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SELECTION FUNCTIONS FOR RECURSIVE FUNCTIONALS*

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Many of the interesting results in the study of recursive functions of finite higher types owe their existence to the way that partial recursive functions can be defined inductively by schemes. However, the profusion of notation sometimes clouds the techniques involved in working with higher types. For greater clarity and possible generalization it is natural to reinvestigate these results in the setting of inductively defined sets. In particular, the selection functions of Gandy [1, p. 14], Moschovakis [3, (I) of p. 270] and Platek [4, pp. 178-182] are basically functions that choose from a countable set of trees of computation (indexed by numbers) those trees that are the shortest. More abstractly, they are functions that choose from some objects (some of) which belong to an inductively defined set those objects that are placed in the set at the earliest stage. We prove the existence of such functions in Theorems 1 and 2. Our Theorems 3 and 4 are analogues (in the setting of inductively defined sets) of two results related to selection functions; namely, that sets r.e. in E^{n+2} ($n > 0$) are not closed under type-n quantification (proved by Moschovakis [3, Theorem 10] for the case $n = 1$), and that predicates r.e. in E^{n+2} can be characterized by a bounded existential-quantifier form (proved by Gandy [1, p. 17] for the case $n = 0$ and by Moschovakis [3, Theorem 9] for the case $n = 1$). By showing in Theorem 5 that the inductive definition of partial recursive functions fits the hypotheses of Theorems 1 through 4, we are able to obtain the known theorems related to selection functions (Theorems 6, 7 and 8) for all higher types, plus a generalization to selections from uncountable sets (Theorem 6). This generalization enables us to characterize the types of quantification under which r.e. predicates are closed (Theorem 9).

We assume that the reader is familiar with the notation of the first half

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of Kleene's 1959 paper [2] on recursive functions of higher types. E^{n+2} will denote the type-$(n+2)$ object that introduces type-n quantification. Since we will use variables of types n and $n-1$ rather often, we reserve nonsuperscripted α, β, γ, δ, ε for type-n variables, nonsuperscripted ζ, η, ξ for type-$(n-1)$ variables and S, T for sets of type-n objects. We use ρ, σ, τ for ordinals. We regard the predicate $P(\mathfrak{a}) \vee Q(\mathfrak{a})$ as being computed from left to right so that $Q(\mathfrak{a})$ need not be computed (and hence need not be defined) when $P(\mathfrak{a})$ is true. A similar remark holds for $P(\mathfrak{a})$ & $Q(\mathfrak{a})$ and $P(\mathfrak{a}) \rightarrow Q(\mathfrak{a})$.

Let J be an operation that maps sets of type-n objects into sets of type-n objects. For each ordinal σ, we define a set S_σ of type-n objects inductively by the formula

$$S_\sigma = \bigcup_{\tau<\sigma} \mathsf{J}(S_\tau) \ .$$

The set $S = \bigcup_\sigma S_\sigma$ is called *the set inductively defined with respect to* J, and S_σ is called the σ*th stage of* S. An ordinal $|\alpha|$ can be assigned to each type-n object α as follows: if $\alpha \epsilon S$, let $|\alpha|$ be the smallest ordinal σ such that $\alpha \epsilon S_{\sigma+1}$; if $\alpha \notin S$, let $|\alpha|$ be the smallest ordinal σ such that $S_\sigma = S_{\sigma+1}$. The simplest type of selection function for an inductively defined set S is a partial function that selects from two objects the one (if any) that is put into S at an earlier stage. Thus a partial function $\phi(\alpha, \beta)$ is a selection function for the set S if

$$\text{(1)} \qquad \phi(\alpha,\beta) \simeq \begin{cases} 0 & \textit{if} \ \ |\alpha| \leq |\beta| \ \& \ \alpha \epsilon S, \\ 1 & \textit{if} \ \ |\beta| < |\alpha| \ \& \ \beta \epsilon S. \end{cases}$$

We now prove the existence of relatively simple selection functions of this sort.

