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AXIOMATIC QUASI-NATURAL DEDUCTION

JOHN R. GREGG

I. Two distinct methods are available for constructing proofs in quantification theory: deduction from axioms and "natural" deduction from premises. The first method has the formal advantage that each line of a proof is valid; but proofs turn less upon strategy than upon brute ingenuity, and often are hard to come by. The second method, generally speaking, is the more convenient and perspicuous, proofs turning more upon strategy and somewhat less upon insight; but these niceties are paid for, sometimes dearly, in more or less artificial restrictions upon proof format and upon the use of rules of inference.

The burden of this paper is to describe a new axiom system G; to show that adoption of some simple, transparent and purely typographical conventions yields a method of proof—called *axiomatic quasi-natural deduction*—in which are combined all advantages of deduction from axioms with those of deduction from premises; and to show that the system is both complete and sound in the sense that all and only valid quantificational formulae are among its theorems.

II. The primitives of G are the constant (inconsistency) 'f', sentence letters ('p', 'q', 'r', 's' and their subscripted variants), n-place predicate letters ('F^n', 'G^n', 'H^n' and their subscripted variants), variables ('w', 'x', 'y', 'z' and their subscripted variants), parentheses, the conditional sign '$\supset$' and the universal quantifier sign '$\forall$'.

The formulae of G are all and only expressions identified recursively by these rules: (i) 'f' is a formula, (ii) a sentence letter is a formula, (iii) an n-place predicate letter followed by a string of n variables is a formula, (iv) any result of putting a formula for 'p' and one for 'q' in '$(p \supset q)$' is a formula, (v) any result of putting a variable for 'x' and a formula for 'p' in '$\forall xp$' is a formula.

Henceforth, 'P', 'Q', 'R' and their subscripted variants will refer ambiguously to formulae, 'X' and 'Y' to variables. Thus, $(P \supset Q)$ is the conditional whose antecedent is P and whose consequent is Q, $\forall XP$ is the universal quantification of P with respect to X, and so on.

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An occurrence of X in P is free in P if it lies within no part of P of the form $\forall XQ$, else it is bound (not free) in P. X itself is free or bound in P according as it has free or bound occurrences therein. The formula that is like P except for having a free occurrence of Y at each place where P has a free occurrence of X will be called $(P{:}\,Y/X)$. If P lacks free occurrences of X, or if Y is X, $(P{:}\,Y/X)$ is P.

An expression of the form $(P_1 \supset (P_2 \supset \ldots \supset (P_n \supset \_\_) \ldots))$, $(n \geq 0)$, will be called a *context* with *antecedents* $P_1, P_2, \ldots, P_n$. Antecedents are to be distinguished from occurrences thereof; for example, the context '$(p \supset (q \supset (p \supset \_\_)))$' has two antecedents, one of them occurring twice. Ambiguous reference to contexts will be made with the help of the letter 'C' and its subscripted variants. If C_1 and C_2 are contexts, not necessarily distinct, '(C_1C_2)' will refer indiscriminately to contexts whose antecedents are just those of C_1 together with those of C_2. For example, if C_1 is $(Q \supset (Q \supset \_\_))$ and C_2 is $(P \supset \_\_)$, (C_1C_2) may be taken as $(P \supset (Q \supset \_\_))$ or as $(Q \supset (P \supset \_\_))$ or as $(P \supset (Q \supset (P \supset \_\_)))$ or as any of a denumerable infinity of further instances. The result of putting a formula P for the blank in a context C will be called CP. Obviously, if C has zero antecedents, CP is P.

Often a formula may be construed as being of any one of several forms expressible in contextual notation. Thus, $(P \supset (Q \supset R))$ may be construed as being of the form CR, taking C as $(P \supset (Q \supset \_\_))$; or as being of the form $C(Q \supset R)$, taking C as $(P \supset \_\_)$; or as being of the form $C(P \supset (Q \supset R))$, taking C as the context with zero antecedents. In all such cases, one is free to choose in accord with the matter at hand.

Parentheses sometimes will be dropped or supplanted by dots, the conventions being those of Quine.[1] Furthermore, superscripts on predicate letters will be omitted when no confusion threatens.

