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TWO VIEWS OF VARIABLES

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This paper has been prompted by the article "Logic and Existence," by Czesław Lejewski, which appeared in the *British Journal for the Philosophy of Science* 5 (1954). In his article, Dr. Lejewski has considered how to give a logical analysis of statements where we say that something does or does not exist.

In making such statements in a formalized language (i.e., one based on an axiomatic formal system), it is common to employ quantifiers. Dr. Lejewski distinguished two interpretations of quantifiers. One of these, the more common interpretation of quantifiers, he calls the restricted interpretation. Ordinarily we are satisfied that the sentences

Pegasus does not exist.
The Prime Minister of Great Britain exists.

(are true or) express true propositions. But if we rewrite these sentences in symbolic form, we would get

$\sim(\exists x)\ P(x)$ - where 'P' represents the predicate 'is Pegasus'

$(\exists x)\ M(x)$ - where 'M' represents the predicate 'is the Prime Minister of Great Britain.'

Now, in English, 'Pegasus' and 'the Prime Minister of Great Britain' function as proper names. Why not have a predicate to indicate existence, and write

$\sim$R(Pegasus) - where 'R' represents the predicate 'exists'

R(the Prime Minister of Great Britain) ?

But if we have these two statements, then, according to the customary procedures of inference, we could obtain

$$(\exists x)\sim R(x)$$

$$(\exists x)R(x).$$

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There is not much that is objectionable about the second of these formulas, but the first seems to say that something exists which does not exist. Dr. Lejewski claims that this difficulty is occasioned by the ordinary, restricted interpretation of the quantifiers. We customarily read '$(\exists x)$' as "There exists an entity x such that"; and we read '(x)' as "for all entities x." But if one adopts the unrestricted interpretation of quantifiers, then the existential, or particular, quantifier is not used to state that some entity exists. Instead, Lejewski would read

$$(\exists x) \sim \mathrm{R}(x)$$

as "For some x, x does not exist." If he were pressed to tell us what it is that does not exist, I think that he would reply that the particular quantifier is used to talk about expressions. It is possible to find an expression to put in place of 'x' so that

$$\sim \mathrm{R}(x)$$

is true.[1] The unrestricted interpretation of quantifiers gives us "pure" quantifiers, unmixed with existential claims. On the restricted interpretation, quantification and existence are jumbled together in a confused way. The unrestricted interpretation makes it possible to recognize an important distinction.

Another shortcoming of the restricted interpretation of quantifiers is that it requires us to adopt logical laws that are not universally valid. The following

$$(x)f(x) \supset (\exists x)f(x)$$

is a theorem of most systems of predicate (functional) calculus. But this "law" does not hold for empty domains. It has been argued that

$$(x)f(x) \supset (\exists x)f(x)$$

is only contingently "valid," and so ought not be considered a logical law. Dr. Lejewski claims that the contingent character of this law is a consequence of the restricted interpretation of quantifiers. If the unrestricted interpretation is adopted, this law is universally valid. For in that case the law means something like

> If an expression of the form
>
> $$f(x)$$
>
> is true no matter what name expression is put in place of 'x,' then there is an expression which can replace 'x' so that
>
> $$f(x)$$
>
> is true.[2]

However, if the unrestricted interpretation is adopted, the ordinary systems of predicate calculus are not entirely satisfactory, because there must be some way to distinguish what exists from what does not. When the unrestricted interpretation is adopted, Dr. Lejewski feels that some system other than predicate calculus is called for.[3]

I agree that the difference between the restricted and unrestricted interpretations is an important one. However, the two interpretations are consequences of a more fundamental difference. There are not just two ways to interpret quantifiers. More basic than this are two ways of looking at, or regarding, variables. These different ways of considering variables make an important difference to one's interpretation of a formalized language, and the difference is worth exploring at some length.

I feel that one can distinguish two fundamentally different ways of regarding variables--I will call these two views of variables. The first view I call the Russell-Quine view; the second is the Frege-Leśniewski view (these will be abbreviated as R-Q and F-L, respectively).[4] These two are not the only possible views, but I feel that they are the two basic views; other views will be variants of one or the other, or perhaps combinations of the two.

