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THE NON-EXISTENCE OF A CERTAIN COMBINATORIAL DESIGN ON AN INFINITE SET*

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In [1] the notion of a combinatorial design on an infinite set M was based on a covering relation of the following kind.

Definition 1. Let F and G be two families of subsets of M and let p be a non-zero cardinal number. G is said to be a p-Steiner cover of F if and only if every member of F is contained (as a subset) in exactly p members of the family G.

We showed in [1], roughly speaking, that a rather large class of families F possess p-Steiner covers of a specified nature. To be more exact, we introduce the following additional definitions.

Definition 2. Let k be a non-zero cardinal number such that $k \leq \overline{\overline{M}}$. A family F of subsets of M is called a k-tuple family of M if and only if i) *if $x, y \in F$ such that $x \neq y$ then $x \not\subset y$,* ii) *if $x \in F$ then $\overline{\overline{x}} = k$ and* iii) $\overline{\overline{F}} \leq \overline{\overline{M}}$.

In terms of Definitions 1 and 2 we can state the main result of [1] as

Theorem 3.[1] *Let v, k, n and p be non-zero cardinal numbers such that* i) *v is non-finite,* ii) *$k < n < v$, and* iii) *$p \leq v$. Then if M is a set of cardinality v every k-tuple family F of M possesses a p-Steiner cover G such that every member $y \in G$ is a subset of M of cardinality n.*

A natural question arises as to whether Theorem 3 would be true if restriction iii) of Definition 2 were removed. The present paper's aim is to show this restriction is necessary.

All results achieved in the present paper are formalizable within Zermelo-Fraenkel set theory with the axiom of choice. For the most part the notation will be standard. If x is a set, $\overline{\overline{x}}$ will represent the cardinal number of x. Moreover, if n is any cardinal number then $[x]^n = \{y \subset x: \overline{\overline{y}} = n\}$.[2] The expression "$x \subset y$" means "$x$ is a subset of y" improper inclusion not being excluded. If α is an ordinal ω_α is the smallest ordinal number whose cardinality is $\aleph_\alpha$. As usual we write ω for ω_0.

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The generalization of Definition 2 is now formally stated.

Definition 4. A family F of subsets of M is called a k-tuple family of M, in the wider sense, if and only if it satisfies i) *and* ii) *of Definition 2.*

Definition 5. For each ordinal number α we define a cardinal $\mathfrak{a}_\alpha$, by transfinite induction, as follows: i) $\mathfrak{a}_0 = \aleph_0$, ii) *if* $\alpha = \alpha_0 + 1$ *then* $\mathfrak{a}_\alpha = 2^{\mathfrak{a}_{\alpha_0}}$, iii) *if α is a limit number, then* $\mathfrak{a}_\alpha = \sum_{\beta<\alpha} \mathfrak{a}_\beta$.

It is now possible to state the main result of the present work.

Theorem 6. There is a set M of cardinality $\mathfrak{a}_\omega$ and an $\aleph_0$-tuple family (in the wider sense) F of M which does not possess a 1-Steiner cover G such that $G \subset [M]^{\aleph_1}$.

Before directly proceeding with a proof of Theorem 6 we establish some propositions of a general nature.

Definition 7. Let F be a family of subsets of a set M and n a nonzero cardinal number. A family G is called an n-spoiler of F if and only if for every $x \in F$ and every $y \in [M]^n$ there is a $z \in G$ such that $z \subset x \cup y$.

Proposition 8. Let k and n be non finite cardinal numbers and let F be a k-tuple family (in the wider sense) of an infinite set M. Suppose there exists subfamilies $F_1, F_2 \subset F$ such that i) $F_1 \cap F_2 = 0$, ii) F_2 is an n-spoiler of F_1 and iii) $n^k\overline{\overline{F}}_2 < \overline{\overline{F}}_1$. Then F does not possess a 1-Steiner cover contained in $[M]^n$.

Proof: To the contrary suppose there is a *1*-Steiner cover G of F such that $G \subset [M]^n$. Thus every member of F is contained in exactly one member of G. Now define a relation $\sim$ on F as follows.