The axioms of G are all formulae of the form

2.1 $C(P \supset P)$.

There are four rules of inference; together with the axioms, two of them spell out the truth functional component of G, the others the logic of quantifiers.

2.2 *From* $C(P \supset Q. \supset \mathrm{f})$ *one may infer* CP.
2.3 *From* C_1P *and* $C_2(P \supset Q)$ *one may infer* $(C_1C_2)Q$.
2.4 *From* $C\forall XP$ *one may infer* $C(P{:}\,Y/X)$.
2.5 *From* $C(P{:}\,Y/X)$ *one may infer* $C\forall XP$, *provided that* Y *has no free occurrence in* $C\forall XP$.

A proof in G is a sequence of m formulae (1), (2), . . . , (m) such that, for each $k \leq m$, (k) is an axiom or (k) is inferred from some prior formula(e) by one of 2.2–2.5.

A formula P is provable in (is a theorem of) G if and only if there exist a proof in G whose terminal formula is P.

Familiar truth functional connectives other than '$\supset$', and existential quantifiers, are introduced by the definitions following.

2.6 $\sim P$ *for* $(P \supset \mathrm{f})$.
2.7 $(P.Q)$ *for* $(P \supset. Q \supset \mathrm{f}: \supset \mathrm{f})$.
2.8 $(P \vee Q)$ *for* $(P \supset \mathrm{f}. \supset Q)$.
2.9 $(P \equiv Q)$ *for* $(P \supset Q. \supset: Q \supset P. \supset \mathrm{f}.: \supset \mathrm{f})$.
2.10 $\exists XP$ *for* $(\forall X(P \supset \mathrm{f}) \supset \mathrm{f})$.

Definitional abbreviations notwithstanding, all formulae of G are of the form $(P_1 \supset (P_2 \supset \ldots \supset (P_n \supset Q) \ldots))$, $(n \geq 0)$. If some such formula is suspected to be a theorem, there are three major strategies available for proving it.

The first of these is: set down n axioms $(P_1 \supset P_1)$, $(P_2 \supset P_2)$, . . . , $(P_n \supset P_n)$; then try to derive the whole. The procedure is exemplified by the following proof of '$(p \supset :p \supset q. \supset q)$':

(1) $p \supset p$	2.1
(2) $p \supset q. \supset .p \supset q$	2.1
(3) $p \supset :p \supset q. \supset q$	(1), (2), 2.3

To see that the proof accords with the rules of inference, one may note that (1) is of the form C_1P, with C_1 as '$(p \supset \_\_)$' and P as 'p'; that (2) is of the form $C_2(P \supset Q)$, with C_2 as '$(p \supset q. \supset \_\_)$' and $(P \supset Q)$ as '$(p \supset q)$'; and that (3) is of the form $(C_1C_2)Q$, with (C_1C_2) as '$(p \supset :p \supset q. \supset \_\_)$' and Q as 'q'.

The second strategy is: set down n axioms as before, add a further axiom $(Q \supset \mathrm{f}. \supset .Q \supset \mathrm{f})$; then try to derive the whole. The method is illustrated by a proof of '$(p \vee q. \supset r: \supset .p \supset r)$'.

(1) $p \vee q. \supset r: \supset :p \vee q. \supset r$	2.1
(2) $p \supset p$	2.1
(3) $r \supset \mathrm{f}. \supset .r \supset \mathrm{f}$	2.1
(4) $p \vee q. \supset r: \supset :. r \supset \mathrm{f}. \supset :p \vee q. \supset \mathrm{f}$	(1), (3), 2.3
(5) $p \vee q. \supset r: \supset ::r \supset \mathrm{f}. \supset :. p \supset \mathrm{f}. \supset q: \supset \mathrm{f}$	(4), 2.8
(6) $p \vee q. \supset r: \supset :r \supset \mathrm{f}. \supset .p \supset \mathrm{f}$	(5), 2.2
(7) $p \vee q. \supset r: \supset :. p \supset :r \supset \mathrm{f}. \supset \mathrm{f}$	(2), (6), 2.3
(8) $p \vee q. \supset r: \supset .p \supset r$	(7), 2.2

The third strategy available to prove a suspected theorem P is this: set down the axiom $(P \supset \mathrm{f}. \supset .P \supset \mathrm{f})$, try to derive $(P \supset \mathrm{f}. \supset \mathrm{f})$, then apply 2.2 to obtain P. A proof of '$(\mathrm{f} \supset p)$' will illustrate.