At first I will give an informal account to explain the difference between the views. The value of this account is primarily a heuristic value--I do not intend to make any controversial claims on the basis of this account. The Russell-Quine (R-Q) point of view will be considered first. Imagine a game for which there are three different kinds of pieces, or men. In the course of the game these pieces are arranged in rows of varying lengths. Rows may also cross one another (as in Scrabble), so that a single piece may belong to two rows. If a language were developed to report on this game, it would be natural to develop a language which contains three (sub-) alphabets. Three symbols are not enough, because we want a way to indicate that one physical piece occurs in two distinct rows (and we want this written language to be essentially one-dimensional). Strings of symbols can represent rows of pieces in an obvious way. The occurrence of a symbol from one alphabet in a string represents the occurrence of a piece of one kind in a row. While the occurrence of a symbol in a string indicates that a piece of a certain kind occurs in a row, the symbol does not indicate which piece, for in the game there is no reason to distinguish the individual pieces. Now the symbols in the language can be considered as variables, at least on the R-Q view of variables. Variables are used to make statements about things of a certain kind, or things generally, without saying what things are involved (except in a predicative way, by talking about the things that are so-and-so).

To understand the Frege-Leśniewski view of variables, it is necessary to begin with some (written) language. When one considers sentences, or, more generally, expressions, of the language, it is clear that different expressions can have the same form. That is, several expressions can be just alike, except for the occurrence of some component expressions. The following sentences have the same form:

Jones is rich.
Smith is rich.
Brown is rich.

If there is some reason to be interested in the forms that different expressions have in common, then one can take the different expressions and delete the distinguishing expressions. If this is done to the above sentences, the result is

_______ is rich.

Dealing with expressions that contain gaps is inconvenient, for expressions might contain more than one gap; in such a case it would not always be clear if different expressions were to go in different gaps. For this reason, letters are used to fill the gaps. These letters are variables. On the F-L view, any expressions at all can be replaced by variables--they need not be names.

The R-Q view of variables could be called the pronoun view of variables. Professor Quine has compared the use of variables with many uses of pronouns in English; however, he views pronouns as more fundamental than nouns.[5] For Professor Quine, a variable is a symbol that has a range of values—each value is an entity of some kind. Variables in no sense depend upon names, for names can be dispensed with (in principle). "Whatever we say with the help of names can be said in a language which shuns names altogether."[6] On the R-Q view, quantifiers are used to talk about all entities or some entity.

On the F-L view, a variable is seen as a replacement for an expression. Variables do not have ranges of values, where each value is an entity of some kind. One could talk about expressions of a certain grammatical category as values of a variable, but there need be no commitment to timeless, abstract expressions. Or if a variable is used for expressions in the category of names, there might be some reason to consider the entities named and call them values of the variable; but then not all variables would have values (a variable replacing an expression which is not a name would not have a value). On the Frege-Leśniewski view, quantifiers are not used to make statements about all entities or some entities. Leśniewski writes,

> in times when I did not know how to operate by means of quantifiers, but in the colloquial language which I used needed something to correspond to expressions of the type "$(\exists a)\cdot f(a)$," "$(\exists X, a)\cdot f(X, a)$," etc., which are expressions of the symbolic language, I used corresponding expressions of the type "For some significant word "a," $f(a)$;" "For some significant words "X" and "a," $f(X,a)$."[7]

On the F-L view, variables are put in blanks to identify the blanks. Quantifiers are used to make statements from expressions that contain blanks. They make it possible to say that an expression of a certain form is always true, or is sometimes true (always expresses a true proposition or sometimes expresses a true proposition). On the F-L view, the use of variables in a quantifier does not commit one to recognizing entities to which the variables "refer."

The two interpretations of quantifiers that Dr. Lejewski distinguishes are simply consequences of these two views of variables. On the R-Q view, a variable is used for talking about entities. Consequently, quantifiers enable us to talk about all or some entities. But on the F-L view, variables are blank-fillers. We use quantifiers to talk about all expressions of a given form. And an expression whose form is

$$(\exists x)\,\phi(x)$$

does not mean that there is an entity x such that $\phi(x)$. Instead, it means that there is an expression which can replace 'x' to make

$$\phi(x)$$

a true statement (the expression might be 'Pegasus'). There are not really different interpretations of the quantifiers. There are instead different views of variables—how one understands a quantifier depends on the way he regards the quantified variable.