Theorem 1. *Let S be inductively defined with respect to an operation* J, *and let $P(\alpha, \beta)$ have the property that, if $\alpha \epsilon S$, then*

$$\text{(2)} \qquad |\alpha| = \sup\{|\beta| + 1 : P(\alpha, \beta)\}.$$

Then there exists a partial function $\phi(\alpha, \beta)$, partial recursive in P, J *and* E^{n+2}, *such that equation* (1) *holds.*

Proof. Let $[\alpha]$ be the ordinal defined by the equation

$$[\alpha] = \sup\{|\beta| + 1 : P(\alpha, \beta)\}.$$

Thus $[\alpha] = |\alpha|$ whenever $\alpha \epsilon S$. Suppose that $\phi(\alpha, \beta)$ has been defined by a recursion (which we will exhibit presently) so that $\phi(\alpha, \beta)$ satisfies (1) for all α, β for which $\inf(|\alpha|, |\beta|) < \sigma$. Let one of α and β be in S and let $\inf(|\alpha|, |\beta|) = \sigma$. We will show that (1) holds. If $[\alpha] \leq \sigma$, then, by the induction hypothesis, we see that

$$\text{(3)} \qquad \begin{aligned} \delta \epsilon S_{[\alpha]} &\longleftrightarrow |\delta| < [\alpha] \\ &\longleftrightarrow \exists\gamma[P(\alpha, \gamma) \ \& \ |\delta| \leq |\gamma|] \\ &\longleftrightarrow \exists\gamma[P(\alpha, \gamma) \ \& \ \phi(\delta, \gamma) = 0]. \end{aligned}$$

Note that $\phi(\delta, \gamma)$ is computed only when $P(\alpha, \gamma)$ is true and hence only when $|\gamma| < [\alpha] \leq \sigma$. Also, by the induction hypothesis, we obtain

$$(4) \qquad \begin{aligned}[\alpha] \leq [\beta] &\longleftrightarrow \forall\gamma[P(\alpha, \gamma) \to \exists\delta(P(\beta, \delta) \;\&\; |\gamma| \leq |\delta|)] \\ &\longleftrightarrow \forall\gamma[P(\alpha, \gamma) \to \exists\delta(P(\beta, \delta) \;\&\; \phi(\gamma, \delta) = 0)].\end{aligned}$$

Since

$$|\alpha| \leq |\beta| \longleftrightarrow [\alpha] \leq [\beta] \;\&\; \alpha \epsilon \mathrm{J}(S_{[\alpha]}),$$

$\phi(\alpha, \beta)$ can be defined by the recursion

$$\beta) \simeq \begin{cases} 0 & \text{if } [\alpha] \leq [\beta] \;\&\; \alpha \epsilon \mathrm{J}(S_{[\alpha]}), \\ 1 & \text{if not,} \end{cases}$$

where for the expressions $[\alpha] \leq [\beta]$ and $S_{[\alpha]}$ we use the equivalent expressions given in (3) and (4) involving ϕ, P and type-n quantifiers. The only thing left to verify is that equivalence (3) is used only when $[\alpha] \leq \sigma$ (since, otherwise, the expression replacing $S_{[\alpha]}$ may be undefined). However, we compute $\mathrm{J}(S_{[\alpha]})$ only when $[\alpha] \leq [\beta]$, and hence only when $[\alpha] \leq \inf(|\alpha|, |\beta|) = \sigma$. This completes the proof.

The preceding theorem proves the existence of selection functions that select from two objects the one that is put into an inductively defined set at an earlier stage. Now we will prove the existence of selection functions that select from larger collections. In the following α_ζ will be an abbreviation for $\lambda_\eta \alpha(<\zeta, \eta>)$. This allows us to think of the type-n object α as an abbreviation for an infinite collection of objects of the form α_ζ.

Theorem 2. *Let $n > 0$. Let S be inductively defined with respect to an operation J, and let $P(\alpha, \beta)$ have the property that, if $\alpha \epsilon S$, then*

$$(5) \qquad |\alpha| = \sup\{\inf_\zeta |\beta_\zeta| + 1 : P(\alpha, \beta)\}.$$

Then there exists a partial function $\psi(\zeta, \alpha)$, partial recursive in P, J and E^{n+2}, such that, if $\alpha_\zeta \epsilon S$ for some ζ, then

$$\psi(\xi, \alpha) \simeq \begin{cases} 0 & \text{if } |\alpha_\xi| \leq \inf_\zeta |\alpha_\zeta|, \\ 1 & \text{if not.} \end{cases}$$

Thus ψ specifies which α_ξ are put into S at the earliest stage.

