106
Notre Dame Journal of Formal Logic
Volume XI, Number 1, January 1970

EVERY FUNCTIONALLY COMPLETE m-VALUED LOGIC HAS A POST-COMPLETE AXIOMATIZATION

NUEL D. BELNAP, Jr., and STORRS McCALL

Let $\mathfrak{M}$ be an m-valued functionally complete matrix with truth-values 1, 2, . . . m, and let D be the class of designated values and U the class of undesignated values of $\mathfrak{M}$. We shall assume that $\mathfrak{M}$ is Post-consistent, i.e. that U is not empty. Then the truth-functions definable using $\mathfrak{M}$ will include the functions Cpq, Np, and F_ip, $1 \leq i \leq m$, with the following properties, where '$|\alpha|$' denotes the truth-value of α:

(1) If, for all assignments of values to the variables of α and β, $|\alpha| \in$ D and $|C\alpha\beta| \in$ D, then $|\beta| \in$ D for all similar assignments.

(2) For all values of p and q, $|CpCNpq| \in$ D.

(3) For all values of the variables in α, if $|\alpha| \in$ U then $|N\alpha| \in$ D.

(4) The F_ip are constant functions such that, for all values of p, $|F_1p| = 1$, $|F_2p| = 2, \ldots, |F_mp| = m$.

Consider now the deductive system M having as theses all $\mathfrak{M}$-tautologies, and having among its rules of inference the rule of substitution of wffs for proposition variables and the rule of *modus ponens* for C in the sense that β is derivable from α and $C\alpha\beta$.[1] We shall show that adding any non-$\mathfrak{M}$-tautology γ to M as a thesis makes M Post-inconsistent. Since γ is not an $\mathfrak{M}$-tautology, there will be an assignment of values to its variables such that $|\gamma| \in$ U. Let γ' be the result of substituting appropriate constant functions F_ip for the variables of γ so that γ' is itself a constant function and $|\gamma'| \in$ U. We shall write '$\vdash\alpha$' to denote that α is a thesis of M. Since $\vdash\gamma$, we obtain $\vdash\gamma'$ by substitution. Also, since $|\gamma'| \in$ U, $|N\gamma'| \in$ D, and hence $\vdash N\gamma'$. Using the appropriate substitution of $\vdash CpCNpq$ and two applications of the rule of *modus ponens* we obtain $\vdash q$. Hence M plus $\vdash\gamma$ is Post-inconsistent, and M is Post-complete.

---

1. Note that M may or may not be finitely axiomatizable.

University of Pittsburgh
Pittsburgh, Pennsylvania

Received January 7, 1969