Definition 9. Let $x, x' \in F$. $x \sim x'$ if and only if x and x' are contained in the same member of G.

It is immediate that $\sim$ defines an equivalence relation on F. Let $[x]^\sim$ represent the equivalence class which contains x.

Lemma 10. $(\exists x_0 \in F_1)\ (\forall x' \in F_2)\ [(x_0 \not\sim x')]$

Proof. Observe that since every member of G is a set of cardinality n and since any such set contains exactly n^k subsets of cardinality k we must have

(1) for each $z \in F$, $\overline{\overline{[z]^\sim}} \le n^k$.

Consequently (1) and iii) of Proposition 8 yield

(2) $\overline{\overline{\bigcup\{[z]^\sim \mid z \in F_2\}}} \le n^k\overline{\overline{F}}_2 < \overline{\overline{F}}_1$.

In view of (2)

(3) $(\exists x_0 \in F_1)\ (\forall z \in F_2)\ [x_0 \notin [z]^\sim]$.

Hence there is some x_0 in F_1 such that it is not the case that $x_0 \sim z$ for each $z \in F_2$. This proves Lemma 10.

Definition 11. Let y_0 be that unique member of G which contains x_0.

But since F_2 is an n-spoiler of F_1 and $x_0 \in F_1$ and $y_0 \in G \subset [M]^n$ we have

(4) $(\exists x^* \in F_2)\ [x^* \subset x_0 \cup y_0]$

which together with Definition 11 yields

(5) $x^* \subset y_0$.

But Definitions 9, 11 and (5) imply

(6) $x_0 \in [x^*]^{\sim}$

which says $x_0 \sim x^*$. But (6) and (4) contradict Lemma 10. This proves Proposition 8.

Proof of Theorem 6. Let M be any set of cardinality $\aleph_\omega$. By Definition 5 there exists for each n, $0 < n < \omega$, a set M_n such that

(7) $M = \bigcup \{M_n \mid 0 < n < \omega\}$

(8) $M_n \cap M_m = 0$ if $n \neq m$

and

(9) $\overline{\overline{M_n}} = \aleph_n$.

We begin our construction of a $\aleph_0$-tuple family (in the wider sense) of M with the following.

Lemma 12. For each n, $0 < n < \omega$, there exists a $\aleph_0$-tuple family (in the wider sense) F_n of M_n such that $(\forall y \in [M_n])^{\aleph_1})\ (\exists x \in F_n)\ [x \subset y]$.

Proof. By the well ordering theorem the family $[M_n]^{\aleph_1}$ may be expressed as follows

(10) $[M_n]^{\aleph_1} = \{y_\xi \mid \xi < \mu\}$.

The construction of the family F_n will be accomplished by transfinite induction in the following manner. Let $\gamma < \mu$. Suppose we have found a $\aleph_0$-tuple family (in the wider sense) F of M_n such that

(11) $(\forall \xi < \gamma)\ (\exists x \in F)\ [x \subset y_\xi]$.

The construction will be complete if we can establish the existence of $\aleph_0$-tuple family F_n such that

(12) $(\forall \xi \leq \gamma)\ (\exists x \in F_n)\ [x \subset y_\xi]$

We distinguish the following cases.

Case 1°. $(\exists x \in F)\ [x \subset y_\gamma]$

Here we may let $F_n = F$ and (12) follows immediately from (11).

Case 2°. $(\forall x \in F)\ [x \not\subset y_\gamma]$

Since $y_\gamma \in [M_n]^{\aleph_1}$ there exists x^* such that

(13) $\overline{\overline{x^*}} = \aleph_0$

and

(14) $x^* \subset y_\gamma$.

Definition 13. *Let* $F_n = (F - \{x \in F | x^* \subset x\}) \cup \{x^*\}$

We now must show F_n is i) an $\aleph_0$-tuple family (in the wider sense) of M_n and ii) (12) is satisfied. With regard to i) let x and y be such that

(15) $x, y \in F_n$

and

(16) $x \neq y$.