Proof. Let α^* be some object such that $\alpha^*_\zeta = \alpha$ for all ζ; for instance, let $\alpha^* = \lambda_\zeta \alpha((\zeta)_1)$. It is easily seen that the set $S' = \{\alpha : \exists\zeta(\alpha_\zeta \epsilon S)\}$ is inductively defined with respect to the operation J' defined by the equivalence

$$\alpha \epsilon \mathrm{J}'(T) \longleftrightarrow \exists\zeta(\alpha_\zeta \epsilon \mathrm{J}(\{\beta : \beta^* \epsilon T\}))$$

in such a way that the σth stage S'_σ is the set $\{\alpha : \exists\zeta(\alpha_\zeta \epsilon S_\sigma)\}$. Thus, for all α, $|\alpha|' = \inf_\zeta |\alpha_\zeta|$ where $|\alpha|$ (respectively, $|\alpha|'$) denotes the least ordinal σ such that $\alpha \epsilon S_{\sigma+1}$ (respectively, $\alpha \epsilon S'_{\sigma+1}$). Using the fact that any function $F(\zeta, \beta)$ into the ordinals satisfies

$$\inf_\zeta \sup_\beta F(\zeta, \beta) = \sup_\beta \inf_\zeta F(\zeta, \beta_\zeta),$$

we obtain

$$\begin{aligned}|\alpha|' &= \inf_\zeta |\alpha_\zeta| \\ &= \inf_\zeta \sup \{\inf_\xi |\beta_\xi| + 1 : P(\alpha_\zeta, \beta)\} \\ &= \sup \{\inf_\zeta \inf_\xi |\gamma_{\zeta\xi}| + 1 : \forall\zeta P(\alpha_\zeta, \gamma_\zeta)\} \\ &= \sup \{|\delta|' + 1 : P'(\alpha, \delta)\},\end{aligned}$$

where

$$P'(\alpha, \delta) \longleftrightarrow \exists\gamma[\forall\zeta P(\alpha_\zeta, \gamma_\zeta) \ \&\ \forall\zeta\forall\xi(\gamma_{\zeta\xi} = \delta_{<\zeta,\xi>})].$$

By the preceding theorem, there exists a partial function $\phi(\alpha, \beta)$, partial recursive in P', J' and $\mathbf{E}^{n+2}$ (and hence partial recursive in P, J and $\mathbf{E}^{n+2}$) such that

$$\phi(\alpha, \beta) \simeq \begin{cases} 0 & \text{if } |\alpha|' \leq |\beta|' \ \&\ \alpha\epsilon S', \\ 1 & \text{if } |\beta|' < |\alpha|' \ \&\ \beta\epsilon S'. \end{cases}$$

Since $|\alpha|' = \inf_\zeta |\alpha_\zeta|$ and $|\alpha_\xi*|' = |\alpha_\xi|$, the function $\psi(\xi, \alpha) \simeq \phi(\alpha_\xi*, \alpha)$ is the desired function.

The next theorem shows that, in a few cases, the complement of an inductively defined set S can be reduced to S by using type-n quantifiers. This sort of reduction was proved by Moschovakis [3, Theorem 10] in the case where S is r.e. in $\mathbf{E}^3$. Recall that a predicate $P(\mathfrak{a})$ is r.e. in $\mathfrak{b}$ if $P(\mathfrak{a}) \longleftrightarrow [\{e\}(\mathfrak{a}, \mathfrak{b})$ is defined$]$ for some e. He showed that the negation of a predicate r.e. in $\mathbf{E}^3$ has the form $\exists\alpha^1\ P(\alpha^1, \cdot)$ where P is r.e. in $\mathbf{E}^3$.

Theorem 3. *Let $n > 0$. Let S, J and P be as in the preceding theorem. There exists a predicate $R(\alpha, \beta)$,* r.e. *in J, P and $\mathbf{E}^{n+2}$, such that*

$$\alpha\notin S \longleftrightarrow \exists\beta R(\alpha, \beta).$$

Proof. For each α, let

$$[\alpha] = \sup \{\inf_\zeta |\beta_\zeta| + 1 : P(\alpha, \beta)\}.$$

Thus $[\alpha] = |\alpha|$ whenever $\alpha\epsilon S$. Let

$$N(\alpha) \longleftrightarrow [\alpha\epsilon \mathsf{J}(S_{[\alpha]}) \textit{ is defined and false}],$$

where for $S_{[\alpha]}$ we use the following equivalent form, which by the preceding theorem is partial recursive in P, J, $\mathbf{E}^{n+2}$ and is defined when $[\alpha] < \sup_{\beta\epsilon S} |\beta|$:

$$\delta\epsilon S_{[\alpha]} \longleftrightarrow \exists\gamma[P(\alpha, \gamma) \ \&\ |\delta| \leq \inf_\zeta |\gamma_\zeta|].$$

