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ON INFINITE MATRICES AND THE PARADOXES OF MATERIAL IMPLICATION

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The purpose of this note is to lend some justification to Łukasiewicz's claim that, when one has gone beyond 3-valued logics, the next interesting case is infinitely many-valued logics.[1]

One of the things that drives people into many-valued logics is the feeling of being 'cramped' in the classical, 2-valued logic. This feeling of being cramped particularly makes itself felt when we are working with the conditional, or implication, and to some extent furnishes an alibi for the paradoxes of material implication. We find ourselves saying something like this: "Ordinarily we would not say that a conditional with a false antecedent (or whatever) was true, but it is in a 2-valued logic . . . ." We feel forced to put different animals in the same cage. Propositions which strike us as being significantly different are assigned the same truth-value.

This does not occur with alternation, conjunction, or any of the commutative functors—those in which the order of the arguments is irrelevant to the truth-value of the function of which they are arguments. As can be seen, we have as many values, in such cases, as there are ways in which the arguments can differ in value. Consider these cases[2]:

2-valued	3-valued	n-valued
1. $p = q$	1. $p = q$	1. $p = q$
2. $p \langle 1 \rangle q$	2. $p \langle 1 \rangle q$	2. $p \langle 1 \rangle q$
	3. $p \langle 2 \rangle q$	3. $p \langle 2 \rangle q$
		n. $p \langle n - 1 \rangle q$

When the order of the arguments becomes important, however, as in the conditional, then if our choice of values for the function are to be based on the difference in value of the arguments, we are always going, in a finitely many-valued logic, to be short of function-values.

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2-valued

1. $p = q$
2. $p = q + 1$
3. $q = p + 1$

3-valued

1. $p = q$
2. $p = q + 1$
3. $p = q + 2$
4. $q = p + 1$
5. $q = p + 2$

5-valued

1. $p = q$
2. $p = q + 1$
3. $p = q + 2$
4. $p = q + 3$
5. $p = q + 4$
6. $q = p + 1$
7. $q = p + 2$
8. $q = p + 3$
9. $q = p + 4$

The ratio between truth-values available, and those needed to avoid cramping, can be seen to increase 2:3, 3:5, . . . , 5:9, . . . and in general can be determined by the following formula, where a is the number of values available, and b the number needed: $a + n : b + 2n$.

This difficulty can be overcome, however, if we take an infinitely many-valued logic. Let these infinitely many values be the even integers: 2, 4, 6, ... There are, of course, an infinite number of such integers. The values of a function, φ, for its arguments p and q which take the values 2, 4, . . . , will be indicated by (*but will not be*) the odd integers: 1, 3, 5, ... The following matrix, or any one of a number like it, can then be constructed:

φ	2	4	6	8	10	...
2	1	3	7	11	15	.
4	5	1	3	7	11	.
6	9	5	1	3	7	.
8	13	9	5	1	3	.
10	17	13	9	5	1	.
...	.	.	.	.	.	1

We can now take the function-valued (the odd integers) and, because they can be placed in a one-to-one correspondence with the even integers, each of them will determine a unique truth-value, and our cramp is relieved. We could even, if we so desired, assign each function-value a unique odd integer, and, thereby, a unique truth-value. This takes into consideration not just the order of the arguments and the difference in their values, but what those values are. The following matrix shows one way in which this might be done:

φ	2	4	6	...
2	1	$k + 2$	.	.
4	.	3	$k + 4$	.
6	.	.	5	.
...	.	.	.	k

FOOTNOTES

1. Jan Łukasiewicz, "Philosophical Remarks on Many-Valued Systems of Propositional Logic," trans. H. Weber, in *Polish Logic*, ed. Storrs McCall, Oxford (1967), pp. 60-61.

2. The symbol $\langle n \rangle$ means that either p is greater than q by n, or q is greater than p by n.

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