If $x, y \in F$ then it is clear, from the fact that F is an $\aleph_0$-tuple family (in the wider sense) that $x \not\subset y$ and $y \not\subset x$. Now suppose either x or y is x^*. In fact, assume

(17) $x = x^*$

which with (15), (16) and Definition 13 implies

(18) $y \in F - \{x \in F | x^* \subset x\}$.

From (18) it is clear that

(19) $x = x^* \not\subset y$.

Moreover, suppose

(20) $y \subset x$.

But (20) together with (17) and (14) give

(21) $y \subset y_\gamma$.

Yet (18) and (21) contradict the assumption of Case 2°. Thus (20) cannot obtain which shows F_n is an $\aleph_0$-family (in the wider sense) of M_n.

To see F_n satisfies (12) let $\xi \leq \gamma$. If $\xi = \gamma$ then (14) and Definition 13 show that $x^* \in F_n$ and $x^* \subset y_\xi$. Now suppose $\xi < \gamma$. By (11) there must be $x \in F$ such that

(22) $x \subset y_\xi$.

Suppose $x \subset x^*$. But this would imply by (14)

(23) $x \subset y_\gamma$

again contradicting the assumption of Case 2°. Consequently we have $x \not\subset x^*$ which implies with Definition 13

(24) $x \in F_n$.

This shows F_n satisfies (12) and consequently completes the proof of Lemma 12.

Definition 14. $F^\# = \bigcup \{F_n \mid 0 < n < \omega\}$.

Remark. Since each F_n is an $\aleph_0$-tuple family (in the wider sense) of M_n (and therefore of M) and since they are pairwise disjoint it follows that $F^{\#}$ is an $\aleph_0$-tuple family (in the wider sense) of M.

Lemma 15. $\overline{\overline{F^{\#}}} \leq \alpha_\omega$.

Proof. Since $F_n \subset [M_n]^{\aleph_0}$ and since

(25) $\overline{\overline{[M_n]^{\aleph_0}}} = \alpha_n{}^{\aleph_0}$

we arrive at, in view of Definition 14

(26) $\overline{\overline{F^{\#}}} \leq \sum_{0<n<\omega} \alpha_n^{\aleph_0}$.

But for each n, $0 < n < \omega$, we have

(27) $\alpha_n^{\aleph_0} = (2^{\alpha_{n-1}})^{\aleph_0} = 2^{\alpha_{n-1}\aleph_0} = 2^{\alpha_{n-1}} = \alpha_n$.

Thus (26) and (27) yield

(28) $\overline{\overline{F^{\#}}} \leq \sum_{0<n<\omega} \alpha_n = \alpha_\omega$

which proves Lemma 15.

Definition 16. $F^* = \{y \in [M]^{\aleph_0} \mid$ *for each* n, $\overline{\overline{y \cap M_n}} = 1\}$.

Remark. Since the M_n's are disjoint it is immediate from Definition 16 that F^* is an $\aleph_0$-tuple family of M. (Note that if $y_1, y_2 \in F^*$ and $y_1 \neq y_2$, there must exist some n such that $y_1 \cap M_n \neq y_2 \cap M_n$. Let $y_1 \cap M_n = \{p_1\}$ and $y_2 \cap M_n = \{p_2\}$. Clearly $p_1 \in y_1 - y_2$ and $p_2 \in y_2 - y_1$ showing $y_2 \not\subset y_1$ and $y_1 \not\subset y_2$).

Lemma 17. $\overline{\overline{F^*}} > \alpha_\omega$.

Proof. It is clear from Definition 16 that the family F^* is equinumerous with the generalized Cartesian product $\bigtimes_{0<n<\omega} M_n$. Hence (9) gives

(29) $\overline{\overline{F^*}} = \prod_{0<n<\omega} \overline{\overline{M_n}} = \prod_{0<n<\omega} \alpha_n$.

But by an immediate corollary[3] to a theorem by J. König and the fact that the sequence of cardinals $\{\alpha_n\}_{n<\omega}$ is strictly increasing we obtain

(30) $\sum_{0<n<\omega} \alpha_n < \prod_{0<n<\omega} \alpha_n$

which together with (29) and Definition 5 yield $\overline{\overline{F^*}} > \alpha_\omega$ which proves Lemma 17.