If $n > 1$, in analogy with the familiar notation $x * y$ and $\mathrm{seq}(x)$, we define $\zeta * \eta = \lambda\epsilon(\zeta(\epsilon) * \eta(\epsilon))$ and $\mathrm{seq}(\zeta) \longleftrightarrow \zeta = <(\zeta)_0, \ldots, (\zeta)_{\mathrm{lh}(\zeta)}>$ where $\mathrm{lh}(\zeta) = \mathrm{lh}(\zeta(\lambda 0))$. Suppose $\alpha\notin S$. For each ζ satisfying $\mathrm{seq}(\zeta)$, we can define an object $\beta_\zeta\notin S$ inductively as follows. Let $\beta_{<>} = \alpha$. Suppose β_ζ has been defined so that $\beta_\zeta\notin S$. If $N(\beta_\zeta)$ is true, let $\beta_{\zeta*<\eta>} = \beta_\zeta$ for all η. Otherwise, $S_{[\beta_\zeta]}$ must not be defined, which implies that there exists some γ such that $P(\beta_\zeta, \gamma) \ \&\ \forall_\eta(\gamma_\eta\notin S)$; so let $\beta_{\zeta*<\eta>} = \gamma_\eta$ for all η. We see that the object β we have defined satisfies the predicate

$$R(\alpha, \beta) \longleftrightarrow \beta_{<>} = \alpha \;\&\; \forall\zeta[\mathrm{seq}(\zeta) \rightarrow N(\beta_\zeta) \vee \exists\gamma(P(\beta_\zeta, \gamma) \;\&\; \forall\eta(\beta_{\zeta*<\eta>} = \gamma_\eta))].$$

Therefore, $\alpha \notin S \rightarrow \exists\beta R(\alpha, \beta)$.

Conversely, suppose that $\gamma \in S_\tau \rightarrow \neg\exists\beta R(\gamma, \beta)$ for all γ and all $\tau < \sigma$. Let $\alpha \in S_\sigma$, but suppose that $R(\alpha, \beta)$ is true for some β. By the last clause of $R(\alpha, \beta)$ and by equation (5), there exists an object η and an ordinal $\tau < \sigma$ such that $\beta_{<\eta>} \in S_\tau$. Let δ be any object satisfying $\delta_\zeta = \beta_{<\eta>*\zeta}$ for all ζ. It follows that $R(\beta_{<\eta>}\ \delta)$ is true, contradicting the induction hypotheis. Therefore, $\alpha \notin S \longleftrightarrow \exists\beta R(\alpha, \beta)$. Since $N(\alpha)$ is r.e. in P, J and $\mathbf{E}^{n+2}$, $R(\alpha, \beta)$ is also.

Sets with strong closure properties can often be expressed in terms of a bounded existential quantifier form. Thus the $\prod_1^1$ predicates are precisely those predicates of the form $\exists\alpha^1_{\mathrm{HA}}\ P(\alpha^1, \cdot)$ where P is arithmetical. The next theorem leads to results of this sort for higher types. The idea of the theorem is to find some simple conditions that characterize the stages of an inductively defined set S. We cannot quite do this because in order to identify some stage we need to know the history of previous stages. Rather, we will find some simple conditions that characterize the sets

$$S^*_\sigma = \{<\alpha, \beta>: |\alpha| < |\beta| < \sigma\}.$$

Theorem 4. *Let S be inductively defined with respect to an operation* J. *Let T_α be an abbreviation for the set $\{\beta: <\beta, \alpha> \in T\}$ and let $U_T = \{(\alpha)_0, (\alpha)_1 : \alpha \in T\}$. The sets S^*_σ are precisely the sets T satisfying*:

(a) *T consists of objects only of the form $<\alpha, \beta>$;*

(b) *if $\alpha \in U_T$, then $\mathrm{J}(T_\alpha) \subseteq U_T$;*

(c) *the relation $<$ on U_T given by the rule $\alpha < \beta \longleftrightarrow \alpha \in T_\beta$ is well-founded, and the relation $\leq$ on U_T given by the rule $\alpha \leq \beta \longleftrightarrow \alpha \in T_\beta \cup \mathrm{J}(T_\beta)$ satisfies $\alpha \leq \alpha$ and $\alpha \leq \beta \;\&\; \beta < \gamma \rightarrow \alpha < \gamma$.*

Proof. It is evident that S^*_σ satisfies (a), (b) and (c). Suppose T satisfies (a), (b) and (c). It is sufficient to show that

$$\alpha \in U_T \rightarrow \alpha \in S \;\&\; S_{|\alpha|} = T_\alpha. \tag{6}$$