(1) $\mathrm{f} \supset p. \supset \mathrm{f}: \supset :\mathrm{f} \supset p. \supset \mathrm{f}$	2.1
(2) $\mathrm{f} \supset p. \supset \mathrm{f}: \supset \mathrm{f}$	(1), 2.2
(3) $\mathrm{f} \supset p$	(2), 2.2

These examples have been deliberately kept simple, but readers who have grasped them are unlikely to have difficulty constructing some of greater complexity. Nevertheless, in view of the spare primitive machinery of G, there is no arguing against a stock of derived rules of inference as a source of convenient deductive power. Four such, presently to be useful, will now be established.

2.11 *From CP one may infer CP.*

(1)	CP	Suppose given
(2)	$C(P \supset P)$	2.1
(3)	CP	(1), (2), 2.3

The reader justifiably may argue that tacit use of 2.11 already has been made two proofs back in the passage from (4) to (5)—supported merely by citing 2.8. On another view, adopted here, 2.8 and other definitions explicitly sanction steps typified by that from (4) to (5).

2.12 *From* $C(P \supset Q. \supset R)$ *one may infer* $C(Q \supset R)$.

(1)	$C(P \supset Q. \supset R)$	Suppose given
(2)	$Q \supset R. \supset \mathrm{f}: \supset :Q \supset R. \supset \mathrm{f}$	2.1
(3)	$Q \supset R. \supset \mathrm{f}: \supset Q$	(2), 2.2
(4)	$P \supset .Q \supset Q$	2.1
(5)	$Q \supset P. \supset \mathrm{f}: \supset .P \supset Q$	(3), (4), 2.3
(6)	$C(Q \supset R. \supset \mathrm{f}: \supset R)$	(1), (5), 2.3
(7)	$Q \supset .R \supset R$	2.1
(8)	$C(Q \supset R. \supset \mathrm{f}: \supset .Q \supset R)$	(6), (7), 2.3
(9)	$C(Q \supset R. \supset \mathrm{f}: \supset \mathrm{f})$	(2), (8), 2.3
(10)	$C(Q \supset R)$	(9), 2.2

2.13 *From* $C(P{:}Y/X)$ *one may infer* $C\exists XP$.

(1)	$C(P{:}Y/X)$	Suppose given
(2)	$\exists XP \supset \mathrm{f}. \supset .\exists XP \supset \mathrm{f}$	2.1
(3)	$\exists XP \supset \mathrm{f}. \supset :\forall X(P \supset \mathrm{f}) \supset \mathrm{f}. \supset \mathrm{f}$	(2), 2.10
(4)	$\exists XP \supset \mathrm{f}. \supset \forall X(P \supset \mathrm{f})$	(3), 2.2
(5)	$\exists XP \supset \mathrm{f}. \supset .(P{:}Y/X) \supset \mathrm{f}$	(4), 2.4
(6)	$C(\exists XP \supset \mathrm{f}. \supset \mathrm{f})$	(1), (5), 2.3
(7)	$C\exists XP$	(6), 2.2

2.14 *From* $C_1\exists XP$ *and* $C_2((P{:}Y/X) \supset Q)$ *one may infer* $(C_1C_2)Q$, *provided that* Y *has no free occurrence in* Q *or in any antecedent of* C_2 *or in* $\exists XP$.