Lemma 18. $F^{\#} \cap F^* = 0$.

Proof. Immediate.

Lemma 19. $(\forall y \in [M]^{\aleph_1})\ (\exists n < \omega)\ [\overline{\overline{y \cap M_n}} \geq \aleph_1]$.

Proof. Let $y \in [M]^{\aleph_1}$. Now suppose to the contrary that

(31) $(\forall n < \omega)\ [\overline{\overline{y \cap M_n}} < \aleph_1]$

which immediately implies

(32) $(\forall n < \omega)\ [\overline{\overline{y \cap M_n}} \leq \aleph_0]$

But it is clear that

(33) $y = \bigcup \{y \cap M_n \mid 0 < n < \omega\}$

which with (32) yields

(34) $y \leq \overline{\omega}\aleph_0 = \aleph_0 \aleph_0 = \aleph_0$

contradicting the fact that $y \in [M]^{\aleph_1}$. This establishes Lemma 19.

Lemma 20. $F^{\#}$ is an $\aleph_1$-spoiler of F^*.

Proof. Let $x \in F^*$ and $y \in [M]^{\aleph_1}$. Using Lemma 19 there is an n_0, $0 < n_0 < \omega$, such that

(35) $\overline{\overline{y \cap M_{n_0}}} \geq \aleph_1$

which implies, since $\overline{\overline{y}} = \aleph_1$

(36) $\overline{\overline{y \cap M_{n_0}}} = \aleph_1$.

Consequently $(y \cap M_{n_0}) \in [M_{n_0}]^{\aleph_1}$. Using Lemma 12 we know there is an x_0 such that

(37) $x_0 \in F_{n_0}$

and

(38) $x_0 \subset y \cap M_{n_0}$.

But (37) and Definition 14 give

(39) $x_0 \in F^{\#}$

and (38) gives

(40) $x_0 \subset x \cup y$.

Consequently, in terms of Definition 7, $F^{\#}$ is seen to be an $\aleph_1$-spoiler of F^*, which establishes Lemma 20.

Lemma 21. $\aleph_1^{\aleph_0}\overline{\overline{F^{\#}}} < \overline{\overline{F^*}}$

Proof. Since $\aleph_1 \leq 2^{\aleph_0} = \alpha_1$ it is clear that

(41) $\aleph_1^{\aleph_0} \leq \alpha_1^{\aleph_0} = (2^{\alpha_0})^{\aleph_0} = 2^{\alpha_0} = \alpha_1$.

Using Lemma 15 and (41) we obtain

(42) $\aleph_1^{\aleph_0}\overline{\overline{F^{\#}}} \leq \aleph_1^{\aleph_0}\alpha_\omega \leq \alpha_1 \alpha_\omega = \alpha_\omega$.

But (42) and Lemma 17 yield

(43) $\aleph_1^{\aleph_0}\overline{\overline{F^{\#}}} \leq \alpha_\omega < \overline{\overline{F^*}}$

which was to be proved.

If we let $F = F^{\#} \cup F^{*}$ we now see, that the conditions of Proposition 8 are satisfied. (Let $F_1 = F^{*}$, $F_2 = F^{\#}$, $k = \aleph_0$ and $n = \aleph_1$. Then i), ii), and iii) are satisfied in virtue of Lemmas 18, 20, and 21, respectively.) Thus the $\aleph_0$-tuple family (in the wider sense) F of the set M does not possess a 1-Steiner cover G contained in $[M]^{\aleph_1}$. This concludes the proof of Theorem 6.

NOTES

1. This appears as Theorem III.12 in [1].
2. Moreover, we make use of the result that if x is a nonfinite set and if n is a non-zero cardinal number such that $n \leq \overline{\overline{x}}$ then $\overline{\overline{[x]^n}} = \overline{\overline{x}}^n$. For a proof of this see [2], p. 291.
3. Ibid., p. 204.

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