If this is the case, then $T = S^*_\sigma$ where $\sigma = \sup_{\alpha\in U_T}(|\alpha| + 1)$. (Condition (b) is needed to show that $S^*_\sigma \subseteq T$.) Suppose (6) is false for some α minimal in the ordering $<$. If $\beta < \alpha$, then $\beta \in S_{|\beta|+1}$ and

$$S_{|\beta|+1} = S_{|\beta|} \cup \mathrm{J}(S_{|\beta|}) = T_\beta \cup \mathrm{J}(T_\beta) \subseteq T_\alpha.$$

Thus

$$T_\alpha = \bigcup\nolimits_{\beta<\alpha} S_{|\beta|+1},$$

which implies that $T_\alpha = S_\tau$ for some τ. Since $\alpha \in T_\alpha \cup \mathrm{J}(T_\alpha) = S_{\tau+1}$ and since $\alpha \in T_\alpha = S_\tau$, it follows that $S_{|\alpha|} = S_\tau$, contradicting our supposition that (6) is false for α.

Corollary. *Let* $n > 0$, *and let* S, J *and* P *be as in* Theorem 2. *There exists a predicate* $Q(\alpha, T)$, *formed from logical connectives, quantifiers of type* n, *the operation* J *and recursive predicates, such that*

$$\exists T Q(\alpha, T) \rightarrow \alpha \epsilon S \rightarrow Q(\alpha, S^*_{|\alpha|+1}).$$

Hence by Theorem 2,

$$\alpha \epsilon S \longleftrightarrow \exists T Q(\alpha, T) \longleftrightarrow \exists T\,[Q(\alpha, T)\ \&\ T \text{ is recursive in } \alpha, P, \mathrm{J}, \mathbf{E}^{n+2}].$$

Proof. Let $Q(\alpha, T)$ say that (a), (b) and (c) of Theorem 4 hold and that $\alpha \epsilon U_T$.

We will now apply the previous results to recursive functionals. Let $\mathbf{F}$ be any type-$(n + 2)$ object and let $\mathbf{E}$ be an abbreviation for $\mathbf{E}^{n+2}$. To each defined $\{z\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]$, we associate an ordinal $||\{z\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]||$ equal to the supremum of all ordinals of the form $||\{y\}^{\mathbf{E},\mathbf{F}}[\beta^0, \ldots, \beta^n]|| + 1$, where $\{y\}^{\mathbf{E},\mathbf{F}}[\beta^0, \ldots, \beta^n]$ is in the tree of computation of $\{z\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]$. The idea of the following theorem is to inductively define the set

$$S = \{< w, z, \alpha^0, \ldots, \alpha^n >: \{z\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n] \simeq w\}$$

with respect to a relatively simple operation so that

$$|< w, z, \alpha^0, \ldots, \alpha^n >| = ||\{z\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]||.$$

One minor complication arises from schemes S4 and S5. These schemes force us to work with pairings, and so we modify the definition of S accordingly as follows.

Theorem 5. *Let* $n > 0$. *The set*

$$S = \{<w, y, \alpha^0, \ldots, \alpha^n, x, z, \beta^0, \ldots, \beta^n>: \\ w \simeq \{y\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]\ \&\ x \simeq \{z\}^{\mathbf{E},\mathbf{F}}[\beta^0, \ldots, \beta^n]\}.$$

can be inductively defined with respect to an operation J *that is* Δ^n_2 *in* $\mathbf{E}^2$ *and* $\mathbf{F}$, *uniformly in* $\mathbf{F}$, *in such a way that some* $\sum^n_1$ *predicate* P *satisfies*

$$|\gamma| = \sup\{\inf_x |\delta_x| + 1 : P(\gamma, \delta)\} \\ = ||\{y\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]|| \vee ||\{z\}^{\mathbf{E},\mathbf{F}}[\beta^0, \ldots, \beta^n]|| \tag{7}$$

whenever

$$\gamma = < w, y, \alpha^0, \ldots, \alpha^n, x, z, \beta^0, \ldots, \beta^n > \epsilon S. \tag{8}$$

(The notation δ_x is defined like δ_ξ; i.e., $\delta_x = \lambda_n \delta(< x, \eta >) \cdot$)

Since the proof of this theorem is tedious and somewhat uninteresting, we will append it to the end of this paper. In view of this theorem, the following three theorems are direct consequences of Theorems 2 through 4.

