GENERALIZABILITY OF THE PROPOSITIONAL AND PREDICATE CALCULI TO INFINITE-VALUED CALCULI

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INTRODUCTION: A logical calculus which is to be applied in the development of axiomatic systems must provide an adequate vocabulary of defined logical terms which, when incorporated in suitable theorems, will permit the explication of the needed logical concepts. Some authors, e.g. Woodger [7] and Carnap [1], have utilized for this purpose modifications of Whitehead and Russell's [6] extensive list of defined terms. An infinite-valued logic intended for similar applications must likewise have an adequate vocabulary of complex logical terms. Such a vocabulary may be readily devised by generalizing the definitions of PM [6] (or some other sufficiently complex two-valued logical system) provided that the theorems in which the defined terms occur can all be generalized to theorems in the new logic. The question thus arises as to whether there is any infinite-valued logic in which every theorem of PM has a valid generalization. More precisely: Is there an infinite-valued logic into which PM generalizes in the sense defined below in 1.18? The principle theses of the present paper answer this question in the affirmative for the propositional and first-order pure functional calculi in terms of an infinite-valued logic in which two binary connectives represent each binary connective of PM (equivalence excluded). The question of the generalizability of that part of PM which utilizes higher functional calculi is left open.

1. GENERALIZING AND DEGENERATING CALCULI.

1.1 Explication. A language is taken to be a set of wff defined by a given set of symbols and rules of formation.

1.2 Explication. The symbols of a language constitute a mutually exclusive set of sets of symbol tokens in that language. A token of a symbol a is an element of the set a.

1.3 Definition. A token in A is a symbol token in language A or an unbroken, finite, ordered sequence the terms of which are symbol tokens in A.

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1.4 Definition. A token in $A$ is similar to another token in $A$ iff, for each symbol token of $A$ in the one there is a token of the same symbol of $A$ in the other.

1.5 Definition. A formula in $A$ is a set each element of which is a token in $A$ which is similar in $A$ to every other element of the set and not similar in $A$ to any token in $A$ which is not an element of the set. It follows that the symbols of $A$ constitute a subset of the formulae in $A$. A token of a formula is an element of the set which constitutes the formula.

Note: When no ambiguity is likely to result repeated references to the language, $A$ in the preceding definitions, are generally omitted.

1.6 Definition. A token $x$ is part of a token $y$ iff $x$ is identical with $y$, a term of $y$ or a subsequence of $y$.

1.7 Definition. A token $x$ occurs at position $n$ in a token $y$ iff $x$ is part of $y$ and the first term of $x$ is the $n$th term of $y$.

1.8 Definition. A formula $a$ occurs at position $n$ in a formula $\beta$ iff a token of $a$ occurs at position $n$ in each token of $\beta$.

1.9 Definition. A formula $a$ occurs in $\beta$ iff for some $n$ a occurs at position $n$ in $\beta$.

1.10 Definition. The occurrence of $a$ in $\beta$ at position $n$ is the set $y$ of tokens of $a$ such that each member of $y$ occurs at position $n$ in a token of $\beta$.

1.11 Definition. A formula $a$ is the result of replacing the occurrences of formula $\delta$ at positions $n_1, n_2, \ldots, n_k$ in $\beta$ by occurrences of $\gamma$ iff there are tokens $x$ of $a$, $y$ of $\beta$, $v_1, v_2, \ldots, v_k$ of $\gamma$, and $w_1, w_2, \ldots, w_k$ of $\delta$ such that for $1 \leq i \leq k$ each $v_i$ occurs at position $m_i$ in $x$ and each $w_i$ occurs at position $n_i$ of $y$ and those parts of $x$ of which no part of any $v_i$ is a part are similar, in the order in which they occur to those parts of $y$ of which no part of any $w_i$ is a part.

1.12 Definition. A formula $a$ consists of $\beta_1, \beta_2, \ldots, \beta_n$ iff the $\beta_i$ are occurrences of formulae in $a$ such that each token of $a$ is the sum, in the sense of PM*160 and/or *161, of $n$ tokens one selected from each of the $\beta_i$ in the order listed.

1.13 Definition. Formulas $a$ and $\beta$ occur in corresponding positions in $\gamma$ and $\delta$ iff there is a number $n$ such that $a$ occurs at position $n$ in $\gamma$ and $\beta$ occurs at position $n$ in $\delta$.

1.14 Definition. A token is wf in $A$ iff it is a token of $A$ constructed in accordance with the rules of formation of $A$. A formula $a$ is wf in $A$ iff its tokens of $A$ are wf. A wff of $A$ is a formula $a$ which is wf in $A$.

1.15 Definition. A language $B$ is a generalization of a language $A$ iff there is a mapping of $A$ onto $B$ such that: (i) every symbol in $B$ is an image of one and only one symbol in $A$ and (ii) for every wff $Q$ of $B$ there is one and only one wff $P$ of $A$ such that each symbol in $B$ which occurs in $Q$ is an image of a symbol in $A$ occurring in a corresponding position in $P$. 

1.16 Definition. A calculus $X$ in a language $\mathcal{L}$ is a set of wff of $\mathcal{L}$, the theorems of $X$, which is completely specified by (i) a finite set of schemes the union of the instances of which is a subset of theorems, the axioms of $X$, and (ii) one or more rules of inference of $X$ whereby new theorems of $X$ can be obtained from established theorems.

1.17 Definition. A calculus $Y$ in language $\mathcal{K}$ degenerates into a calculus $X$ in language $\mathcal{L}$ iff $\mathcal{K}$ is a generalization of $\mathcal{L}$ by a mapping such that for every theorem $Q$ of $Y$ there is a theorem $P$ of $X$ such that each symbol which occurs in $Q$ is an image of a symbol occurring in a corresponding position in $P$.

1.18 Definition. A calculus $X$ in language $\mathcal{L}$ generalizes into a calculus $Y$ in language $\mathcal{K}$ iff $\mathcal{K}$ is a generalization of $\mathcal{L}$ and by a mapping such that for every theorem $P$ of $X$ there is a theorem $Q$ of $Y$ such that each symbol which occurs in $Q$ is an image of a symbol occurring in a corresponding position in $P$.

2. CHARACTERIZATION OF THE LANGUAGES.

The languages and the calculi which are being compared are, for clarity as well as brevity, displayed wherever practical in parallel columns with the generalized language or calculus on the right. Statements about the languages or the calculi are frequently arranged in the form of single sentences with such alternate or additional words as apply specifically to the generalized language or calculus appearing within parentheses.

2.1 Designation. Prototype tokens of the logical symbols of language $\mathcal{S} (\mathcal{S}_\mathcal{G})$ are listed below followed by the terms used to designate the symbol occurrences. The same terms written with initial capitals are used to designate the symbols themselves. Thus Implicator is a symbol of $\mathcal{S}$. Its tokens resemble the horseshoe-shaped mark appearing below. Each of its occurrences in a formula is called an implicator. In fact, Implicator is the union of the set of implicators.

<table>
<thead>
<tr>
<th>$\mathcal{S}$</th>
<th>$\mathcal{S}_\mathcal{G}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sim$ negator</td>
<td>$\neg$ negator</td>
</tr>
<tr>
<td>. dot</td>
<td>. dot</td>
</tr>
<tr>
<td>$\sqsubset$ implicator</td>
<td>$\rightarrow$ strong implicator</td>
</tr>
<tr>
<td>$\forall$ disjunctors</td>
<td>$\exists$ weak implicator</td>
</tr>
<tr>
<td>$&amp;$ conjunctors</td>
<td>$\lor$ strong disjunctors</td>
</tr>
<tr>
<td></td>
<td>$\lor$ weak disjunctors</td>
</tr>
<tr>
<td></td>
<td>$\land$ strong conjunctors</td>
</tr>
<tr>
<td></td>
<td>$\land$ weak conjunctors</td>
</tr>
</tbody>
</table>

The logical symbols of $\mathcal{S} (\mathcal{S}_\mathcal{G})$ other than Dot and Negator constitute the binary connectives of $\mathcal{S} (\mathcal{S}_\mathcal{G})$.

2.2 Designation. The nonlogical symbols of $\mathcal{S} (\mathcal{S}_\mathcal{G})$ are an unspecified number of propositional variables (the same in both languages).
2.3 Definition. The rules of formation of \( S(G) \) are the following:

(i) Propositional variables are \( \text{wff} \) and constitute the elementary \( \text{wffs} \).
(ii) A formula consisting of a negator followed by a \( \text{wff} \) occurrence is \( \text{wff} \) and constitutes the negation of the \( \text{wff} \).
(iii) A formula consisting of a dot followed by two \( \text{wff} \) occurrences separated by a (strong/weak) implicator is \( \text{wff} \) and constitutes the (strong/weak) implication of the \( \text{wff} \) occurring second by the \( \text{wff} \) occurring first.
(iv) A formula consisting of a dot followed by two \( \text{wff} \) occurrences separated by a (strong/weak) disjunctior is \( \text{wff} \) and constitutes the (strong/weak) disjunction of the \( \text{wff} \) occurring first with the \( \text{wff} \) occurring second.
(v) A formula consisting of a dot followed by two \( \text{wff} \) occurrences separated by a (strong/weak) conjunctior is \( \text{wff} \) and constitutes the (strong/weak) conjunction of the \( \text{wff} \) occurring first with the \( \text{wff} \) occurring second.