Assuming that Y meets the proviso on the rule:

(1)	$C_1\exists XP$	Suppose given
(2)	$C_2((P{:}Y/X) \supset Q)$	Suppose given
(3)	$Q \supset \mathrm{f}. \supset .Q \supset \mathrm{f}$	2.1
(4)	$C_2(Q \supset \mathrm{f}. \supset .(P{:}Y/X) \supset \mathrm{f})$	(2), (3), 2.3
(5)	$C_2(Q \supset \mathrm{f}. \supset \forall X(P \supset \mathrm{f})$	(4), 2.5
(6)	$C_1(\forall X(P \supset \mathrm{f}) \supset \mathrm{f})$	(1), 2.10
(7)	$(C_1C_2)(Q \supset \mathrm{f}. \supset \mathrm{f})$	(5), (6), 2.3
(8)	$(C_1C_2)Q$	(7), 2.2

Note that if Y were not as assumed, the passage from (4) to (5) would exemplify fallacious usage of 2.5.

Next, it will be shown that G may be worked straightforwardly into something that verges on being a system of natural deduction. The conventions required can be stated in full generality, but are as well conveyed by example.

First, consider the formula

(i) $(p \supset (q \supset (r \supset s)))$.

Nearly enough, it typifies the general form of formulae in G. Suppose that one agrees to write it in any of the following ways:

(ii)	p	$(q \supset (r \supset s))$
(iii)	p,q	$(r \supset s)$
(iv)	p,q,r	s

in which case, one may agree that (i)—(iv) are nothing more than typographically distinct versions of one and the same formula. In each instance, a sequence of zero or more formulae (a *prefix*) is peeled off to the left, leaving a formula (a *suffix*) off to the right. For example, 'p, q' is the prefix and '$(r \supset s)$' the suffix of (iii). Thus, the first convention may be stated as follows: *one may write any formula in prefix-suffix form.*

Second, consider any axiom of G; for example,

$$(p \supset (q \supset (r \supset r))).$$

By the convention above, it may be written in the form

$$p,q,r \quad r$$

in which there is an occurrence of the suffix among the formulae of the prefix. In practice, much writing is saved if one agrees to a second convention: *if an axiom in prefix-suffix form has an occurrence of the suffix among the formulae of the prefix, one may represent that occurrence by the numeral of the line in which the axiom is to appear; furthermore, one may preserve the numerical representation in the prefixes of succeeding lines.* Thus, if the axiom above is to appear as the third line of a proof, one may write it in the form

$$p,q,3 \quad r$$

in which '3' represents 'r'.

Two examples will suffice to exemplify use of these conventions. The first is a proof of '$(p \supset q. \supset r: \supset :s \supset .q \supset r)$'.

(1)	1	$p \supset q. \supset r$	2.1
(2)	2	$s \supset .q \supset r: \supset \mathrm{f}$	2.1
(3)	1	$q \supset r$	(1), 2.12
(4)	2	$q \supset r. \supset \mathrm{f}$	(2), 2.12
(5)	1,2	f	(3), (4), 2.3
(6)	1	$s \supset .q \supset r: \supset \mathrm{f}.: \supset \mathrm{f}$	(5), 2.11
(7)	1	$s \supset .q \supset r$	(6), 2.2

The proof is now complete: (7) is just a way of writing the theorem. Of course, another application of 2.11 would yield an eighth line displaying the theorem explicitly; but nothing further would be gained. The second example is a proof of '$(\exists y \forall x Fxy \supset \forall x \exists y Fxy)$'.

(1)	1	$\exists y \forall x Fxy$	2.1
(2)	2	$\forall x Fxy$	2.1
(3)	2	Fxy	(2), 2.4
(4)	2	$\exists y Fxy$	(3), 2.13
(5)		$\forall x Fxy \supset \exists y Fxy$	(4), 2.11
(6)	1	$\exists y Fxy$	(1), (5), 2.14
(7)	1	$\forall x \exists y Fxy$	(6), 2.5

Without instruction to the contrary, one viewing these proofs might well suppose them to be proofs by natural deduction; that 2.1 is a rule for introducing premises; that 2.11 is a rule of conditionalization; that the prefix of each line is a device for keeping track of the premises upon which the suffix depends; that but one of the fourteen lines is valid. The reader knows better. But some substance now attaches to the claim that axiomatic quasi-natural deduction combines the advantages of deduction from axioms with the advantages of deduction from premises.