Theorem 6. *Let* $n > 0$, *and let* $\mathfrak{b}$ *be a list of variables of types* $\leq n + 2$. *There exists* $\theta(z, \xi, \mathfrak{b})$, *partial recursive in* $\mathbf{E}$, *such that, if* $\{z\}(\xi, \mathfrak{b})$ *is defined for some* ξ, *then*

$$\theta(z, \xi, \mathfrak{b}) \simeq \begin{cases} 0 & \text{if } \|\{z\}(\xi, \mathfrak{b})\| \leq \inf_\zeta \|\{z\}(\zeta, \mathfrak{b})\|, \\ 1 & \text{if not.} \end{cases}$$

Thus the set $U(z, \mathfrak{b}) = \{\xi : \theta(z, \xi, \mathfrak{b}) = 0\}$ *satisfies the implication*

$\exists \xi\, [\{z\}(\xi, \mathfrak{b})$ *is defined*$] \rightarrow$
$U(z, \mathfrak{b})$ *is defined and nonempty* & $\forall \xi \in U(z, \mathfrak{b})\ [\{z\}(\xi, \mathfrak{b})$ *is defined*$]$.

Proof. It suffices to show that there is a partial function $\theta(\xi)$, partial recursive in E and F, uniformly in F, such that, if $\{e\}^{\mathsf{E},\mathsf{F}}(\xi)$ is defined for some ξ, then

$$\theta(\xi) = \begin{cases} 0 & \text{if } \|\{e\}^{\mathsf{E},\mathsf{F}}(\xi)\| \leq \inf_\zeta \|\{e\}^{\mathsf{E},\mathsf{F}}(\zeta)\|, \\ 1 & \text{if not.} \end{cases}$$

Assume that $\{e\}^{\mathsf{E},\mathsf{F}}(\xi)$ is 0 whenever it is defined. Let S be as in Theorem 5 and ψ as in Theorem 2. Let α be the recursive object

$$\begin{aligned} \lambda_\zeta < 0,\ e, \langle\ \rangle^0, \ldots, \langle\ \rangle^{n-2}, \langle(\zeta)_0\rangle, \langle\ \rangle^n, \\ 0,\ e, \langle\ \rangle^0, \ldots, \langle\ \rangle^{n-2}, \langle(\zeta)_0\rangle, \langle\ \rangle^n > ((\zeta)_1), \end{aligned}$$

which satisfies

$$\alpha_\xi \in S \longleftrightarrow \{e\}^{\mathsf{E},\mathsf{F}}(\xi) \text{ is defined } \rightarrow |\alpha_\xi| = \|\{e\}^{\mathsf{E},\mathsf{F}}(\xi)\|.$$

Then $\theta(\xi) \simeq \psi(\xi, \alpha)$ is the desired partial function.

Corollary 1. *Let* $<$ *be a well-ordering of type-k objects, where* $k + 1 \leq n$, *and let* $\mathfrak{b}$ *be a list of variables of types* $\leq n + 2$. *If there exists an object* ε^k *such that* $\{z\}(\varepsilon^k, \mathfrak{b})$ *is defined, then there exists one that is recursive in* $<$, E *and* $\mathfrak{b}$, *uniformly in* $\mathfrak{b}$.

Proof. Consider the case when $k = n - 1$. Let ε^k be the $<$-smallest element of $U(z, \mathfrak{b})$, which is defined in the preceding theorem.

In the particular case when $<$ is the natural ordering of the natural numbers, we obtain the selection-function theorems of Moschovakis and Platek.

Corollary 2. *Let* $n > 0$. *The predicates, with arguments of types* $\leq n + 2$, *that are* r.e. *in* E *are closed under existential quantification of types* $\leq n - 1$.

Proof. It suffices to show that, if $P(\xi, \nu^{n+2})$ is r.e. in E, then $\exists \xi P(\xi, \nu)$ is r.e. in E. Let e be an index such that

$$P(\xi, \nu^{n+2}) \longleftrightarrow \{e\}^{\mathsf{E},\nu}(\xi) \simeq 0.$$

and let $\theta(\xi)$ be as in the proof of Theorem 6. Let

$$\chi(\xi, \nu^{n+2}) \simeq \begin{cases} \{e\}^{\mathsf{E},\nu}(\xi) & \text{if } \theta(\xi) = 0. \\ 1 & \text{if not.} \end{cases}$$

Then

$$\exists \xi P(\xi, \nu^{n+2}) \longleftrightarrow \mathsf{E}^{n+1}(\lambda_\xi\, \chi(\xi, \nu^{n+2})) \simeq 0.$$

As we have already mentioned, the next theorem was proved in the case $n = 1$ by Moschovakis.