2.4 Definition. The term junction will be used as a metalanguage variable ranging over the designate of the six terms: (strong/weak) disjunction, (strong/weak) conjunction. As with other variables, when junction is used more than once in a single sentence form the intended sentences are obtained by substituting the same term for each such use.

2.5 Definition. (i) If \( P \) is a \( \text{wff} \) of \( S(G) \) then the junction of \( P \) is \( P \).
(ii) If \( P_1, P_2, \ldots, P_n \) are \( \text{wff} \) of \( S(G) \) then the junction of \( P_1, P_2, \ldots, P_n \) is the junction of \( P_1 \) with the junction of \( P_2, \ldots, P_n \).

2.6 Designation. The logical symbols of language \( S(G) \) are the logical symbols of \( S(G) \) plus the following:

\[ S(G) \]

\[ \exists \text{ existential quantifier} \]
\[ \forall \text{ universal operator} \]
\[ \forall \text{ universal operator} \]

Note: Since each of the operators is represented by a single operator in \( S(G) \) the modifiers, enclosed in brackets above, are generally omitted.

2.7 Designation. The nonlogical symbols of language \( S(G) \) are:

(i) the propositional variables of \( S(G) \);
(ii) a set of individual variables;
(iii) for each positive integer \( n \) less than some unspecified number, a set of \( n \)-place predicate variables.

2.8 Definition: A formula in \( S(G) \) is an existential quantifier or a universal quantifier, respectively, \( \text{iff} \) it consists of an existential operator followed by an occurrence of an individual variable.

2.9 Definition. The rules of formation for \( S(G) \) are the following:

(i) to (v): a reiteration of 2.3 (i) to (v);
(vi) A formula which consists of the occurrence of an \( n \)-place predicate variable followed by \( n \) occurrences of individual variables is \( \text{wff} \) and is an elementary \( \text{wff} \).
A formula consisting of the occurrence of an existential or universal quantifier followed by the occurrence of a wff is wf.

2.10 Definition. An occurrence β is the scope of α in a wff γ of $S_j(S_{jG})$ iff α is an occurrence of a universal or an existential quantifier and γ consists of α followed by β.

2.11 Definition. A free occurrence of an individual variable α in a formula β is an occurrence of α in some formula γ which occurs in β but not in a quantifier nor in the scope any quantifier in which β occurs.

2.12 Definition. A bound occurrence of an individual variable α in a formula β is an occurrence of α in the scope of a quantifier which occurs in β and in which α occurs.

2.13 Definition. An individual variable α occurs free in a formula β iff there is a free occurrence of α in β.

2.14 Definition. An individual variable α occurs bound in a formula β iff there is a bound occurrence of α in β.

2.15 Definition. The relation represents performs a many-one mapping of the symbols of S and $S_j$ onto the symbols of $S_G$ and $S_{jG}$, respectively, in the following way:

(i) Each nonlogical symbol of $S_G$ or $S_{jG}$ represents a typographically similar symbol of S or, respectively, $S_G$;
(ii) Negator represents Negator;
(iii) Strong Implicator and Weak Implicator represent Implicator;
(iv) Strong Disjunctor and Weak Disjunctor represent Disjunctor;
(v) Strong Conjunctor and Weak Conjunctor represent Conjunctor;
(vi) Dot represents Dot;
(vii) [Strong] Existential Operator represents Existential Operator;
(ix) [Weak] Universal Operator represents Universal Operator.

2.16 Definition. A formula β in $S_G$ or in $S_{jG}$ is said to be a generalization of a formula α in S or in $S_j$ iff each symbol occurring in β represents a symbol occurring in α in a corresponding position.

2.17 Theorem. $S_G$ is a generalization of S and $S_{jG}$ is a generalization of $S_j$.

Proof: Represents, as defined in 2.15, satisfies the requirements of 1.15(i) while generalization of, as defined in 2.16, together with the rules of formation for S and $S_G$ as set forth in 2.3 and those for $S_j$ and $S_{jG}$ as set forth in 2.9 satisfy the requirements of 1.15(ii).

2.18 Conventions. In such metalanguage formulations as axiom and theorem schemes the symbols P, Q, and R with and without subscripts and the formulae $F(x)$, $F(y)$, $G(x)$, are used as metavariables, i.e. variables of the metalanguage, ranging over wff of S, $S_j$, $S_G$, and $S_{jG}$ while the symbols x, y, and z with and without subscripts are used as metavariables ranging
over individual variables. The logical symbols of $S$, etc. are designated by symbols of like design, however, a pair of successive dots is sometimes designated by a colon and a sequence of $n$ dots by a single dot with the numeral $n$ as a superscript.

2.19 Definition. A wff $P$ is a junction iff there are wffs $Q$ and $R$ such that $P$ is the junction of $Q$ with $R$.

3. THE CALCULI.

3.1 Definition. The axioms of calculus $\mathcal{F}(\mathcal{F}_0)$ are all of the wffs of $S(S^\alpha_0)$ which are instances of the axiom schemes A1 through A8b (GA1 through GA8b).

3.2 Definition. The axioms of calculus $\mathcal{F}\mathcal{E}(\mathcal{F}_0)$ are all of the wffs of $S^\alpha(S^\alpha_0)$ which are instances of the axiom schemes A1 through A14b (GA1 through GA14b).

Note: The axiom schemes are selected and arranged so that the instances of each scheme in the right hand column are generalizations of the instances of the scheme appearing opposite it in the left hand column. The axioms are not all independent and those schemes in the left hand column marked with asterisks are readily derivable from the others.

3.3 Definition. The rule of inference for calculus $\mathcal{F}(\mathcal{F}_0)$ is the following:

R1 (GR1). If $P$ is a theorem and the implication (weak implication) of $Q$ by $P$ is a theorem, then $Q$ is a theorem.

3.4 Definition. The rules of inference for calculus $\mathcal{F}\mathcal{E}(\mathcal{F}_0)$ are the following:

R1 (GR1). If $P$ is a theorem and the implication (weak implication) of $Q$ by $P$ is a theorem, then $Q$ is a theorem.

R2 (GR2). If $P$ is a theorem then any wff of $S^\alpha(S^\alpha_0)$ consisting of an occurrence of a universal quantifier followed by an occurrence of $P$ is a theorem.

R3 (GR3). Let $P_i$ be, for $1 \leq i \leq n$, the result of replacing all occurrences of the individual variable $x$ in a wff $P$ of $S^\alpha(S^\alpha_0)$ by occurrences of the variable $y_i$, and let $y_1, y_2, \ldots, y_n$ be all of the individual variable occurring free in $P$, then, if $\exists x P(\forall x P)$ is a theorem, there is an $i$, $1 \leq i \leq n$, such that $P_i$ is a theorem.

3.5 Designation. The axiom schemes referred to in definitions 3.1 and 3.2 are listed below:

<table>
<thead>
<tr>
<th>Axiom Schemes for $\mathcal{F}$ and $\mathcal{F}_3$</th>
<th>Axiom Schemes for $\mathcal{F}_0$ and $\mathcal{F}_3\mathcal{F}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1 : $P \supset Q \supset P$</td>
<td>GA1 : $P \rightarrow Q \rightarrow P$</td>
</tr>
<tr>
<td>A2 : $P \supset Q \supset :Q \supset R \supset :P \supset R$</td>
<td>GA2 : $P \rightarrow Q \rightarrow :Q \rightarrow R \rightarrow :P \rightarrow R$</td>
</tr>
<tr>
<td>A3a : $P \lor Q \supset :P \supset Q \supset Q$</td>
<td>GA3a : $P \lor Q \supset :P \rightarrow Q \rightarrow :Q \lor P$</td>
</tr>
<tr>
<td>A3b : $P \supset Q \supset :Q \supset :P \lor P$</td>
<td>GA3b : $P \supset Q \rightarrow :Q \rightarrow :Q \lor P$</td>
</tr>
</tbody>
</table>
A4 \quad P \supset Q \supset .Q \supset P

* A5a \quad P \supset Q \supset .P \lor Q

* A5b \quad P \lor Q \supset .P \lor Q

* A6a \quad P \lor Q \supset .P \lor Q

* A6b \quad P \supset Q \supset .P \lor Q

A7a \quad P \& Q \supset .P \lor Q

A7b \quad .P \lor Q \supset P \& Q

* A8a \quad P \& Q \supset .P \lor Q

* A8b \quad P \lor Q \supset .P \& Q

Axiom Schemes for \( \mathfrak{I}_6 \)

A9 \quad .F(y) \supset \exists x F(x)

Axiom Schemes for \( \mathfrak{I}_{36} \)

A9 \quad .F(y) \rightarrow \forall x F(x)

where \( F(y) \) is the result of replacing all free occurrences of \( x \) in \( F(x) \) by occurrences of \( y \) and no free occurrence of \( x \) in \( F(x) \) is in the scope of any quantifier in which \( y \) occurs.