III. Church's system F^1 is known to be complete in the sense that all valid quantificational formulae are among its theorems.[2] Therefore, to show that G is complete, it is sufficient to show that all theorems of F^1 are theorems of G.

The axioms of F^1 are given by the schemata following:

(i) $P \supset .Q \supset P$
(ii) $P \supset .Q \supset R : \supset : P \supset Q . \supset . P \supset R$
(iii) $\sim P \supset \sim Q . \supset . Q \supset P$
(iv) $\forall X(P \supset Q) \supset . P \supset \forall XQ$, *if X is not free in P.*
(v) $\forall XP \supset (P: Y/X)$

The rules of inference are:

(vi) *From P and* $(P \supset Q)$ *one may infer Q.*
(vii) *From P one may infer* $\forall XP$.

A proof in F^1 is a sequence of m formulae (1), (2), . . . , (m) such that, for each $k \leq m$, (k) is an axiom or (k) is inferred from some prior formula(e) by (vi) or by (vii). A formula P is provable in (is a theorem of) F^1 if and only if there exists a proof in F^1 whose terminal formula is P.

Suppose of a proof in F^1 that all lines prior to some arbitrarily chosen line (k) are provable in G. What is to be shown is that (k) is provable in G. There are seven cases to consider.

Case (i): (k) is an axiom ($P \supset .Q \supset P$). Then (k) is provable in G (using the shorthand introduced in the last section) as follows:

(1)	1	P	2.1
(2)	Q	$P \supset P$	2.1
(3)	1,Q	P	(1), (2), 2.3

Case (ii): (k) is an axiom ($P \supset .Q \supset R : \supset : P \supset Q . \supset . P \supset R$). Then ($k$) is provable in G, as follows:

(1)	1	$P \supset .Q \supset R$	2.1
(2)	2	$P \supset Q$	2.1
(3)	1,2	$P \supset R$	(1), (2), 2.4

Case (iii): (k) is an axiom ($\sim P \supset \sim Q. \supset .Q \supset P$). Then ($k$) is provable in G, as follows:

(1)	1	$\sim P \supset \sim Q$	2.1
(2)	1	$P \supset \mathrm{f}. \supset .Q \supset \mathrm{f}$	(1), 2.6
(3)	3	Q	2.1
(4)	1,3	$P \supset \mathrm{f}. \supset \mathrm{f}$	(2), (3), 2.3
(5)	1,3	P	(4), 2.2

Case (iv): (k) is an axiom ($\forall X(P \supset Q) \supset .P \supset \forall XQ$). By the proviso on (iv), X is not free in P. Then (k) is provable in G, as follows:

(1)	1	$\forall X(P \supset Q)$	2.1
(2)	1	$P \supset (Q{:}X/X)$	(1), 2.4
(3)	1	$P \supset \forall XQ$	(2), 2.5

Case (v): (k) is an axiom ($\forall XP \supset (P{:}Y/X)$). Then ($k$) is provable in G, as follows:

(1)	1	$\forall XP$	2.1
(2)	1	$(P{:}Y/X)$	(1), 4.2

Case (vi): (k) is Q and is inferred from prior lines P and ($P \supset Q$). By hypothesis, both P and ($P \supset Q$) are provable in G. Hence there exists a proof in G (a proof of P continued by a proof of ($P \supset Q$), say) in which both P and ($P \supset Q$) are lines; whence one may infer Q as a further line by 2.3. So (k) is provable in G.

Case (vii): (k) is $\forall XP$ and is inferred from a prior line P. By hypothesis, P is provable in G. Hence there exists a proof in G whose terminal line is P—that is, whose terminal line is ($P{:}X/X$)—whence one may infer $\forall XP$ as a further line by 2.5 So (k) is provable in G.

Thus, in all cases, (k) is provable in G. But (k) was any line of a proof in F^1. That all formulae provable in F^1 are provable in G follows by course of values induction.

What has been shown is that all theorems of F^1 are theorems of G. But all valid quantificational formulae are theorems of F^1, hence all valid quantificational formulae are theorems of G. That is, G is complete.

