Notre Dame Journal of Formal Logic
Volume X, Number 3, July 1969

A SEMI-COMPLETENESS THEOREM*

STEPHEN L. BLOOM

In [2] and [3] the hyperprojective hierarchy of subsets of $\mathfrak{N}$ (the functions from **N**, the natural numbers, into **N**) was constructed using the hyperarithmetic hierarchy as a model. As a consequence, many properties of the hyperarithmetic sets have perfect analogues in the hyperprojective sets. In this paper we consider the problem of finding a "projective analogue" for the following important fact connected with the hyperarithmetic hierarchy:

The set of Gödel numbers of the recursive well-orderings is a complete π_1^1 set of natural numbers (see [5]).

In [2] it was shown that the set of indices of the projective well-orderings[1] of subsets of $\mathfrak{N}$ is a Δ_1^2 set[2] and thus cannot be a "complete π_1^2 set" in any natural sense, as our analogy would have it. The difficulty is that in order to express the notion "there is no countable descending chain of functions such that . . ." one needs only a function quantifier, not a quantifier over functions from $\mathfrak{N}$ into **N**. Thus we are led to consider the collection W^* of indices of those projective linear orderings having no *uncountable* descending chains.

In this paper we will show that W^* has a semi-completeness property with respect to a subclass of the π_1^2 sets. Our proof will assume the existence of a projective well-ordering $<^*$ of all of $\mathfrak{N}$ such that $\mathfrak{N}$ in this ordering is order-isomorphic to the first uncountable ordinal Ω. This assumption is consistent with the usual axioms for set theory, since the existence of a Δ_2^1 well-ordering of $\mathfrak{N}$ (of length Ω) follows from the Axiom of Constructibility [1].

Definition. If a subset B of $\mathfrak{N}$ is linearly ordered by some relation $<B$,

*This result is contained in the author's Ph.D. dissertation, [2], done under the direction of Hilary Putnam. The preparation of this paper was partially supported by a grant from the Stevens Institute of Technology.

Received August 17, 1968

we say B has an *uncountable descending chain* (**UDC**) if there is a function f from Ω into B such that if $\mu < \nu$ $(<\Omega)$ then $f(\nu) <_B f(\mu)$; i.e. f is order-reversing.

The real numbers **R** in their usual order have no **UDC**. For suppose that f: $\Omega \rightarrow \mathbf{R}$ is order-reversing. For each $\mu < \Omega$ let r_μ be a rational number between $f(\mu)$ and $f(\mu + 1)$. Since $\mu \neq \nu$ implies $r_\mu \neq r_\nu$, the collection $\{r_\mu\}$ of rationals must be uncountable, an impossibility.

Since we are assuming $\mathfrak{N}$ and Ω are equivalent as well-ordered sets, we can say that B has a **UDC** if there is function from $\mathfrak{N}$ into B which is order-reversing. Using this observation (and the fact cited in note 2) it is not difficult to show that the set W^* of the indices of projective linear orderings having no **UDC** is π_1^2.

Definition. A π_1^2 set B is said to be *semi-complete* (complete) with respect to a subclass $\mathcal{C}$ of π_1^2 sets if for each set C in $\mathcal{C}$ there is a projective function f such that $\alpha \,\varepsilon\, C$ iff $f(\alpha) \,\varepsilon\, B$ (and B is in $\mathcal{C}$ as well).

Definition. A primitive recursive predicate $R(\alpha, \delta, F)$ is called *bounded* if, for fixed α, the arguments of F are all $\leq^* \alpha$.

For example, if $R(\alpha, \delta, F)$ is the predicate

$$F(\alpha) = \delta(0)$$

then R is bounded; but if $R(\alpha, \delta, F)$ is the predicate

$$F(\delta) = \alpha(0)$$

R is not bounded. Note that if R is bounded, so is the negation of R.

Let $\mathcal{C}_0$ be the collection of all sets C definable in the form

$$\delta \,\varepsilon\, C \longleftrightarrow (\forall F)(\exists \alpha)\, R(\alpha, \delta, F)$$

where $R(\alpha, \delta, F)$ is a primitive recursive bounded predicate. We can now state our main result.