It is worth noting that there are accounts of proper names which are related to these different views of variables. One can take Russell's treatment of definite descriptions, and use it to dispense with names altogether. Professor Quine has done this; he speaks of himself as "construing names as general terms."[8] Variables are used for referring to entities; predicates or general terms are used for saying things about entities. Although it was Russell who formulated the theory of definite descriptions, Russell did not eliminate names entirely. He distinguished names from descriptions on the basis of an epistemological distinction; for this reason he did not assimilate names to descriptions. Russell's example shows that the R-Q view of variables does not require that proper names be eliminated, though it seems to suggest such elimination.

An account of proper names that is in harmony with the F-L view of variables would be one that takes the category of names as a basic category of expressions. Frege's account, where he distinguishes sense and reference, is such an account. However, there is no need to accept his particular formulation, nor is there any requirement to regard declarative sentences as names.

Accepting one or the other view of variables makes some difference with respect to logical systems, or formalized languages. Let us consider a standard system of first order predicate calculus, when the variables are construed as mere gap-fillers. The system given by Hilbert and Ackerman in *Mathematical Logic* contains these two axioms

$$(x)F(x) \supset F(y)$$

$$F(y) \supset (\exists x)F(x)$$

(but expressed in a different notation), as well as axioms common to propositional calculus. On the F-L view these axioms are valid.[9] The first axiom can be read: If for all expressions 'x,' $F(x)$, then $F(y)$. The second can be read: If $F(y)$, then for some expression 'x,' $F(x)$. The variables belong to the category of proper names. If we allow names which have no referents (e.g., 'Pegasus'), a substitution instance of the second axiom would be

$$\text{Pegasus is unreal} \supset (\exists x).\ x \text{ is unreal.}$$

The particular quantifier is used to indicate that there is an expression which does a certain job (i.e., makes 'x is unreal' into a true statement); there is no requirement to admit even a possible Pegasus.

The axioms of predicate calculus remain valid when variables are considered from the F-L standpoint. But the system of predicate calculus is now defective. For it lacks a device for distinguishing what exists from what does not (for distinguishing non-empty from empty names). A predicate is wanted for indicating that something exists, or is real. Such a predicate would have to be introduced axiomatically. However, instead of introducing such a predicate, Dr. Lejewski turns his attention to an entirely different formal system.[10] Dr. Lejewski considers Leśniewski's system of Ontology.

Leśniewski's system of Ontology is a more natural system than predicate calculus, when the F-L view of variables is adopted. In this system it is possible to formulate a statement to the effect that something exists, or that something does not exist (but this does not mean that some entity does not exist). In Ontology the basic grammatical category (Leśniewski calls it a semantical category) is the category of names. But these are not proper names which refer to at most one object; the category includes names for more than one object (like common nouns), names of just one object, and names which have no referent. In Ontology, the expressions 'visible natural satellite of Earth' and 'the Moon' would be of the same grammatical type. Nor is there a grammatical distinction between 'Pegasus' and 'horse.'

In Leśniewski's presentation of Ontology, the primitive constant is 'ε'; this symbol is used to join two names. The axiom (or axioms—for there are alternative systems containing different number of axioms) of Ontology characterizes the meaning of an expression having the form

$$a\varepsilon b \text{ (or } \varepsilon\{a\ b\}).$$

Such a statement means that the single a is b. If there is more than a, or if there are none, then it is false that $a\varepsilon b$. To indicate that an individual exists, one can write,

$$c\varepsilon c.$$

If the single c is c, then there is just one c. To indicate that there is something, that something exists, the following expression can be used:

$$(\exists a).a\varepsilon a.$$

It is not the quantifier, but the symbol 'ε,' that is used to claim that something exists. In Ontology it is not possible to prove

$$(\exists a).a\varepsilon a.$$

This is an advantage of Ontology, for it does not seem to be a logical matter to determine whether anything exists.