A10 \quad .\exists x F(x) \supset \exists y F(y)

where \( F(y) \) is the result of replacing all free occurrences of \( x \) in \( F(x) \) by occurrences of \( y \) and \( F(x) \) is the result of replacing all free occurrences of \( y \) in \( F(y) \) by occurrences of \( x \).

A11 \quad \forall x .P \supset F(x) \supset .P \lor \forall x F(x)

where \( x \) does not occur free in \( P \).

* A12 \quad P \supset \exists x F(x) \supset \exists x .P \lor F(x)

where \( x \) does not occur free in \( P \).

* A13 \quad \exists x F(x) \& \exists x F(x) \supset \exists x .F(x) \& F(x)

A14a \quad \exists x F(x) \supset \forall x \sim P

A14b \quad .\forall x \sim P \supset \exists x F(x)

3.6 Remark. A comparison of the axiom schemes given above for \( \mathfrak{I}_6 \) and \( \mathfrak{I}_{36} \) with the axiom schemes and definitions of Rose and Rosser [5] and of Hay [4] reveals a correspondence between symbols indicated in the following table:

<table>
<thead>
<tr>
<th>( \mathfrak{I}<em>6 ) and ( \mathfrak{I}</em>{36} )</th>
<th>( \sim )</th>
<th>( + )</th>
<th>( \land )</th>
<th>( \lor )</th>
<th>( \forall )</th>
<th>( \exists )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rose and Rosser</td>
<td>C</td>
<td>AN</td>
<td>A</td>
<td>B</td>
<td>K</td>
<td>L</td>
</tr>
<tr>
<td>Hay</td>
<td>( \supset )</td>
<td>( + )</td>
<td>( \land )</td>
<td>( \lor )</td>
<td>( \forall )</td>
<td>( \exists )</td>
</tr>
</tbody>
</table>

Our Strong Implicator (\( \sim \)), for which the cited authors have no special symbol, corresponds to the \( C’ \) of Dienes [3]. The rules of formation of the Polish notation which is used by Rose and Rosser require that the capital letters representing binary connectives occur in the positions assigned to dots in \( S_6 \) and \( S_{36} \).

3.7 Citation. References to [6], [2], [5] and [4] will be cited by use of the authors’ original numbers with the bold prefixes PM, C, R and H, respectively.
3.8 Theorem. *Calci J and Jις each contain the statement calculus R2 and R3.\*

*Proof:* A1, A2, A4, R1 (GA1, GA2, GA4, GR1) correspond to RA1, RA2, RA4, R Rule C, respectively, while a theorem corresponding to RA3 is derivable from A2, A3a and A3b by R1 (from GA2, GA3a and GA3b by GR1). Because of this metatheorem, theorems of R2 and R3 will frequently be cited in proof of theorems of J(JI) and Jς(Jις).\*

3.9 Theorem. *Calculus J contains the classical two-valued calculus.*

*Proof:* The three Łukasiewicz axiom schemes for the two-valued propositional calculus are (1) R3.32, (2) A1, (3) a theorem scheme derivable from A6b, A2 and R2.16.

3.10 Theorem. *Conversely, calculus J is contained in the classical two-valued calculus.*

*Proof:* The axiom schemes of J can be derived from any of the equivalent sets of axioms of the classical two-valued propositional calculus, e.g. the axiom schemes A1 to A8b correspond, respectively, to the following theorems of PM: *2.02, *2.06, *1.62, a theorem derivable from *2.06, *2.68, *1.4, *2.17, *1.01, *1.01, *2.53, *2.54, *3.1, *3.11, *4.63 and *4.63.

3.11 Theorem. *Calculus Jις is contained in the infinite-valued statement calculus of R2.3.*

*Proof:* All of the axioms of Jις correspond to axioms or definitions of R excepting GA3a,b which follow from RA3 and the definition of A, and GA5a,b which introduce a new connective. Complete correspondence is obtained if a new symbol say D is introduced into R which is defined as AN and hence corresponds to our Strong Implicator.


*Proof:* Calculus CFlp has in addition to the schemes embodying the propositional calculus only the axiom schemes C*305, which is A11, and C*306, which is readily derived from A9, R3.4, A2, and A4 by R1.

3.13 Remark. *All of the axioms of Jς are theorems of CFlp:* Axiom schemes A9 to A13 correspond respectively to C*330, C*378, C*305, C*382 and an instance of the law of conjunctive tautology combined with C*301 and C34.5. However, no rule equivalent to R3, which might be called the rule of exhaustive instantiation, appears in Church's formulation. This rule is only applicable to theorems containing neither individual nor functional constants, hence we compare $S_\exists$ to CFlp rather than to CFl.

3.14 Remark. The condition in R3 that all of the free variables of $\exists x P$ be among the $y_i$ is necessitated by A9, and the condition that just one additional variable $x$ be among the $y_i$ suffices because of R2.
3.15 Theorem. The generalized functional calculus \( \mathfrak{L}_{g6} \) contains the infinite-valued predicate calculus axiomatized by Hay [4].

Proof: The propositional calculus and auxiliary definitions of \( \mathbf{H} \) are equivalent to those of \( \mathbf{R} \) and hence by 3.8 and 3.11 to those of \( \mathfrak{L}_{g6} \). The axiom schemes HA5, HA6, HA7 and HA9 correspond, respectively, to GA13, GA9, GA10 and GA12. Axiom scheme HA8 corresponds to the lemma L4, below, which is derivable in \( \mathfrak{L}_{g6} \) as follows: (The citations within parentheses are to proof schemes, the others to axiom or theorem schemes needed in the proofs)

L1. \( \forall x F(x) \rightarrow F(y) \)  
L2. \( \forall x . F(x) \rightarrow Q \rightarrow \forall x F(x) \rightarrow Q \)  
L3. \( \forall x . F(x) \rightarrow G(x) \rightarrow \forall x F(x) \rightarrow \forall x G(x) \)  
L4. \( \forall x . F(x) \rightarrow Q \rightarrow \forall x F(x) \rightarrow Q \)  

where \( x \) does not occur free in \( Q \). Rules H1 and H2 correspond to GR1 and GR2.

3.16 Remark. Proof that the calculus of \( \mathbf{H} \) is contained in \( \mathfrak{L}_{g6} \) requires a derivation of GA11 in \( \mathbf{H} \). Since \( \mathbf{H} \) is complete and GA11 is valid under the truth functions assigned to the connectives in \( \mathbf{H} \), such a derivation must be possible.

3.17 Definition. Unlike the other logical connectives, equivalence in \( S(S_{g}) \) and in \( S_{g}(S_{g}) \) is treated as an abbreviation in the metalanguage:

\[ .P = Q \text{ for } :P \supset Q \& .Q \supset P \text{ and } .P \leftrightarrow Q \text{ for } :P \supset Q \times .Q \supset P \]

3.18 Theorem. If \( P \) and \( Q \) are wff of \( S \) or \( S_{g} \), \( P' \) and \( Q' \) are wff of \( S_{g} \) or \( S_{g} \), \( P' \) is a generalizatum of \( P \) and \( Q' \) is a generalizatum of \( Q \), then \( .P' = Q' \) is a generalizatum of \( .P = Q \).

Proof: Follows directly from the definitions 2.15, 2.16 and 3.17.

Thus in a sense the symbol \( \leftrightarrow \) can be said to represent the symbol \( = \).

3.19 Theorem. The conservation of equivalence over a transformation by the replacement of equivalent parts is affirmed by an auxilliary rule of inference:

R4 (GR4) If \( M, N, P \) and \( Q \) are wff of \( S(S_{g}) \) or \( S_{g}(S_{g}) \) and if \( Q \) is the result of replacing zero or more occurrences of \( M \) in \( P \) by occurrences of \( N \) and if \( M \equiv N \) (\( M \leftrightarrow N \)) is a theorem of \( \mathfrak{L}(\mathfrak{L}_{g6}) \) or \( \mathfrak{L}_{g}(\mathfrak{L}_{g6}) \), then \( P \equiv Q(P \leftrightarrow Q) \) is a theorem and, if \( P \) is a theorem, \( Q \) is also a theorem.