Theorem. W^* is semi-complete with respect to $\mathcal{C}_0$.

Before beginning the proof, we introduce some notation. For any function β, let β_n be the function defined by[3]

$$\beta_n(x) = \beta(<n,x>) \quad ;$$

also, let $\beta_n^+(x) = \beta_n(x + 1)$.

Let F be any functional (i.e. a function from $\mathfrak{N}$ into **N**). For a fixed function α, we define a functional F/α and a function $\overline{F}\alpha$ as follows:

$$F/\alpha(\gamma) = \begin{cases} F(\gamma) \text{ if } \gamma \leq^* \alpha; \\ 0 \text{ otherwise.} \end{cases}$$

$\overline{F}\alpha$ is the $<^*$-least function β having the following three properties:

(i) If $\gamma \leq {}^*\alpha$, then $\gamma = \beta_n^+$, some n.
(ii) For each n, $\beta_n^+ \leq {}^*\alpha$
(iii) $F(\beta_n^+) = \beta_n(0)$.

Thus F/α is a functional agreeing with F on the segment of functions $\gamma \leq {}^*\alpha$; $\overline{F}\alpha$ is a function which codes the values of F (and F/α) on this segment. Note that this segment contains at most a countable number of functions.

For later reference, we note an important property of bounded predicates.

(a) If F is a functional such that

$$(\forall \beta)_{\leq^* \alpha} R(\beta, \delta, F/\alpha)$$

then

$$(\forall \beta)_{\leq^* \alpha} R(\beta, \delta, F)$$

and conversely.

We now begin the proof. Let $R(\alpha, \delta, F)$ be any bounded primitive recursive predicate, and let the function δ be fixed. We define a set $U(\delta)$ by

$$U(\delta) = \{\overline{F}\alpha : (\forall \beta)_{\leq^* \alpha} R(\beta, \delta, F/\alpha)\} .$$

$U(\delta)$ is the set of functions coding the functionals that "work" up to α.

We impose a Kleene-Brouwer ordering on $U(\delta)$[4];

(b) $\overline{F}\alpha \ <_{\mathrm{KB}} \overline{G}\beta$ iff

(i) $\beta <^* \alpha \ \gamma \ G/\beta = F/\beta$,

or

(ii) $F(\gamma_0) < G(\gamma_0)$, where γ_0 is the $<^*$-least function on which F and G disagree.

The important clause in this definition is (i); if $\overline{F}\alpha$ codes a longer segment of the functional coded by $\overline{G}\beta$, then $\overline{F}\alpha <_{\mathrm{KB}} \overline{G}\beta$. We will denote an **UDC** in $U(\delta)$ by $\{\overline{F}_\alpha \beta_\alpha\}_\alpha$; i.e. for each $\alpha \varepsilon \mathfrak{N}$, $\overline{F}_\alpha \beta_\alpha$ is in $U(\delta)$ and if $\alpha_1 <^* \alpha_2$ then $\overline{F}_{\alpha_2} \beta_{\alpha_2} <_{\mathrm{KB}} \overline{F}_{\alpha_1} \beta_{\alpha_1}$. Our proof will be completed by the following lemmas.

Lemma 1. $U(\delta)$ is a projective set and $<_{\mathrm{KB}}$ is a projective linear ordering of $U(\delta)$. Moreover, there is a projective function kb(δ) *giving the index*[1] *of $<_{\mathrm{KB}}$ as a function of δ; i.e. $<_{\mathrm{KB}}$ on $U(\delta)$ = $\mathcal{A}_{\mathrm{kb}(\delta)}$.*

Lemma. 2. If $(\exists F)\ (\alpha) R(\alpha, \delta, F)$ then there is an **UDC** *in $U(\delta)$.*

Lemma 3. If there is any **UDC** *in $U(\delta)$, then there is one of the form $\{\overline{F}\beta_\alpha\}_\alpha$, where each function in the chain codes is a segment of the same functional.*

Lemma 4. If there is an **UDC** *in $U(\delta)$, then $(\exists F)\ (\alpha) R(\alpha, \delta, F)$.*

Proofs. The proof of lemma 1 is tedious but routine, and is omitted (see [2], section 9). Suppose that F is a functional such that $(\alpha)\ R(\alpha, \delta, F)$. Then $\{\overline{F}\alpha\}_\alpha$ is a **UDC** in $U(\delta)$; indeed, if $\alpha_1 <^* \alpha_2$ then $F_{\alpha_2} <_{\mathsf{KB}} \overline{F}_{\alpha_1}$ by (b), clause (i). This proves lemma 2.