In "Logic and Existence," Dr. Lejewski presents an alternative form of Ontology whose primitive constant is '$\subset$.' He does this because he feels that "ordinary inclusion seems to be more intuitive to an English speaking reader than Leśniewski's singular inclusion.''[11] When '$\subset$' is the primitive term, it is possible to state that the single a exists by writing

$$(\exists b) \sim(a \subset b) \ \& \ (b, c, d). \sim(c \subset d) \ \& \ (b \subset a) \ \& \ (c \subset a) \supset (b \subset c).$$ [12]

The way of regarding variables that one adopts will determine which formal systems seem most natural to him. Ontology is more natural on the F-L view than is the first order predicate calculus. However, Ontology and predicate calculus are not so different as they first appear. In an ordinary system of predicate calculus, even if names are allowed, a general name (common noun) cannot replace the 'x' in

$$f(x).$$

But in Leśniewski's system the category of names includes general names. Professor Quine writes that "Leśniewski..is best construed as assimilating names to general terms, though he does not so phrase the matter.''[13] In a standard system of predicate calculus, the predicates are the expressions that come the closest to general terms. Keeping Professor Quine's remark in mind, it is possible to formulate a rough analogue to Ontology in second order predicate calculus.[14] A symbol 'ε' can be defined which is an analogue to the 'ε' of Ontology. In order to accomplish this, some preliminary definitions are needed.

$$\mathrm{Sol}(f) =_{(def)} (x)(y).f(x) \supset f(y) \supset x=y$$

'$\mathrm{Sol}(f)$' means that there is at most one thing of which 'f' can be predicated.

$$!(f) =_{(def)} (\exists x)f(x)$$

'$!(f)$ means that there is something of which 'f' can be predicated.

$$\subset(fg) =_{(def)} (x).f(x) \supset g(x)$$

'$\subset$' is the sign of non-existential (Boolean) inclusion.

$$\sqsubset(fg) =_{(def)} \subset(fg) \ \& \ !(f)$$

'$\sqsubset$' is the sign of existential inclusion.

$$\varepsilon(fg) =_{(def)} \sqsubset(fg) \ \& \ \mathrm{sol}(f)$$

However, on the basis of these definitions, it is not possible to prove a theorem which is the analogue to the axiom of Ontology. (The axiom given by Leśniewski is

$$(a \ b).(a \varepsilon b) \equiv .(\exists c)(c \varepsilon a) \ \& \ (c \ d) \ [(c \varepsilon a) \ \& \ (d \varepsilon a) \supset (c \varepsilon d)] \ \& \ (c).(c \varepsilon a) \supset (c \varepsilon b) \ .)$$ [15]

I have found that the following formula,

$$f(x) \supset (\exists h).\mathrm{Sol}(h) \ \& \ h(x),$$

if added to second order predicate calculus as an axiom, makes it possible to prove an analogue to the axiom of Ontology.[16] The formula that must be added is not an implausible one--from the standpoint of the F-L view, it would be perfectly acceptable. For example, if we can truly say, John Jones is a human being.

then we can formulate a predicate which can be predicated of at most one individual and which can be predicated of John Jones:

John Jones is called John Jones.

Of course, the point of defining 'ε' for second order predicate calculus was to show that Ontology is not so different from predicate calculus as *normally understood.* And the normal understanding of predicate calculus seems to be one that involves accepting the R-Q view of variables. However, it will be explained below how one can adopt the F-L view of variables, and still grant the formulas of predicate calculus their normal force.

In the analogue of Ontology that has been formulated in predicate calculus (including the axiom that was added), it is possible to prove that at least one individual exists. The Ontological theorem which gives this result is

$$(f)[\varepsilon(ff) \supset \varepsilon(fg)] \supset (\exists f)\varepsilon(fg).$$

If there is one predicate which can be predicated of everything, then there is an expression (a predicate) which names just one individual and the universal predicate can be predicated of this individual. In Ontology, there is a universal term 'V.' If an analogue to this term is allowed in second order predicate calculus, then the following can be proved

$$(\exists f)\varepsilon(fV).$$

This result ultimately depends on the features that make it possible to prove the theorem

$$(x)f(x) \supset (\exists x)f(x).$$

Because it is possible in first order predicate calculus to prove that at least one individual exists, it is no surprise that a similar result can be obtained in this analogue to Ontology.