Proof: By 3.9 the proofs of C*158 and C*159 constitute proof of R4 for \( \mathfrak{L} \) and by 3.12 the proofs of C*341 and C*342 constitute proof of R4 for \( \mathfrak{L}_{g6} \). Rule GR4 may be proved by the methods of C34 using L1 of 3.15, R3.38, GA1, GA2, GR1, GR2 and the following lemmas of \( \mathfrak{L}_{g6} \):

L5. \( \forall x . F(x) \leftrightarrow G(x) \rightarrow \forall x F(x) \leftrightarrow \forall x G(x) \)  
(C*334), R3.34, L3, GA2, R3.28
\[ L6 \quad :P \leftrightarrow Q \leftrightarrow \forall P \leftrightarrow \forall Q \quad (PM^{84.11}), \, R3.5, \, R3.36, \, R3.28, \, R3.10, \, GA4 \]

\[ L7 \quad \land x \cdot F(x) \leftrightarrow G(x) \leftrightarrow \forall x F(x) \leftrightarrow \forall x G(x) \]

\[ \text{L6, GR2, R3.14, L3, L5, L6, GA14ab} \]

\[ L8 \quad :P \rightarrow :P \rightarrow Q \rightarrow :P \rightarrow S \leftrightarrow T \rightarrow :P \rightarrow Q \land S \leftrightarrow R \lor T \quad (PM^{*4.39}), \, R2.19, \, R3.35, \, R3.28, \, R3.30, \, R3.10; \, R3.35, \, R2.8, \, R3.10 \]

\[ L9 \quad :P \rightarrow :P \rightarrow Q \leftrightarrow R \rightarrow :P \rightarrow S \leftrightarrow T \rightarrow :P \rightarrow Q \land S \leftrightarrow R \land T \quad (L8), \, R3.25 \text{ in place of R2.19} \]

\[ L10 \quad :P \rightarrow :P \rightarrow Q \leftrightarrow R \rightarrow :P \rightarrow S \leftrightarrow T \rightarrow :P \rightarrow Q \land S \leftrightarrow R \land T \quad (L9), \, R3.28 \]

\[ L11 \quad :P \rightarrow :P \rightarrow Q \leftrightarrow R \rightarrow :P \rightarrow S \leftrightarrow T \rightarrow :P \rightarrow Q \land S \leftrightarrow R \land T \quad (L9), \, R3.28 \]

\[ L12 \quad :P \leftrightarrow Q \leftrightarrow \forall P \leftrightarrow \forall Q \quad (PM^{*4.11}), \, R3.5, \, R3.36, \, R3.28, \, R3.10 \]

\[ L13 \quad :P \rightarrow :P \rightarrow Q \leftrightarrow R \rightarrow :P \rightarrow S \leftrightarrow T \rightarrow :P \rightarrow Q \land S \leftrightarrow R \land T \quad (PM^{*4.39}), \, R2.19, \, R3.35, \, R3.28, \, R3.30, \, R3.10; \, GA5ab, \, L10, \, R3.35, \, R2.8, \, R2.7 \]

\[ L14 \quad :P \rightarrow :P \rightarrow Q \leftrightarrow R \rightarrow :P \rightarrow S \leftrightarrow T \rightarrow :P \rightarrow Q \land S \leftrightarrow R \land T \quad (L13), \, R3.25 \text{ for R2.19}; \, GA6ab \text{ and R3.4 for GA5ab} \]

\[ L15 \quad :P \rightarrow Q \leftrightarrow R \rightarrow :P \rightarrow \forall Q \leftrightarrow \forall R \quad L10, \, R2.8, \, R2.7 \]

No generalizatum of C*340 is obtained since no suitable generalizatum of the tautology employed in Case 1 of Church's proof is available. However a generalized version of C*341 may be derived using the proof scheme of C*340 but with the weaker theorems L8, L9, L10, L11, L13 and L14, expanding Case 1 to the six subcases required by the primitive status of the six binary connectives of $\land$ and $\lor$, and expanding case 3 to the two subcases required by the primitive status of the quantification operators of $\exists x$ using GR1 and GR2 in place of C*300 and C*301. The generalization C*342 follows using R3.34 and GR1. GR4 incorporates both generalizations.

The following definitions serve to simplify the diction in some metatheorems and proofs:

3.20 Definition. A wff $P$ is a *transformation* of a wff $Q$ iff $P \equiv Q(P \leftrightarrow Q)$ is a theorem and the same propositional and predicate variables occur, and the same individual variables occur free, in $P$ as in $Q$.

3.21 Definition. A wff $P$ is *transformed into* a wff $Q$ iff there exists a proof that $Q$ is a transformation of $P$.

3.22 Theorem. If $P$ is a transformation of $Q$, then $P$ is a theorem iff $Q$ is a theorem.

Proof: Follows from 3.17, 3.20, R3.3, R3.4 and R1 (GR1) for all four calculi.

4. PROLOGEMINA.

In this section such object language theorem schemes and metalanguage lemmas are presented as are required to prove the generalizability of the propositional and pure first order functional calculi. There is also a metatheorem on degeneration.
4.1 Lemma. The following are theorem schemes for the calculi as indicated:

**Theorem Schemes for $\mathbb{K}$ and $\boldsymbol{\mathbb{K}}_n$**

<table>
<thead>
<tr>
<th>Theorem Scheme</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>$P \supset Q \equiv \sim P \vee Q$</td>
</tr>
<tr>
<td>T2</td>
<td>$\sim (P \lor Q) \equiv \sim P \land \sim Q$</td>
</tr>
<tr>
<td>T3</td>
<td>$\sim P \land Q \equiv \sim P \lor \sim Q$</td>
</tr>
<tr>
<td>T4</td>
<td>$\sim \sim P \equiv P$</td>
</tr>
<tr>
<td>T5</td>
<td>$P \lor Q \equiv Q \lor P$</td>
</tr>
<tr>
<td>T6</td>
<td>$(P \lor Q) \land (P \lor R) \equiv P \lor (Q \land R)$</td>
</tr>
<tr>
<td>T7</td>
<td>$(P \lor Q) \land R \equiv (P \lor Q) \land R$</td>
</tr>
<tr>
<td>T8</td>
<td>$P \land Q \equiv Q \land P$</td>
</tr>
<tr>
<td>T9</td>
<td>$(P \land Q) \land R \equiv (P \land Q) \land R$</td>
</tr>
<tr>
<td>T10</td>
<td>$P \lor \sim P$</td>
</tr>
<tr>
<td>T11</td>
<td>$P \supset .Q \supset .P \land Q$</td>
</tr>
<tr>
<td>T12</td>
<td>$P \supset Q \supset P$</td>
</tr>
<tr>
<td>T13</td>
<td>$P \supset .P \lor Q$</td>
</tr>
<tr>
<td>T14</td>
<td>$P \supset .P \lor Q$</td>
</tr>
</tbody>
</table>

**Theorem Schemes for $\mathbb{K}_6$ and $\boldsymbol{\mathbb{K}}_{36}$**

<table>
<thead>
<tr>
<th>Theorem Scheme</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>GT1a</td>
<td>$P \rightarrow Q \iff \sim P + Q$</td>
</tr>
<tr>
<td>GT1b</td>
<td>$P \Rightarrow Q \iff \sim P \lor Q$</td>
</tr>
<tr>
<td>GT2a</td>
<td>$\sim P + Q \rightarrow \sim P \times Q$</td>
</tr>
<tr>
<td>GT2b</td>
<td>$\sim P \lor Q \rightarrow \sim P \land Q$</td>
</tr>
<tr>
<td>GT3a</td>
<td>$\sim P \times Q \rightarrow \sim P \lor Q$</td>
</tr>
<tr>
<td>GT3b</td>
<td>$\sim P \land Q \rightarrow \sim P \lor Q$</td>
</tr>
<tr>
<td>GT4</td>
<td>$\sim P \lor Q \leftrightarrow \sim P \land Q$</td>
</tr>
<tr>
<td>GT5</td>
<td>$P \lor Q \leftrightarrow Q + P$</td>
</tr>
<tr>
<td>GT6</td>
<td>$P + Q \land P + R \leftrightarrow P + Q + R$</td>
</tr>
<tr>
<td>GT7</td>
<td>$(P + Q + R \leftrightarrow P + Q + R)$</td>
</tr>
<tr>
<td>GT8</td>
<td>$P \land Q \leftrightarrow Q \land P$</td>
</tr>
<tr>
<td>GT9</td>
<td>$(P \land Q \land R \leftrightarrow P \land Q \land R)$</td>
</tr>
<tr>
<td>GT10</td>
<td>$P \lor \sim P$</td>
</tr>
<tr>
<td>GT11a</td>
<td>$P \rightarrow Q \rightarrow P \land Q$</td>
</tr>
<tr>
<td>GT11b</td>
<td>$P \rightarrow Q \rightarrow P \times Q$</td>
</tr>
<tr>
<td>GT12a</td>
<td>$P \land Q \rightarrow P$</td>
</tr>
<tr>
<td>GT12b</td>
<td>$P \times Q \rightarrow P$</td>
</tr>
<tr>
<td>GT13</td>
<td>$P \rightarrow P \lor Q$</td>
</tr>
<tr>
<td>GT14</td>
<td>$P \times Q \rightarrow P \land Q$</td>
</tr>
<tr>
<td>GT15</td>
<td>$\sim \forall x F(x) \leftrightarrow \forall y \sim F(y)$</td>
</tr>
<tr>
<td>GT16</td>
<td>$\forall x F(x) \leftrightarrow \forall y \sim F(x)$</td>
</tr>
<tr>
<td>GT17</td>
<td>$\forall x F(x)$</td>
</tr>
<tr>
<td>GT18a</td>
<td>$P \land \forall x F(x) \leftrightarrow \forall x .P \land F(x)$</td>
</tr>
<tr>
<td>GT18b</td>
<td>$P \lor \forall x F(x) \equiv \forall x .P \lor F(x)$</td>
</tr>
</tbody>
</table>
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GT18b :\( P \lor \land F(x) \leftrightarrow \land \lor P \lor F(x) \)