Lemma 3 is the heart of the argument and we postpone its proof until the end. To prove lemma 4, assume that $\{\overline{F}\beta_\alpha\}_\alpha$ is a **UDC** in $U(\delta)$. This is no loss of generality by lemma 3. But now we may define a functional F_0 by

$$F_0(\gamma) = m \longleftrightarrow (\exists\beta_\alpha)\ (\gamma <^* \beta_\alpha\ \&\ \overline{F}\beta_\alpha(\gamma) = m)$$

where

$$\overline{F}\beta_\alpha(\gamma) = m \longleftrightarrow (\exists n)\ (x)\ (\overline{F}\beta_\alpha(<n, x+1>) = \gamma(x)\ \&\ \overline{F}\beta_\alpha(<n, 0>) = m).$$

F_0 is well-defined, since the functions $\overline{F}\beta_\alpha$ agree on their common domain. F_0 is defined everywhere since for any fixed γ the set of functions $\leq^* \gamma$ is countable. Thus there must be a β_α such that $\gamma <^* \beta_\alpha$.

We claim that $(\alpha)\ R(\alpha, \delta, F_0)$. Indeed, let γ be any fixed function. As noted above, there is an β_α such that $\gamma <^* \beta_\alpha$. Since $\overline{F}\beta_\alpha$ is in $U(\delta)$, $(\beta)_{\leqslant^* \beta_\alpha} R(\beta, \delta, F/\beta_\alpha)$; in particular then, $R(\gamma, \delta, F/\beta_\alpha)$. But since R is bounded and $F_0/\beta_\alpha = F/\beta_\alpha$, we have, by (a), $R(\gamma, \delta, F_0)$. Since γ is arbitrary, our claim is proved.

Proof of the Theorem. Let C be some set in $\mathcal{C}_0$. Then for some bounded primitive recursive R,

$$\delta \not\in C \longleftrightarrow (\exists F)\ (\alpha)\ R(\alpha, \delta, F)$$

Let $\mathsf{kb}(\delta)$ be the function of lemma 1. By lemmas 2 and 4

$$\delta \not\in C \longleftrightarrow \mathsf{kb}(\delta) \not\in W^*,$$

or

$$\delta \in C \longleftrightarrow \mathsf{kb}(\delta) \in W^*$$

which shows that W^* is semi-complete with respect to $\mathcal{C}_0$.

Proof of lemma 3. We note first that

(c) if $\overline{F}\beta$ is in $U(\delta)$ and $\gamma <^* \beta$, then $\overline{F}\gamma$ is in $U(\delta)$ also.

Now suppose that $\mathcal{K} = \{\overline{F}_\alpha\beta_\alpha\}_\alpha$ is some **UDC** in $U(\delta)$. We will write $\overline{F}_\alpha$ for $\overline{F}_\alpha\beta_\alpha$ and $\overline{F}_\alpha(\gamma)$ for the value of the functional F_α/β_α on γ. We will prove the following assertion by induction on $<^*$:

(d) For each α there is a function τ_α such that for every γ, $\tau_\alpha \leq^* \gamma$, the functionals coded by $\overline{F}_\gamma$ agree on the segment of $\mathfrak{N} \leq^* \alpha$; i.e. if $\tau_\alpha <^* \gamma$ and $\xi \leq^* \alpha$, then $\overline{F}_{\tau_\alpha}(\xi) = \overline{F}_\gamma(\xi)$.

Now suppose that (d) holds for all $\beta <^* \alpha$. We show that (d) holds for α as well. There are three possibilities.