IV. In showing that all formulae provable in G are valid, free use will be made of well known laws of validity, implication and equivalence.[3] An auxiliary notation will be useful: if C is a context with more than zero antecedents, '$*C$' will refer ambiguously to conjunctions of all and only antecedents of C; otherwise, '$*C$' will refer to '$(p \supset p)$'. Thus, by laws alluded to above, CP is equivalent to ($*C \supset P$) and $*(C_1C_2)$ is equivalent to ($*C_1.*C_2$).

Suppose of a proof in G that all lines prior to some arbitrarily chosen line (k) are valid. What is to be shown is that (k) is valid. There are five cases to consider.

Case (i): (k) is an axiom $C(P \supset P)$. But $(P \supset P)$ is valid, so is implied by any formula; in particular, by $*C$. That is, $(*C \supset (P \supset P))$ is valid; hence so is its equivalent $C(P \supset P)$. That is, (k) is valid.
Case (ii): (k) is CP and is inferred from a prior line $C(P \supset Q . \supset \mathrm{f})$. By hypothesis, $C(P \supset Q . \supset \mathrm{f})$ is valid, hence so is its equivalent $(*C \supset :P \supset Q . \supset \mathrm{f})$. That is, $*C$ implies $(P \supset Q . \supset \mathrm{f})$. But $(P \supset Q . \supset \mathrm{f})$ implies P, so $*C$ implies P. That is, $(*C \supset P)$ is valid. Hence so is its equivalent CP. That is, (k) is valid. Case (iii): (k) is $(C_1C_2)Q$ and is inferred from prior lines C_1P and $C_2(P \supset Q)$. By hypothesis, C_1P and $C_2(P \supset Q)$ are both valid; hence their respective equivalents $(*C_1 \supset P)$ and $(*C_2 \supset .P \supset Q)$ are both valid. That is, $*C_1$ implies P and $*C_2$ implies $(P \supset Q)$. Hence $(*C_1 . *C_2)$, also its equivalent $*(C_1C_2)$, implies $(P . (P \supset Q))$. But $(P . (P \supset Q))$ implies Q, so $*(C_1C_2)$ implies Q. That is, $(*(C_1C_2) \supset Q)$ is valid. Hence so is its equivalent $(C_1C_2)Q$. That is, (k) is valid.
Case (iv): (k) is $C(P{:}Y/X)$ and is inferred from a prior line $C\forall XP$. By hypothesis, $C\forall XP$ is valid, hence so is its equivalent $(*C \supset \forall XP))$. That is, $*C$ implies $\forall XP$. But $\forall XP$ implies $(P{:}Y/X)$, so $*C$ implies $(P{:}Y/X)$. That is, $(*C \supset (P{:}Y/X))$ is valid, hence so is its equivalent $C(P{:}Y/X)$. That is, (k) is valid.
Case (v): (k) is $C\forall XP$ and is inferred from a prior line $C(P{:}Y/X)$ by 2.5. By the proviso on 2.5, Y has no free occurrence in $C\forall XP$, hence none in P nor in any antecedent of C nor in any conjunct of $*C$. By hypothesis, $C(P{:}Y/X)$ is valid; hence so is its equivalent $(*C \supset (P{:}Y/X))$, hence so is $\forall Y(*C \supset (P{:}Y/X))$. But the latter implies $(*C \supset \forall Y(P{:}Y/X))$, which is equivlent to $(*C \supset \forall XP)$; so $(*C \supset \forall XP)$, also its equivalent $C\forall XP$, is valid. That is, (k) is valid.

Thus, in all cases, (k) is valid. But (k) was any line of a proof in G. That all formulae provable in G are valid follows by course of values induction. That is, G is sound.

FOOTNOTES

1. W. V. Quine, *Mathematical Logic*, rev. ed., Harvard University Press, Cambridge, Massachusetts (1951).
2. Alonzo Church, *Introduction to Mathematical Logic*, vol. 1, Princeton University Press, Princeton, New Jersey (1956).
3. E.g., see: W. V. Quine, *Methods of Logic*, rev. ed., Holt, Rinehart and Winston, New York, New York (1959).

Duke University
Durham, North Carolina