L1, R2.18, GR2, GA11, GA12, GT1a, R3.36

T19 :\( P \lor \exists xF(x) \equiv \forall x . P \lor F(x) \)

GT19a :\( P \lor \lor F(x) \leftrightarrow \lor \lor P \lor F(x) \)

GA9, R3.23, GR2, L4, GA12, GT1a, R3.36

T20 :\( P \land \forall xF(x) \equiv \forall x . P \land F(x) \)

GT20a :\( P \land \land F(x) \leftrightarrow \land \land P \land F(x) \)

L1, R3.14.13, GR2, GA11, R3.19; L1, R3.17, GA2, GA11; R3.36

T21 :\( P \land \exists xF(x) \equiv \exists x . P \land F(x) \)

GT21a :\( P \land \lor F(x) \leftrightarrow \lor \land P \land F(x) \)

GT18a, GT15, GT3a, GT3b, L6

Note: By 4.3, below, any wff of \( S \) or \( S_S \) is a theorem of \( \Xi \) or \( \Xi_S \) if its

\[ \text{generalizatum is a theorem of } \Xi_S \text{ or } \Xi_{S_S}; \text{ hence proof citations are} \]

indicated only for the theorems of \( \Xi_S \) and \( \Xi_{S_S} \).

4.2 Lemma. The junction of \( P_1, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_n \) is a trans-

formation of the junction of \( P_1, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n \) and also of the

junction of the junction of \( P_1, \ldots, P_n \) with the junction of \( P_{i+1}, \ldots, P_n \).

Proof: Follows from the definitions 2.4 and 2.5 together with T5(GT5) and

T7(GT7), T8(GT8) and T9(GT9), R2.2 and R2.21, and R3.10 and R3.30.

4.3 Lemma. Every theorem of \( \Xi_S \) or of \( \Xi_{S_S} \) is a generalizatum of one and

only one theorem of \( \Xi \) or \( \Xi_S \).

Proof: (1) Every axiom of \( \Xi_S \) or \( \Xi_{S_S} \) is a generalizatum of some axiom of

\( \Xi \) or \( \Xi_S \) as is evident from the structure of the axiom schemes and the

definitions 2.15 and 2.16. (2) From (1), isomorphism of GR1 with R1 and

GR2 with R2, and the many-one character of the relations represents and

generalizatum of it follows that every theorem of \( \Xi_S \) or \( \Xi_{S_S} \) is a generalizatu-

mum of some theorem of \( \Xi \) or \( \Xi_S \). (3) A given wff of \( S_G \) or \( S_G_S \) can be a

generalizatum of only one wff of \( S \) or \( S_S \) by virtue of 2.15 and 2.16.

4.4 Definition. A wff of \( S \) or \( S_S (S_G \) or \( S_G_S \)) is in positive form iff
Implicator does not (Weak Implicator and Strong Implicator do not) occur in it and Negator, if it occurs in it, occurs only immediately preceding occurrences of elementary wff.

4.5 Definition. A wff of $S_g$ or $S_{jg}$ is in weak positive form iff it is in positive form and Strong Disjunctor and Strong Conjunctor do not occur in it.

4.6 Lemma. Each wff of $S(S_g)$ or $S_j(S_{jg})$ can be transformed into a unique wff in positive form in which the same elementary wffs occur with the same frequency, order and grouping.

Proof: By zero or more applications of $R4$ with $T1$ (GR4 with $GT1a$ or $GT1b$) a sequence of wffs is obtained, beginning with the given wff, $P$, and ending with a wff, $Q$, such that each of the wffs is a transformation of the preceeding one with one less (weak or strong) implicator and such that $Q$ is free of (strong or weak) implicators. By means of zero or more applications of $R4$ with $T2$, $T3$, $T4$, $T15$ or $T16$ (GR4 with $GT2a$, $GT2b$, $GT3a$, $GT4$, $GT15$ or $GT16$) a sequence of one or more wffs is obtained, beginning with $Q$ and ending with a wff, $R$, such that each of the wffs except $Q$ is a transformation of its predecessor in which a pair negators has been eliminated, a negator preceeding a quantifier has been replaced by a negator following a quantifier, or a negator preceeding a dot has been replaced by two negators following the dot, and such that in $R$ no negator immediately preceeds another negator, a quantifier or a dot. $R$ is thus a transformation of $P$ in positive form and no permutation, duplication, elimination or regrouping of elementary wffs has occurred in the process.

4.7 Lemma. For every wff, $P$, of $S$ or $S_j$ in positive form there is one and only one wff, $P'$, of $S_g$ or $S_{jg}$ such that $P'$ is a generalizatum of $P$ and is in weak positive form.

Proof: A direct consequence of definitions 2.15, 2.16, 4.4 and 4.5.

4.8 Definition. The positive form of a wff, $P$, of $S$ or $S_j(S_g$ or $S_{jg})$ is that wff which is a transformation of $P$, which is in positive form and in which the elementary wffs occur with the same frequency, order and grouping as in $P$.

4.9 Lemma. A wff of $S_g$ or $S_{jg}$ which is in weak positive form and which is a generalizatum of the positive form of some wff $P$ of $S$ or $S_j$ can be transformed into a unique generalizatum of $P$ of which it is the positive form.

Proof: Let $A_0, A_1, \ldots, A_n$ be a sequence of wffs of $S$ or $S_j$ such that $A_n$ is the positive form of $A_0$ and for each $i$, $1 \leq i \leq n$, $A_i$ is a transformation of $A_{i-1}$ in that the first wff occurring in which an implicator occurs is transformed by applying $R4$ with $T1$ or, if no implicator occurs in $A_{i-1}$ in that the first wff in $A_{i-1}$ beginning with a negator followed by a dot or a quantifier is transformed by applying $R4$ with $T2$, $T3$, $T4$, $T15$ or $T16$. Let $B_0, B_1, \ldots, B_n$ be a sequence of wffs of $S_g$ or $S_{jg}$ such that $B_0$ is a
generalizatum of $A_n$ in weak positive form and for each $i, 1 \leq i \leq n$, $B_i$ is a transformation of $B_{i-1}$ by GT16, GT15, GT4, GT3a, GT3b, GT2a, GT2b, GT1b, or GT1a so chosen that for each $i, 0 \leq i \leq n$, $B_i$ is a generalizatum of $A_{n-1}$. This is always possible since each GT theorem is a generalizatum of one of the T theorems. Although there are two generalizata of some T theorems among the GT theorems only one is applicable to a given $B_{i-1}$, hence the $B$ sequence is completely determined by $B_0$ and the $A$ sequence. The $A$ sequence is completely determined by $A_0$ and the instructions given for its construction. $B_0$ is determined by $A_n$ in accordance with 4.7. Also by 4.7, if a different $A$ sequence had been specified the same generalizatum, $B_n$, of $A_0$ would have been obtained.

4.10 Definition. A wff $Q$ of $S$ or $S_j$ is the *weakest generalizatum* of a wff $P$ of $S_G$ or $S_{jG}$ iff the positive form of $Q$ is in weak positive form and $Q$ is a generalizatum of $P$.

4.11 Definition. A wff of $S$ or $S_j$ is in *conjunctive normal form* iff it is in positive form and if no wff occurring in it is a disjunction of wff any of which is a conjunction.