Case I: α is $<^*$-least. Let $B = \{\overline{F}_\beta(\alpha) : \overline{F}_\beta \in \mathcal{K}\}$. B is a set of non-negative integers and has a least. Let τ_α be a function such that $\overline{F}_{\tau_\alpha}(\alpha)$ is least in B.

If $\tau_\alpha <^* \gamma$, then $\overline{F}_\gamma <_{KB} \overline{F}_{\tau_\alpha}$; thus either $\overline{F}_\gamma$ extends $\overline{F}_{\tau_\alpha}$ (in which case they agree on α) or at the least function γ_0 on which they differ, $\overline{F}_\gamma(\gamma_0) < \overline{F}_{\tau_\alpha}(\gamma_0)$. But since $\alpha \leq^* \gamma_0$ and $\overline{F}_{\tau_\alpha}(\alpha) \leq \overline{F}_\gamma(\alpha)$, $\overline{F}_\gamma$ and $\overline{F}_{\tau_\alpha}$ must agree on α. (By (c), $\overline{F}_{\tau_\alpha}\alpha$ is in $U(\delta)$.)

Case II. α is the $<^*$-successor of α_1. By assumption τ_{α_1} is defined. Suppose α_2 is the $<^*$-successor of τ_{α_1}. Then $\overline{F}_{\alpha_2}$ in $\mathcal{K}$ must be defined at least up to and including α, since $\overline{F}_{\alpha_2} <_{KB} \overline{F}_{\tau_{\alpha_1}}$ and $\overline{F}_{\alpha_2}$ agrees with $\overline{F}_{\tau_{\alpha_1}}$ on all function $\leq^* \alpha_1$. Thus the set B is non-empty, where B is the set $\{\overline{F}_\beta(\alpha) : \overline{F}_\beta \,\varepsilon\, \mathcal{K} \,\&\, \tau_{\alpha_1} <^* \beta\}$. Again choose τ_α such that $\overline{F}_{\tau_\alpha}(\alpha)$ is least in B. Arguing as in case I, we can show that τ_α satisfies (d).

Case III. α is a $<^*$-limit. Let $\{\beta_n\}$ be a countable sequence of functions such that $\beta_1 <^* \beta_2 <^* \ldots$ and $\alpha = \lim_n \beta_n$[5]. For each n, τ_{β_n} is defined. Let $\alpha_0 = \sup_n\{\tau_{\beta_n}\}$[5]. Then if $\alpha_0 <^* \gamma$, $\overline{F}_\gamma$ and F_{α_0} must agree on all function $<^*\alpha$. Indeed, if $\gamma_0 <^* \alpha$, there is an n such that $\gamma_0 <^* \beta_n$. But all functionals $\overline{F}_\gamma$ for $\tau_{\beta_n} <^* \gamma$ agree on γ_0. Thus if α_1 is the $<^*$-successor of α_0, $\overline{F}_{\alpha_1}(\alpha)$ must be defined. As before then, we choose τ_α such that $\overline{F}_{\tau_\alpha}(\alpha)$ is least in the set $\{\overline{F}_\beta(\alpha) : \overline{F}_\beta \,\varepsilon\, \mathcal{K} \,\&\, \alpha_0 \leq^* \beta\}$.

This completes the proof of (d). Finally, we choose the chain $\{\overline{F}_{\tau_\alpha}\alpha\}$. By (c) and (d) respectively we know that each function $\overline{F}_{\tau_\alpha}\alpha$ is in $U(\delta)$ and that if $\alpha_1 <^* \alpha_2$ then $\overline{F}_{\tau_{\alpha_2}}\alpha_2$ extends $\overline{F}_{\tau_{\alpha_1}}\alpha_1$ so that $\overline{F}_{\tau_{\alpha_2}}\alpha_2 <_{KB} \overline{F}_{\tau_{\alpha_1}}\alpha_1$. Thus lemma 3 is proved.