Theorem 7. *Let* $n > 0$. *Every predicate, with arguments of types* $\leq n + 2$, *whose negation is* r.e. *in* E *can be expressed in the form* $\exists \alpha P(\alpha, \cdot)$ *where* P *is* r.e. *in* E.

Proof. This result follows immediately from Theorems 3 and 5 in the same manner that Theorem 6 followed from Theorems 2 and 5.

Corollary. *Let* $n > 0$. *There is a predicate* $P(x, \alpha)$, r.e. *in* E, *such that* $\exists \alpha P(x, \alpha)$ *is not* r.e. *in* E.

The next theorem was proved by Moschovakis for the case $n = 1$. Prior to this, Gandy proved an analogous theorem for the case $n = 0$.

Theorem 8. *Let* $n > 0$. *Let* $\mathfrak{b}$ *be a list of variables of types* $\leq n + 2$. *There exists a predicate* $Q(z, \mathfrak{b}, T)$ *that is* Π_2^n *in* E^2 *such that*

$$\{z\}^{\mathsf{E}}(\mathfrak{b}) \text{ is defined} \longleftrightarrow \exists T\, Q(z, \mathfrak{b}, T)$$
$$\longleftrightarrow \exists T\, [Q(z, \mathfrak{b}, T) \;\&\; T \text{ is rec. in } \mathfrak{b} \text{ and } \mathsf{E}].$$

(Note: the condition "*in* E^2" may be dropped if the set quantifer $\exists T$ is replaced by a function quantifer of the same type.)

Proof. This result is immediate from Theorem 5 and the corollary of Theorem 4. One need only verify that Q in the corollary of Theorem 4 is Π_2^n (in E^2, $\mathfrak{b}$) when J is Δ_2^n (*in* E^2, $\mathfrak{b}$).

Theorem 8 has a converse in that predicates of the form

$$\exists \varepsilon^{n+1} [Q(\mathfrak{b}, \varepsilon^{n+1}) \;\&\; \varepsilon^{n+1} \text{ is recursive in } \mathfrak{b}, \mathsf{E}]$$

are r.e. in E whenever Q is r.e. in E. This follows from Corollary 2 of Theorem 6. We summarize some of the preceding results in the following characterization of the types of quantifiers under which r.e. predicates are closed.

Theorem 9. *Predicates* (*with arguments of types* $\leq n + 2$) *that are* r.e. *in* E^{n+2} *are closed under conjunctions, disjunctions, universal quantification of types* $\leq n$ *and existential quantification of types* $\leq \max(0, n - 1)$, *but not closed under higher type quantification.*

Proof. (a) Closure under universal quantification of types $\leq n$: this follows from a use of scheme S0.

(b) Closure under existential quantification of types $\leq \max(0, n - 1)$: if $n = 0$, this follows from Gandy's selection-function theorem [1, p. 14]; if $n > 0$, this follows from Corollary 2 of Theorem 6.

(c) Non-closure under type-$\max(1, n)$ existential quantification: if

$n = 0$, this follows from Kleene's quantifier forms for r.e. predicates [2, p. 32]; if $n > 0$, this follows from the corollary of Theorem 7.

(d) Non-closure under type-$(n + 1)$ quantification: if $n > 0$, this follows from Kleene's quantifier forms for r.e. predicates [2, p. 32]; if $n = 0$, if we suppose the contrary, we would infer from [2, p. 32] that the Π_1^1 predicates with arguments of types ≤ 2—which do not satisfy the reduction principle by Tugué [5, p. 112]—are identical with the predicates r.e. in $\mathbf{E}^2$—which do satisfy the reduction principle.

Theorem 9 indicates that predicates r.e. in $\mathbf{E}^{n+2}$ have moderately strong closure properties. (Another indication is that sets inductively defined with respect to Π_1^n operations are always r.e. in $\mathbf{E}^{n+2}$, though the same is not true when $n > 0$ and Π_1^n is replaced by Δ_2^n.) On the other hand, mere r.e. predicates are not even closed under disjunctions. For example, let ϕ and ψ satisfy

$$\phi(z, \alpha^2) \simeq \mu y[\alpha^2(\lambda_x\{z\}(x)) = 0]$$
$$\psi(z, \alpha^2) \textit{ is defined} \leftrightarrow \exists x[\{z\}(x) \textit{ is not defined}].$$

Such a ψ exists by Gandy [1, p. 19]. Then no partial recursive function $\theta(z, \alpha^2)$ defined exactly when either $\phi(z, \alpha^2)$ or $\psi(z, \alpha^2)$ is defined. To show this one supposes the contrary and then finds an index e of a total function such that $\theta(e, \lambda_\xi 0)$ and $\theta(e, \mathbf{F}^2)$ have identical trees of computation, where $\mathbf{F}^2$ is a type-2 function that is identically 0 except at the argument $\lambda_x\{e\}(x)$.