4.12 Definition. A wff of $S_G$ or $S_{jG}$ is in *weak conjunctive normal form* iff it is in weak positive form and no wff occurring in it is a weak conjunction.

4.13 Definition. A wff of $S_G$ or $S_{jG}$ is in *strong positive form* iff it is in positive form and neither Weak Disjunctor nor Weak Conjunctor occur in it.

4.14 Lemma. A wff of $S_G$ or of $S_{jG}$ which is in strong positive form and a generalizatum of the positive form of some wff $P$ of $S$ or $S_j$ can be transformed into a unique generalizatum of $P$ of which it is the positive form.

Proof: The proof is exactly the same as that of 4.9 excepting only that $B_0$ is to be in strong positive form.

4.15 Theorem. The calculi $I_6$ and $I_{36}$ degenerate respectively into the calculi $I$ and $I_3$.

Proof: By virtue of 4.3 and 2.17 the relations defined in 2.15 and 2.16 satisfy the requirements of 1.18.

5. GENERALIZABILITY OF THE PROPOSITIONAL CALCULUS.

5.1 Postulate. *If $P$ is a disjunction (weak disjunction) of wffs of $S_G$ ($S_j$) each of which is either an elementary wff or the negation of an elementary wff, then $P$ is a theorem of $I$ ($I_6$) only if there occurs in $P$ the negation of at least one elementary wff which also occurs in $P$ as one of the terms of the disjunction (weak disjunction).*

Justification: Although this follows from the truth table for disjunction (truth function for weak disjunction) and the mutual independence of the truth values assigned to the elementary wffs it cannot be derived from the formal axiom schemes and rules and is hence introduced as a postulate.
5.2 Postulate. If $P$ is a strong disjunction of wffs of $S_G$ each of which is either an elementary wff or the negation of an elementary wff, then $P$ is not a theorem of $\mathcal{L}_G$.

Justification: The postulate also follows from the truth function, in this case for strong disjunction, and the independence of the elementary wffs.

5.3 Lemma. If $P$ is a theorem of $\mathcal{L}$ and is in positive form and $Q$ is a generalizatum of $P$ and is in weak positive form, then $Q$ is a theorem of $\mathcal{L}_G$.

Proof: (1) Let $A_0, A_1, \ldots, A_n$ be a sequence of one or more wffs of $S$ such that $A_0$ is $P$ and such that each member $A_i$ which is not in conjunctive normal form has a successor $A_{i+1}$ which is the result of replacing the first wff occurring in $A_i$ and having the form of a disjunction of some wff with a conjunction by its transformation, by way of $T_6$, into a conjunction of disjunctions or, in case no wff occurs in $A_i$ in the form of a disjunction of some wff with a conjunction, is the result of replacing the first wff occurring in $A_i$ and having the form of a disjunction of a conjunction with some wff by its transformation, by way of $T_5$, into a disjunction of the wff with the conjunction. The sequence terminates in a wff $A_n$ which is in conjunctive normal form and being a transformation of $P$, is by $R_4$ and the hypothesis a theorem of $\mathcal{L}$. (2) The theorem $A_n$ is, or by one or more applications of $T_9$ may be transformed into, a wff of the form $R_0 \& R_1 \& \ldots \& R_r$ where, in either case, each $R_i$ is of the form $S_{i0} \lor S_{i1} \lor \ldots \lor S_{is}$ such that each $S_{ij}$ is either an elementary wff of $S$ or the negation of an elementary wff. The wff $A_n$ is, by $T_{12}$ and $4.2$, a theorem of $\mathcal{L}$ only if each $R_i$ is a theorem of $\mathcal{L}_G$. However, by $5.1$, each $R_i$ is a theorem only if there is an $S_{ij}$ and an $S_{ik}$ such that $S_{ij}$ is the negation of $S_{ik}$. (3) Let $B_0$ be the generalizatum of $A_n$ which is in weak positive form, then, by $4.5$ and $4.12$, $B_0$ is in weak conjunctive form, that is of the form $R_0' \land R_1' \land \ldots \land R_r'$ where each $R_i'$ is of the form $S_{i0}' \lor S_{i1}' \lor \ldots \lor S_{is}'$ in which, for each $i$ and $j$, $S_{ij}'$ is a generalizatum of $S_{ij}$ and hence is either an elementary wff of $S_G$ or the negation of an elementary wff. Wff $B_0$ is a theorem of $\mathcal{L}_G$ by $T_{12}$ if each $R_i'$ is a theorem of $\mathcal{L}_G$. However, by $GT_{10}$ and $4.5$, each $R_i'$ is a theorem if there is an $S_{ij}'$ and an $S_{ik}'$ such that $S_{ik}'$ is the negation of $S_{ij}'$. (4) Consider the sequence $B_0, B_1, \ldots, B_n$ of one or more wff of $\mathcal{L}_G$ such that for each $i$, $1 \leq i \leq n$, $B_i$ is that transformation of $B_{i-1}$ by either $GT_5$ or $GT_6$ which makes it a generalizatum of $A_{n-1}$. This is always possible since $B_0$ is a generalizatum of $A_n$ and $GT_5$ and $GT_6$ are generalizata of $T_5$ and $T_6$. Thus $B_n$ is a generalizatum of $A_n$, i.e. $P$, and, is a theorem of $\mathcal{L}_G$ since by (4) it is a transformation of $B_0$ which by (2) and (3) is a theorem of $\mathcal{L}_G$ if $A_n$ is a theorem of $\mathcal{L}$ which by (1) is the case. Also, since the same connectives occur after transformation by either $GT_5$ or $GT_6$ and the negators remain fixed relative to the elementary wff, the weak positive form of $B_0$ gives rise to a weak positive form for $B_n$. Hence $B_n$ is the required $Q$.

5.4 Lemma. The weakest generalizatum of every theorem of $\mathcal{L}$ is a theorem of $\mathcal{L}_G$. 
Proof: (1) Let \( P \) be a theorem of \( \mathcal{K} \), then, by 4.6, \( P \), the positive form of \( P \), is a theorem of \( \mathcal{K} \). (2) Since \( P \) is in positive form and a theorem of \( \mathcal{K} \), then, by 5.3, \( Q \), that generalizatum of \( P \) which is in weak positive form, is a theorem of \( \mathcal{K} \). (3) By 4.9 and 4.19, \( Q \), the weakest generalizatum of \( P \) is a transformation of \( Q \) and hence, by (2), a theorem of \( \mathcal{K} \).

5.5 Theorem. *The calculus \( \mathcal{K} \) generalizes into the calculus \( \mathcal{K} \).*

Proof: By 2.17 the language \( S_{\mathcal{K}} \) is a generalization of the language \( S \). The relations *represents* defined in 2.15 and *generalizatum of* defined in 2.16 constitute a mapping such that by 5.4 for every theorem \( P \) of \( \mathcal{K} \) there is a theorem \( Q \) of \( \mathcal{K} \) such that each symbol which occurs in \( Q \) represents, i.e. is an image of, a symbol occurring in a corresponding position in \( P \). Thus the definition of generalizes in 1.18 is satisfied.

5.6 Remark. In general, the weakest generalizatum of any theorem \( P \) of \( \mathcal{K} \) will contain either or both representatives of each binary connective which occurs in \( P \) even though it is constructed by means of an intermediate weak positive form in which only two binary connectives occur.

5.7 Lemma. *Disjunctor occurs in the positive form of every theorem of \( \mathcal{K} \).*

Proof: The axioms and \( \mathcal{R}1 \) are such that a binary connective occurs in every theorem. By T12 and 4.2 a conjunction is a theorem only if all of the conjoined wff are theorems. Hence there is at least one disjunctor in every theorem which is in positive form and therefore, by 4.6, 4.8 and 3.22 in the positive form of every theorem of \( \mathcal{K} \).

5.8 Lemma. *Weak Disjunctor occurs in the positive form of every theorem of \( \mathcal{K} \).*

Proof: The axioms of \( \mathcal{K}_6 \) and \( \mathcal{GR}1 \) are such that a binary connective occurs in every theorem of \( \mathcal{K}_6 \). By GT12a and 4.2 a weak conjunction and by GR12b and 4.2 a strong conjunction is a theorem only if all of the conjoined wff are theorems. By 5.2 a wff of \( S_{\mathcal{K}} \) in which Strong Disjunctor is the sole binary connective is not a theorem of \( \mathcal{K}_6 \) and hence, by GT12a, GT12b, R2.17 and R3.23 neither is a wff containing as connectives only Strong Disjunctor together with either or both of the representatives of Conjunctor.