Since the predicate calculus analogue to Ontology contains a result that cannot be proved in Ontology, it is appropriate to ask in what sense this modified predicate calculus is an analogue to Ontology. The analogy consists in this: the meaning of the 'ε' of Ontology is quite similar to the meaning of the demand 'ε' of predicate calculus—or, expressions of the form

$$a\varepsilon b$$

in Ontology can be used for much the same purposes as expressions of the form

$$\varepsilon(fg)$$

in predicate calculus. But in predicate calculus it can be proved that at least one individual exists; this cannot be proved in Ontology.

Dr. Lejewski has suggested that the objectionable theorem

$$(x)f(x) \supset (\exists x)f(x)$$

is objectionable precisely because of the restricted interpretation of quantifiers. Since the important difference behind the two interpretations is the difference between the two views of variables, Dr. Lejewski is suggesting, in effect, that the R-Q view is what makes the theorem objectionable. When the F-L view is adopted, this theorem becomes satisfactory. But given the F-L view of variables, predicate calculus is not an adequate system for distinguishing what exists from what does not. I believe that Dr. Lejewski's suggestion is mistaken. For an analogue (although somewhat stronger in terms of its results) to Ontology has been formulated in second order predicate calculus. Since the variables of Ontology have customarily been construed in the F-L manner, this must also be possible for the variables in the predicate calculus analogue of Ontology. But then this way of viewing variables can be extended to first order formulas. When this is done, individual variables are considered as (replacing expressions) belonging to a category of non-empty proper names. The names of Ontology are not proper names, just as the predicates of predicate calculus are not proper names. To give formulas of first order predicate calculus their normal force, while maintaining the F-L view of variables, one must recognize a category of non-empty proper names which is distinct from the category of names in Ontology.

The difference between the R-Q and the F-L views of variables has little to do with the inclusion of

$$(x)f(x) \supset (\exists x)f(x)$$

as a theorem of predicate calculus. Results obtained by Mostowski and Hailperin[17] have shown that it is possible to reformulate predicate calculus so that its theorems are valid in all domains, even the empty domain. However, they considered different systems than that of Hilbert and Ackermann. To reformulate this system so that its theorems are universally valid, it is convenient to eliminate free individual variables. When this is done, the axioms considered earlier become

$$(y)\,[(x)f(x) \supset f(y)]$$

$$(y)[f(y) \supset (\exists x)f(x)].$$

The rules of the system must be changed, but the changes are fairly minor.[18] The rule of generalization must be dropped, and a new rule must be added--the following is sufficient for this:

From a well-formed formula $(\alpha_1)\ (\alpha_2)\ \ldots\ (\alpha_n) \cdot A(\alpha_i) \supset B(\alpha_i)$ in which both the antecedent and the consequent contain the free variable α_i, the well-formed formula $(\alpha_1)\ \ldots\ (\alpha_{i-1})\ (\alpha_{i+1})\ \ldots\ (\alpha_n) \cdot (\alpha_i)A(\alpha_i) \supset (\alpha_i)B(\alpha_i)$ is obtained.

The original axioms

$$(x)f(x) \supset f(y)$$

$$f(y) \supset (\exists x)f(x)$$

are universally valid because they hold for each individual y (R-Q) or for each non-empty name that can be used to replace 'y' (F-L). But this universal validity can be indicated by an initial universal quantifier binding 'y.' The free 'y' is being used in different ways in the two axioms; what these ways are becomes clear when the initial quantifier is used. For then the two axioms will be equivalent to

$$(x)f(x) \supset (y)f(y)$$

$$(\exists y)f(y) \supset (\exists x)f(x),$$

which are trivial. The occurrence of free variables allows the different roles of 'y' in the two axioms to be confused. It is this confusion, rather than the difference between the two views of variables, that makes it possible to deduce

$$(x)f(x) \supset (\exists x)f(x),$$

which is not universally valid.