Remark. With only a slight modification of the above proof, one can show that W^* is semi-complete with respect to the class $\mathcal{C}_1$ of those π_1^2 sets C definable in the form

$$\delta \varepsilon C \longleftrightarrow (\forall F)\,(\exists \alpha)\, R(\alpha, \delta, F)$$

where R is primitive recursive and weakly bounded. (We say $R(\alpha, \delta, F)$ is *weakly bounded* if there is a projective function $f(\alpha)$ such that, for fixed α, the arguments of the functional F in R are $\leq^* f(\alpha)$.) $\mathcal{C}_1$ is a much larger collection than $\mathcal{C}_0$ but is still a proper subclass of π_1^2. Our method does not apply when we cannot legitimately limit our knowledge of the values of the functional to a proper segment of $\mathfrak{N}$.

NOTES

1. A subset B of $\mathfrak{N}$ is *Projective* if it is analytic in a finite number of functions; i.e. if there are functions $\Psi_1, \Psi_2, \ldots, \Psi_m$ in $\mathfrak{N}$ and a formula Φ of second order number theory having one free function variable v and m function constants $\mathsf{w}_1, \ldots, \mathsf{w}_m$ such that for any function α, $\alpha \in B$ iff Φ is true in the standard model of second-order arithmetic when v is interpreted as α and $\mathsf{w}_1, \ldots, \mathsf{w}_m$ as $\Psi_1, \ldots, \Psi_m$ respectively. An *index* γ of the projective set B is a function coding in some uniform way a Gödel number of a formula Φ defining B and the parameter functions $\Psi_1, \ldots, \Psi_m$. We let $\mathcal{A}_\gamma$ be the projective set with index γ. $\mathcal{A}_\gamma$ is a *projective well-ordering* (*linear ordering*) if the relation $\alpha \leq_\gamma \beta$ (defined by: $\alpha \leq_\gamma \beta \longleftrightarrow \langle \alpha, \beta \rangle \in \mathcal{A}_\gamma$ (see note 3)) well-orders (linearly orders) the set $\{\alpha\colon \alpha \leq_\gamma \alpha\}$.

2. A relation $R \subset \mathfrak{N}^k$ is called π_1^2 if R is definable in the form

$$R(\alpha_1, \ldots, \alpha_k) \longleftrightarrow (\forall G)(\exists \beta)\, S(\alpha_1, \ldots, \alpha_k, \beta, G)$$

where G is a variable over the functions from $\mathfrak{N}$ into $\mathbf{N}$, and where S is primitive recursive [4]. R is called Δ_1^2 if both R and its negation are π_1^2. In [2] it was shown that the relation $J(\alpha, \beta) \longleftrightarrow \alpha \in \mathcal{A}_\beta$ is Δ_1^2.

3. $\langle n, x\rangle = 2^n 3^x$; $\langle \alpha, \beta\rangle$ is the function whose value at x is $\langle \alpha(x), \beta(x)\rangle$.

4. There are three orderings being used at this point: the projective well-ordering $<^*$ of $\mathfrak{N}$ of length Ω; the Kleene-Brouwer ordering $<_{KB}$ of $U(\delta)$; and the normal ordering $<$ of the natural numbers.

5. The sequence $\{\beta_n\}$ and the least upper bound of the set $\{\tau\beta_n : n = 1, 2 \ldots\}$ exist since each function in $\mathfrak{N}$ corresponds to a countable ordinal in the well-ordering $<^*$.

REFERENCES

[1] Addison, J. W., "Some Consequences of the Axiom of Constructibility," *Fundamenta Mathematicae*, vol. 46, pp. 338-357.

[2] Bloom, S. L., "The Hyperprojective Hierarchy," Ph.D. Dissertation, M.I.T., January 1968.

[3] Bloom, S. L., "The Hyperprojective Hierarchy," to appear.

[4] Kleene, S. C., "Recursive Functionals and Quantifiers of Finite Types I," *Transactions of the American Mathematical Society* (1959), pp. 1-52.

[5] Spector, C., "Recursive Well-Orderings," *The Journal of Symbolic Logic*, vol. 20 (1955), pp. 151-163.

Stevens Institute of Technology
Hoboken, New Jersey