5.9 Theorem. *Among the generalizata of every theorem of \( \mathcal{K} \) there is at least one wff of \( S_{\mathcal{K}} \) which is not a theorem of \( \mathcal{K}_6 \).*

Proof: Let \( P \) be a theorem of \( \mathcal{K} \), \( P \), the positive form of \( P \) and \( Q \), a generalizatum of \( P \) in strong positive form. Since, by 4.13, Weak Disjunctor does not occur in \( Q \) it is, by 5.8, not a theorem of \( \mathcal{K}_6 \). However, by 4.14, \( Q \), is the positive form of a wff \( Q \) of \( S_{\mathcal{K}} \) which is a generalizatum of \( P \) and, being a transformation of \( Q \), is also not a theorem of \( \mathcal{K}_6 \).

5.10 Lemma. *If \( P \) and \( Q \) are generalizata of the same wff of \( S \) and are in positive form and if \( Q \) is the result of replacing zero or more strong conjunctors in \( P \) by weak conjunctors, then the weak implication of \( Q \) by \( P \) is a theorem of \( \mathcal{K}_6 \).*
Proof: Since both \( P \) and \( Q \) are in positive form, repeated application of GT14 together with R2.17.18, R3.16.17.23.24.26.27 can be used to construct proofs for all possible cases.

5.11 Theorem. If a theorem of \( \mathfrak{F} \) is such that its positive form is in conjunctive normal form, then every generalizatum of said theorem in which Strong Disjunctor does not occur is a theorem of \( \mathfrak{F}_0 \).

Proof: Let \( P_1 \) be a theorem of \( \mathfrak{F} \) in conjunctive normal form and the positive form of a theorem \( P \), and let \( Q \) be a generalizatum of \( P \) such that in its positive form \( Q_1 \) the only binary connectives occurring are Strong Conjunctor and Weak Disjunctor. By definitions 2.15, 2.16 and 4.4 \( Q_1 \) is a generalizatum of \( P_1 \). Since \( P_1 \) and \( Q_1 \) are in conjunctive normal form the argument used in (2) and (3) of the proof of 5.3 can be used with strong conjunctors in place of weak conjunctors and GT12b in place of GT12a to prove that \( Q_1 \) must be a theorem of \( \mathfrak{F}_0 \) if \( P_1 \) is a theorem of \( \mathfrak{F} \), and hence by 4.6 that \( Q \) is a theorem of \( \mathfrak{F}_0 \) since \( P \) is a theorem of \( \mathfrak{F} \). Consider now another generalizatum of \( P \), the wff \( R \), such that in its positive form \( R_1 \) Weak Conjunctor occurs. By 5.10 if \( Q_1 \) is a theorem so is \( R_1 \) and hence by 4.4 also \( R \). Thus any generalizatum of \( P \) such that Strong Disjunctor does not occur in its positive form is a theorem of \( \mathfrak{F}_0 \).

5.12 Remark. Other conditions can be enunciated which are either sufficient or necessary in order that a generalizatum of a theorem of \( \mathfrak{F} \) be a theorem of \( \mathfrak{F}_0 \); however there is no simple set of conditions which is both sufficient and necessary.

The following lemma establishes the lemmas and theorems of this section as lemmas for Section 7 on the pure predicate calculus of the first order.

5.13 Lemma. Every general statement which holds for wff of \( S(\mathfrak{F}_0) \) and/or theorems of \( \mathfrak{F}(\mathfrak{F}_0) \) also holds for quantifier-free wff of \( S_\gamma(\mathfrak{F}_0) \) and/or quantifier-free theorems of \( \mathfrak{F}_\gamma(\mathfrak{F}_0) \).

Proof: This is a direct result of the common formation rules of \( S(\mathfrak{F}_0) \) and \( S_\gamma(\mathfrak{F}_0) \) and the common axioms and rules of derivation for \( \mathfrak{F}(\mathfrak{F}_0) \) and \( \mathfrak{F}_\gamma(\mathfrak{F}_0) \) applicable to quantifier-free formulae.

6. NONGENERALIZABILITY OF CALCULI WITH EQUIVALENCE.

6.1 Definition. Let \( \mathfrak{F}_c \) be a language differing from \( \mathfrak{F} \) only in that its logical symbols include Equivalor, prototype \( \equiv \), and its formation rules include provision for the formation of wffs called equivalences of two wffs in which Equivalor occurs as a binary connective.

6.2 Definition. Let \( S_{\mathfrak{F}c} \) be a language differing from \( \mathfrak{F}_c \) only in that its logical symbols include representatives of Equivalor and its formation rules include provisions for the formation of wff which are generalizata of equivalences.

6.3 Definition. Let \( \mathfrak{F}_E \) be a calculus differing from \( \mathfrak{F} \) only in that its
language is $S_\ell$ and its axioms include a definition of Equivalor equivalent to 3.14. Similarly let $I_{SE}$ differ from $I_E$ only in that its language is $S_G^\ell$ and its axioms include generalizata of the axioms embodying Equivalor effectively defining the added symbols.

6.4 Definition. A wff of $S_\ell(S_G^\ell)$ is in positive form iff the only binary connectives occurring in it are Disjunctor and Conjunctor and Negator occurs only immediately before elementary wff.

6.5 Definition. A wff of $S_G^\ell$ is in weak positive form iff the only binary connectives occurring in it are Weak Disjunctor and Weak Conjunctor and Negator occurs only immediately before elementary wff.

6.6 Lemma. Lemma 4.6 is not valid for $S_\ell(S_G^\ell)$ in place of $S(S_G)$.

Proof: When an equivalence such as $P = Q$ occurs in a wff which is to be transformed to its positive form it must be replaced by a conjunction of disjunctions, such as $\neg P \lor Q \land \neg Q \lor P$, or a disjunction of conjunctions, such as $P \land Q \lor \neg P \land \neg Q$. The requirement of 4.6 that the elementary wffs occur with the same frequency thus cannot be maintained.

6.7 Lemma. Lemma 4.9 is not valid with $S_G^\ell$ and $S_\ell$ replacing $S_G$ and $S$.

Proof: Consider a wff of $S_\ell$ in which an equivalence occurs between wffs at least one of which is compound. Say $\phi = q \lor r$ in which $p$, $q$ and $r$ are supposed to be elementary wffs. Transformation to positive form can be accomplished in three steps: (1) removal of Equivalor to yield $p \supset q \lor r \land q \lor r \supset p$; (2) removal of Implicator to yield $\neg p \lor q \lor r \land \neg q \lor r \lor p$; (3) shifting of Negator when it precedes a dot to yield $p \lor q \lor r$ and $q \lor r \supset p$.

The weak generalizatum of this in weak positive form is $:\neg p + q + r \land q \lor r \lor p$. Transformation by the generalized reversal of step (3) yields $\neg p + q + r \land q \lor r \lor p$; reversal of step (2) yields $p \lor q \lor r \land q \lor r \lor p$ which however is not transformable to a generalizatum of any wff that can be formed by joining $p$ and $q \lor r$ with a binary connective. Since the dual of each weak disjunction or conjunction is a strong conjunction or disjunction this result is perfectly general.

6.8 Theorem. The calculus $I_E$ does not generalize into the calculus $I_{SE}$.

Proof: The failure of 4.9 and hence the inadmissibility of definition 4.10 for $S_G^\ell$ and $S_G^\ell$ invalidates the use of the proof given in 5.5 for the extension of theorem 5.5 to $I_E$ and $I_{SE}$ but does not directly disprove the theorem or prove 6.8. The easiest proof of 6.8 is by counterexample. Thus, $PM^5.32$ is a theorem of $I_E$ but none of its generalizata are theorems of $I_{SE}$. This is most easily demonstrated by the use of truth functions for the seven possible representatives of Equivalor which can be characterized in terms of the positive forms of equivalences given in 6.6, together with the truth functions for the other connectives as set forth in R1. The truth functions for the seven generalizata of equivalence are as follows:
max(min(1 - p, 1 - q), min(p, q)); max(min(1 - p, 1 - q), p + q - 1)); min(1 - p + q, 1 - q + p); max(1 - p - q, min(p, q)); max(1 - p - q, p + q - 1); min(1 - q + p, max(1 - p, q)); min(1 - p + q, max(1 - q, p)). There are 392 different generalizata in $S_G^E$ of the theorem.

PM*5.32 :P ⊃ .Q ≡ R ≡ :P & Q ≡ .P & R

none of which are theorems of $K_{6E}$ as each can be shown to be contravalid by assigning to P, Q and R one or more of the following sets of truth values: $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 1), (\frac{1}{2}, 1, 1).