I have been arguing that the difference between the two views of variables makes little difference to the understanding of predicate calculus and Ontology (it *need* not make much difference). However, it may be the case that the absence of free individual variables seems more natural from the standpoint of the F-L view of variables—but this is scarcely an important difference. If the view of variables one adopts does not make much difference, then we might ask just what consequences one or the other view does have. My chief argument has been that the formulas of first order predicate calculus can be interpreted from the F-L point of view in such a way that they have their normal force. For example, if an expression whose form is

$$(\exists x)f(x)$$

is true, then there must be at least one individual. But from the F-L viewpoint, the formula might be read: For some expression which can significantly replace 'x,' $f(x)$. In this case, the only expressions which can significantly replace 'x' are non-empty proper names. While one can adopt the Frege-Leśniewski viewpoint without forswearing first order predicate calculus, from this viewpoint Ontology would seem to be a more natural system to use than first order predicate calculus. For in first order predicate calculus the answer to the question of whether there is an expression of a certain kind depends on the existence of an entity for the expression to name. In Ontology, one can start with expressions, and then consider whether or not there is anything that these expressions name.[19]

One can adopt the F-L point of view and still give the formulas of predicate calculus their normal force. But one who adopts the R-Q viewpoint is bound to feel hesitant when faced with the formulas of Ontology. For that matter, such a person should be worried about the formulas of higher order predicate calculus. For quantified variables are used to talk about entities of one sort or another. One may very well wonder what kind of entities these are. Anyone who wants to admit as few kinds of entity as possible will, on the R-Q view, show a marked preference for first order predicate calculus. He will not be willing to employ a system like Ontology.[20]

In considering the two views of variables, I am not trying to argue that either view is the correct one. There is nothing to be correct about. But there are certain advantages and disadvantages connected with these views, and I would like to consider what they are. More particularly, I will argue that the Frege-Leśniewski way of regarding variables is superior to the Russell-Quine approach.

If one adopts the Russell-Quine view of variables, then he must also recognize a different kind of variable. Professor Quine distinguishes authentic variables from schematic letters.

> We can view '$[(p \supset q) \cdot \sim q] \supset \sim p$' . . not as a sentence but as a schemata or diagram such that all actual statements of the depicted form are true. . The schematic letters '*p*,' '*q*,' etc. stand in schemata to take the place of component statements.[21]

What distinguishes an authentic variable from a schematic letter is that the authentic variable can be bound by a quantifier. One might distinguish the Frege-Leśniewski view from the Russell-Quine view by saying that the F-L view regards all variables as schematic letters. Professor Quine feels that anyone who uses authentic variables is making certain ontological commitments. The ontological commitments of a user of propositional and predicate variables (plus individual variables) are much greater than the comitments of one who employs only individual variables.

> The main disadvantage of assimilating schematic letters to bound variables is that it leads to a false accounting of the ontological commitments of most of our discourse.[22]

But if someone adopts the F-L view of variables, he "assimilates" all variables to schematic letters. In that case, the use of variables does not involve ontological commitments. For it is possible to use quantifiers without being commited to recognizing entities of any sort.

The R-Q view of variables is unsatisfactory because it is not sufficient. One cannot treat formalized languages effectively when all variables are required to represent entities. Schematic letters are necessary. But if such a device is necessary, then why not regard all variables in this way? The distinction between authentic and schematic variables appears somewhat arbitrary. Why not quantify over schematic letters? If some logical purposes are served in this way, then surely the practice is legitimate. Of course, it is possible to maintain the R-Q view of variables, and simply admit all sorts of entities. But Professor Quine is clearly correct in arguing that

> When we say that some dogs are white,
>
> $(\exists x)$ (x is a dog $\cdot$ x is white),
>
> we do not commit ourselves to such abstract entities as dogkind or the class of of white things.[23]

It may well be that there are such entities. But one is not required to recognize these entities in order to make the statement in question. Some way is wanted to bring out the ontological commitments of a statement--yet variables should not bear this burden. The use of quantifiers with schematic letters is a desirable practice. It ought not to be prohibited because of a rigid adherance to the Russell-Quine view of variables.

There are certain problems with the F-L point of view that ought to be considered. The first of these concern the readings that are given to formulas which contain variables and quantifiers. Dr. Lejewski writes,

> . . . it would be misleading to read '$(\exists x)$ (Fx)' as 'there exists an x such that Fx.' The noncommital 'for some x, Fx' seems to be more appropriate. Similarly the terms 'existential quantification' and 'existential quantifier' no longer apply and could be conveniently replaced by such expressions as 'particular quantification' and 'particular quantifier.'[24]

The trouble with Dr. Lejewski's reading, as was indicated earlier (see n. 1) is that it is too non-commital. His reading may ultimately be a good one to adopt, but the meaning of this reading is not immediately clear. To make it clear, the reading of '$(\exists x)F(x)$' ought to be expanded to 'For some (significant) expression x, $F(x)$,' or 'There is an expression which can be put in place of 'x' so that $F(x)$.'

