Physical Objects," *Philosophy and Phenomenological Research*, Vol. 17, pp. 441-457 (1956)).

23. Where $P_1 \equiv Q_1 =_{df} (P_1 \supset Q_1) \cdot (Q_1 \supset P_1)$ and where in turn $(P_1 \supset Q_1) =_{df} \sim (P_1 \cdot \sim Q_1)$.

24. Copi's formulation of Leibniz's principle, *op. cit.*, p. 161. Essentially similar definitions can be found in W. V. O. Quine, *Mathematical Logic*.

25. A. N. Whitehead and B. Russell, *op. cit.*

26. *Ibid.*, ***24** and ***25**, pp. 216-230.

27. Demonstration of this is as follows: If one assumes what in the body of this paper we shall subsequently prove, i.e. $(x)(x \approx x)$, then

    1) $x \approx x$
    2) $y \approx y$ } from assumption and Universal Instantiation
    3) $(x \approx x) \cdot (y \approx y)$ from 1 and 2 by Adjunction
    ∴ 4) Vxy from 3 by D-VI and the Principle of Extensionality.

    Further,

    5) $Vx \cdot Vy$ from 1 and 2 by D-V and Adjunction
    6) $(\exists P)(Px \cdot Py)$ from 5 by Existential Generalization

28. In no case will the modifications be so gross as to essentially change our definition.

29. *Op. cit.*

30. Where the existence of a relation can be asserted if, and only if, the existence of the two individuals between which the relation purportedly holds can be asserted.

31. '$(P \equiv Q)$' ≡ '$(P \supset Q) \cdot (Q \supset P)$'.

32. Where $(\iota x)\mathrm{F}x =_{df} (\exists x)\,(\mathrm{F}x \cdot (y)\,(\mathrm{F}y \supset y = x))$.

33. Though *perhaps* not always in practice.

34. Ample demonstration for this can be found in Quine, *op. cit.* or in Copi, *op. cit.*

35. Where '=' is as defined before.

36. Part of the proof of this metatheorem already occurs in the considerations immediately preceding it.

37. If 'P and Q' then '$P \cdot Q$'.

38. Sometimes referred to as the Principle of Exportation, which is a case of C.P. See Copi, *op. cit.*, p. 278.

39. The intended interpretation of $\mathbf{A.S.}_1$ is controlled by "common sense," the characteristics of which are perhaps defined by the explicit and/or tacit rules of use of ordinary language.

40. See our earlier definition of identity, D-III.

41. Where $x \neq y =_{df} \sim (P)\,(Px \equiv Py)$.

42. See D-I.

43. Where the distinction between 'range' and 'domain' is the usual one. Cf. M. Davis, *Computability and Unsolvability*, McGraw-Hill Book Co., New York (1958), XVff.

44. Where the definiendum of such a formula is read: "a is more analogous than b to c." Further, the symbols '<,' '>' are used in their usual way.

45. Whitehead and Russell, *op. cit.*, pp. 26 and 188.

46. Where 'set property' means that concept in terms of which no exemplification of it could fail to be a member of the given set and no member of the given set could fail to be an exemplification of it.

47. I.e., the properties which completely specify the set.

48. Where a formula "is stratified if it is possible to put numerals for its variables (the same numeral for all occurrences of the same variable) in such a way that 'ϵ' comes to be flanked always by consecutive ascending numerals ('$n \epsilon n+1$')." C (f. Quine, *op. cit.*, pp. 157-158).

49. I.e., wffs corresponding to the recursive definitions given at the beginning of our discussion.

50. Quine, *op. cit.*, p. 158.

51. Which corresponds to T-II.

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