6.9 Remark. It is often convenient to consider the theorems asserted in PM as theorems of $K_E$ and to seek among their generalizata in $S_G^E$ theorems of $K_{6E}$. However, by 6.8, such a search may be fruitless. In such cases, if the principal connective is Equivator, the theorem is often best decomposed into two implications, or, if a complex subsidiary equivalence occurs in it, this may be transformed into a conjunction of implications or a disjunction of conjunctions. As an example it may be noted that there are several generalizata of the implications arising from the decomposition of PM*5.32 which are theorems of $K_{6E}$ and retain the subsidiary equivalences,

1. :P \rightarrow .Q \leftrightarrow .R \rightarrow :P \land Q \leftrightarrow .P \land R
2. :.P \lor Q \leftrightarrow .P \land R \rightarrow .P \rightarrow .Q \leftrightarrow R

The following theorem of $K_{6E}$ is a generalizatum of a theorem formed from PM*5.32 by transforming the complex subsidiary equivalence to a conjunction of implications but retaining the principal equivalence:

3. :P \rightarrow .Q \leftrightarrow .R \leftrightarrow :.P \times Q \rightarrow .P \times R \leftrightarrow :P \times Q \rightarrow .P \land Q

Transforming the simple subsidiary equivalence would produce no new independent generalizata.

7. GENERALIZABILITY OF THE PURE PREDICATE CALCULIS.

7.1 Definition. A wff of $S_T(S_T^E)$ is in prenex form iff no dot or negator appears in it preceding any occurrence of a quantifier.

7.2 Definition. The matrix of a wff in prenex form is the longest quantifier-free formula occurring therein. The prefix of a wff in prenex form is the longest formula occurring before any dot or negator.

7.3 Theorem. The matrix of a wff in prenex form is wff. A quantifier-free wff is in prenex form, has no prefix and is identical with its matrix.

Proof. Follows directly from the definitions.

7.4 Lemma. Every wff of $S_T$ in positive (or $S_T^G$ in positive or weak positive) form can be transformed into a wff in prenex form which is also in positive (or weak positive) form.

Proof. By the use of T18, T19, T20, T21 and A10 (GT18ab, GT19ab, GT20, GT21ab and GA10) a sequence of wff may be constructed which begins with the given wff and ends in a wff in prenex form and is such that each
member of the sequence is a transformation of its predecessor resulting from the replacement of the first quantifier occurring after a dot by one occurring before the dot. The connectives are not changed or displaced in such transformations and hence the property of being in positive (positive or weak positive) form is conserved.

7.5 Definition. A wff is in prenex positive (positive or weak positive) form iff it is in positive (positive or weak positive) form and also in prenex form.

7.6 Lemma. If a theorem $P$ is in prenex positive form its generalizatum in weak positive form is a theorem of $P$.

Proof: (1) Consider a theorem $P$ of $P$ which is in prenex positive form with a prefix consisting of $n$ quantifier occurrences. In $n > 0$ then $P$ is either in the form $\forall x Q$ or $\exists x Q$. (2) If $P$ is in the form $\forall x Q$ then there is, by T17, a wff $Q$ which is a theorem; if $P$ is in the form $\exists x Q$ then there is, by R3, a set of wff, $Q_0$, $Q_1$, ... , $Q_m$, at least one of which is a theorem. (3) Hence by (1) and (2) if $n > 0$ there is a set $a_1$ of one or more wffs in positive prenex form at least one of which is a theorem and each of which has the same prefix consisting of $n - 1$ quantifier occurrences. (4) If $n > 1$ there is, corresponding to each member $Q_i$ of $a_1$, a set $Q_i$, $Q_{i2}$, ... , $Q_{ip}$ of wffs in prenex positive form at least one of which is a theorem in the case that $Q_i$ is a theorem. (5) Since by (3) some member $Q_k$ of $a_1$ is a theorem then by (4) some member of $a_2$, the union of the $Q_{ij}$, is a theorem. Also each member of $a_2$ has the same prefix of $n - 2$ quantifier occurrences. (6) In general, each $a_k$ for $0 < k < n$ determines a set of sets of wffs in prenex positive form the union $a_{k+1}$ of which is such that at least one of its members is a theorem and each has the same prefix consisting of $n + k - 1$ quantifier occurrences. (7) The sequence terminates in a set $a_n$ of quantifier-free wffs each in positive form and at least one a theorem. (8) Let $\beta_n$ be the set of weakest generalizata of $a_n$. The members of $\beta_n$ will be wff of $P$ in quantifier-free weak positive form. (9) Since, by (7), at least one member of $a_n$ is a theorem of $P$, it follows by 5.3 and 5.13 that at least one member of $\beta_n$ is a theorem of $P$. (10) The derivation of $a_n$ from $a_{n-1}$ involved the use of T17 or R3, hence, by the use of GR2 or GA9, resp., a set $\beta_{n-1}$ of wffs of $P$ can be derived from $\beta_n$ such that $\beta_{n-1}$ constitutes the generalizata of $a_{n-1}$ in prenex weak positive form and such that each member of $\beta_{n-1}$ which is the generalizatum of a theorem of $P$ is a theorem of $P$. (11) The same single quantifier will occur as the prefix of each member of $\beta_{n-1}$. (12) In general, to each $a_k$ for $0 < k < n$ there corresponds a $\beta_k$ which is derived from $\beta_{k+1}$ by GR2 or GA9, is made up of the weakest generalizata of the $a_k$ each with a prefix consisting of the same $n - k$ quantifiers and is such that every member of $\beta_k$ which is the generalizatum of a theorem of $P$ will be a theorem of $P$. (13) From $\beta_1$ which, being derived stepwise from $\beta_n$ includes among its members at least one theorem of $P$, a wff is obtained by GR2 or GA9 which has a prefix consisting of $n$ quantifiers is a theorem of $P$ and is the weakest generalizatum of $P$. 
7.7 Lemma. The weakest generalizatum of every theorem of $\mathfrak{I}_3$ is a theorem of $\mathfrak{I}_{3\#}$.

Proof: Let $P$ be a theorem of $\mathfrak{I}_3$. By 4.6, 5.13 and 3.22 the positive form $P_1$ of $P$ is a theorem of $\mathfrak{I}_3$; by 7.4 and 3.22 the prenex form $P_2$ of $P_1$ is a theorem of $\mathfrak{I}_{3\#}$; by 7.6 the weakest generalizatum $Q_2$ of $P_2$ is a theorem of $\mathfrak{I}_{3\#}$; hence, by 4.6 and GR4, the generalizatum $Q$ of $P$ of which $Q_2$ is the positive form is a theorem of $\mathfrak{I}_{3\#}$ and by 4.10 is the weakest generalizatum of $P$.

7.8 Theorem. The calculus $\mathfrak{I}_3$ generalizes into the calculus $\mathfrak{I}_{3\#}$.

Proof: By 2.17 the language $\mathcal{S}_{\mathcal{G}}$ is a generalization of the language $\mathcal{S}_3$. The relation, generalizatum of, of 2.16 constitutes a mapping such that by 7.7 for every theorem $P$ of $\mathfrak{I}_3$ there is a theorem $Q$ of $\mathfrak{I}_{3\#}$ such that each symbol which occurs in $Q$ represents and hence is an image of a symbol occurring in a corresponding position in $P$, thus satisfying the definition 1.18 of generalizes.

7.9 Lemma. Weak Disjunctor occurs in the positive form of every theorem of $\mathfrak{I}_{3\#}$.

Proof: (1) Let $Q_1$ be the positive form of a theorem $Q$ of $\mathfrak{I}_{3\#}$ and let $Q_2$ be the prenex form of $Q_1$. (2) By (1) and 7.4 $Q_2$ is a theorem of $\mathfrak{I}_{3\#}$ and the same connectives occur in it as occur in $Q_1$. (3) From $Q_2$, by GR3 and GT17 in a series of steps analogous to steps (1)—(7) in the proof of 7.6, a set of quantifier-free wffs is derivable in each of which the same connectives occur as occur in $Q_2$ and hence by (2) as in $Q_1$ and at least one of which, say $Q_3$ is a theorem of $\mathfrak{I}_{3\#}$. (4) Since $Q_3$ is a theorem then by 5.8 and 5.13 Weak Disjunctor occurs in it, and hence by (3) in $Q_2$ and hence by (2) in $Q_1$.

7.10 Theorem. Among the generalizata of every theorem of $\mathfrak{I}_3$ there is at least one wff of $\mathcal{S}_{\mathcal{G}}$ which is not a theorem of $\mathfrak{I}_{3\#}$.

Proof: Let $P$ be a theorem of $\mathfrak{I}_3$, $P_1$ the positive form of $P$ and $Q_1$ a generalizatum of $P_1$ in strong positive form. Since, by 4.13, Weak Disjunctor does not occur in $Q_1$, $Q_1$ is, by 7.9, not a theorem of $\mathfrak{I}_{3\#}$. However, by 4.14, $Q_1$ is the positive form of a wff $Q$ of $\mathcal{S}_{\mathcal{G}}$ which is a generalizatum of $P$ and, being a transformation of $Q_1$ is also not a theorem of $\mathfrak{I}_{3\#}$.

REFERENCES


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