Consider the universal quantifier as it occurs in the statement

$$(x)(x \text{ is a man} \supset x \text{ is mortal}).$$

This statement might be read: For all expressions which can (significantly) be put in place of 'x,' the following (will be true)

$$x \text{ is a man} \supset x \text{ is mortal}.$$

One question that arises immediately is that concerning the meaning of 'all expressions.' Does this indicate that there is some existing totality of expressions? If it does, the F-L view is up to its neck in ontological commitments. However, there need not be any existing totality of expressions. The universal quantifier is simply used to indicate that any (significant) expression which is put in place of 'x' will produce a true statement--there may not be any such expressions in existence now (although in this example, there are many such expressions). Reading the particular quantifier (or the denial of the universal quantifier) appears to be more difficult. Consider

$$(\exists f)\, f(\text{Brutus Caesar}).$$

If this is read: There is an expression which can replace 'f' so that the following (is true)

$$f(\text{Brutus Caesar}),$$

then what kind of existence claim is made for the expression to replace 'f'? One need not interpret the particular quantifier as making a claim that there is such an expression in existence at this moment. Of course, one can formulate a logical system which permits the statement

$$(\exists x)F(x)$$

only when there is some prior statement

$$F(a).$$

But the particular quantifier can also be justified when the validating expression has not been formed (i.e., it may be permissible to assert '$(\exists x)$ $F(x)$' when no expression '$F(a)$' has been asserted). In such a case, this quantifier should not be understood as claiming that there is an expression--but only that an expression can be formed so that, etc.

Another difficulty with the reading of quantifiers on the F-L view is that this reading seems to require that variables be used both materially and formally (that they be both mentioned and used). For if one reads

$$(\exists f) f(\text{Brutus Caesar})$$

as: For some expression 'f,' the following

$$f(\text{Brutus Caesar}),$$

the variable 'f' occurs both with and without single quotes. However, it makes little sense to talk about the material and formal use of a variable; this distinction is only appropriate for expressions. On the F-L view variables are little more than gaps in an expression; quantifiers are used to indicate how the gaps can be filled.

Difficulties connected with the reading of formulas that contain quantifiers do not pose a serious problem for the F-L view of variables. One expects certain difficulties when he translates formulas into ordinary language. If ordinary language contained the requisite expressions and if it were convenient to make certain distinctions in ordinary language, then formalized languages would serve little purpose. Formalized languages are useful precisely because they make it convenient to say things which cannot be said conveniently in ordinary language. There is no question of making a perfect translation of a formula into an ordinary English sentence. All that is necessary is that there be some means of explaining in English (or some other natural language) how the expressions of the formalized language are to be taken, how they are to be used.[25]

There is another problem with the F-L view of variables; this is raised by Russell's account of definite descriptions. On the F-L view, variables are replacements for expressions. But Russell feels that variables are ultimate, irreplaceable components of any worth-while language. Variables are needed to enable us to talk about things that we do not experience.[26] Only what enters into our experience can be named (defined ostensively), everything else is described. An argument against the F-L view of variables on these grounds commits a kind of genetic fallacy. For it is being claimed that a formalized language must reflect the way one learns the language, that epistemological distinctions must show up as grammatical ones.

There can be no question that variables are needed for the purposes of logic and mathematics.[27] They are an important device for constructing formal systems and formalized languages. Adopting the F-L viewpoint does not require that one regard variables as dispensible. Formalized languages are often set up which contain no constants but only variables in certain categories. It is clear that these variables cannot be replacing constant expressions in the formalized language. But such a formalized language is normally considered to be providing the bare bones of a fleshed-out language. On the F-L view, the variables of such a system are considered in terms of extensions of the formalized languages. Hence, the existence of variables where there are no constants does not render the F-L view of variables untenable. Nor does one who adopts the F-L viewpoint have reason to deny that ordinary language contains expressions whose use is similar to the use of variables in formalized languages.