We conclude with a proof of Theorem 5. To see how to define P and J, suppose that the stages S_τ are properly formed for $\tau \leq \sigma$. Suppose that (8) holds where

$$\sigma = ||\{y\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]|| \vee ||\{z\}^{\mathbf{E},\mathbf{F}}[\beta^0, \ldots, \beta^n]||.$$

Consider the case where $(y)_0 = 4$ (application of scheme S4) and $(z)_0 = 0$ (evaluation of $\mathbf{E}$ or $\mathbf{F}$). By the induction hypothesis, we have

$$||\{y\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]|| = \inf_t |A(t, \gamma)|,$$

where

$$A(t, \gamma) = <(t)_0, (y)_3, \alpha^0, \ldots, \alpha^n, (t)_1, (y)_2, <(t)_0> * \alpha^0, \alpha^1, \ldots, \alpha^n>.$$

Similarly

$$||\{z\}^{\mathbf{E},\mathbf{F}}[\beta^0, \ldots, \beta^n]|| = \sup_\varepsilon \inf_t |B(\varepsilon, t, \gamma)|,$$

where

$$B(\varepsilon, t, \gamma) = <t, (z)_3, \beta^0, \ldots, \beta^{n-1}, <\varepsilon> * \beta^n, t, (z)_3, \beta^0, \ldots, \beta^{n-1}, <\varepsilon> * \beta^n>.$$

Thus

$$\sigma = \sup\{\inf_t |\delta_t| + 1 : P(\gamma, \delta)\},$$

where

$$P(\gamma, \delta) \longleftrightarrow \forall t[\delta_t = A(t, \gamma)] \vee \exists \varepsilon \forall t[\delta_t = B(\varepsilon, t, \gamma)].$$

Additional but similar clauses can be adjoined for the cases where $(y)_0 \neq 4$ or $(z)_0 \neq 0$. In this way a $\sum_1^n P$ can be defined. Next we consider the three tests that should be made before putting γ into $S_{\sigma+1}$. First, we check that y and z are indices; second, that the previous history of γ is already in S (in particular, this tests whether $\{y\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]$ and $\{z\}^{\mathbf{E},\mathbf{F}}[\beta^0, \ldots, \beta^n]$ are defined); third, that w and x are the correct values of $\{y\}^{\mathbf{E},\mathbf{F}}[\alpha^0, \ldots, \alpha^n]$ and $\{z\}^{\mathbf{E},\mathbf{F}}[\beta^0, \ldots, \beta^n]$. Thus we can define J by the rule

$$\begin{aligned} \gamma \in \mathrm{J}(T) \longleftrightarrow & [w \textit{ and } z \textit{ are indices of functions partial recursive in } \mathbf{E}, \mathbf{F}] \\ & \& \ \forall \delta[P(\gamma, \delta) \to \exists t(\delta_t \in T)] \\ & \& \ w = (\mu t(A(t, \gamma) \in T))_1 \\ & \& \ (z)_2 = 2 \to x = \mathbf{F}(\lambda_\varepsilon \mu t(B(\varepsilon, t, \gamma) \in T)) \\ & \& \ (z)_2 = 1 \to x = \begin{cases} 0 \textit{ if } \exists \varepsilon(B(\varepsilon, 0, \gamma) \in T) \\ 1 \textit{ if } \forall \varepsilon \exists t(B(\varepsilon, t+1, \gamma) \in T). \end{cases} \end{aligned}$$

Additional but similar clauses can be adjoined for the cases where $(y)_0 \neq 4$ or $(z)_0 \neq 0$. To prevent the possibility of J being undefined because of some "μt", we may replace each "$\mu t Q$" by "$\mu t(Q \vee \forall t \neg Q)$". Thus J will be $\boldsymbol{\Delta}_2^n$ in $\mathbf{E}^2$ and $\mathbf{F}$, uniformly in $\mathbf{F}$.

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