But, in analyzing a given expression, what if one is left with another expression that contains a particular quantifier? Consider a definite description. Suppose we want to talk about the individual that so-and-so—or the x such that $\phi(x)$. Let us assume that there is just one individual x such that $\phi(x)$. This means that the following is true:

$$(\exists x)[\phi(x) \;\&\; (y).\phi(y) \supset x = y].$$

Suppose also that no more is known about this individual than that it is the x such that $\phi(x)$. How, from the F-L point of view, can we construe the particular quantifier in the formula above? When one considers this formula, the Russell-Quine view seems to be the natural way to regard variables. But it is possible to give an account of the formula from the Frege-Leśniewski viewpoint. On the F-L view, the particular quantifier is used to indicate that an expression can be formed to replace the 'x' and convert the quantified formula into a true statement. This would not require us to claim that a genuine name, in Russell's sense, can be defined. It is clear that not every name can be defined ostensively. But there is no reason for a formalized language to reflect the distinction between knowledge by acquaintance and knowledge by description. In ordinary language, definite descriptions function in about the same way as normal proper names. A formalized language may very well contain a device for converting descriptive sentences into names (a description operator). The formula

$$(\exists x)[\phi(x) \;\&\; (y).\phi(y) \supset x = y]$$

would mean that an expression can be formed to replace 'x' and yield a true statement. But there is no need for this expression to be atomic. The distinction between atomic and complex names is at best a relative one; there is no good reason to embody this distinction as a grammatical distinction.[28]

The difficulties that might seem to be inherent in the F-L view of variables are not significant difficulties. The advantage of this view is that it frees a logical device (or technique) from the limitations that the R-Q view impose upon it. One can use quantified variables without being committed ontologically. Dr. Lejewski claims that by adopting the F-L view of variables, it is "possible to separate the notion of existence from the idea of quantification."[29] He feels that this distinction is customarily overlooked, and it is best brought out by a system such as Ontology (rather than predicate calculus). In view of the fact that an analogue to Ontology can be formulated in higher order predicate calculus, I feel that Dr. Lejewski's claim should be modified. The distinction between "pure" quantification and existence claims can be made in predicate calculus. But the difference between the Russell-Quine and the Frege-Leśniewski views of variables is one that makes a difference to the understanding of formulas in any system.

On the R-Q view, the use of quantifiers in a formalized language reveals the ontological commitments of the language. It is because of these commitments that Professor Quine is reluctant to quantify over schematic letters. On the F-L view, the use of variables is not indicative of ontological commitment. But then what is? The answer to this question depends on the formalized language being considered. In Leśniewski's Ontology, those things exist whose names precede the constant 'ε' in true affirmative statements. Leśniewski did not feel that it is the logician's job to decide just what does exist; his formalized language enables us to talk about whatever does exist. It is possible to adopt the R-Q view of variables and balk at formalized languages that quantify other than individual variables. But it still seems necessary to employ schematic letters. And the restriction on the use of quantifiers is at least a sin against elegance. By adopting the Frege-Leśniewski view, one can employ variables of all sorts without any feelings of guilt. For the criterion for deciding questions of ontological (or ontic) commitment is a different one than that proposed by Professor Quine.

NOTES

1. Dr. Lejewski does not explain the quantifiers in this way in "Logic and Existence." But this is the only reasonable explanation. The following

   For some x, x does not exist

   is not a good English sentence. For what is an x? To fully explain the unrestricted interpretation of the quantifiers, it will be necessary to regard the quantifiers as devices for talking about expressions.

2. Of course, since 'f' is a free variable, this explanation of the reading of the formula needs supplementation. No matter what expression replaces 'x,' the formula

   $$f(x)$$

   will not be true unless (a) 'f' is converted into a bound variable or (b) 'f' is replaced by a predicate.

3. Dr. Lejewski mentions Leśniewski's Ontology (a formal system) in this respect; he also presents a variant of Ontology, which uses a form of Boolean inclusion as its primitive concept.

4. I will not try to make any historical points about either Russell or Frege. In discussing formal systems and formalized languages, each of these men have made statements which suggest the views to which I have attached their names. It may well be that on other occasions they have made statements inconsistent with these views. With respect to Frege, for example, if one takes the account given by Professor Church in the introduction to *Introduction to Mathematical Logic* as a natural development of Frege's own view, then the considered Fregian view of variables is distinct from both the Russell-Quine and the Frege-Leśniewski views.

5. *Cf.* W V. Quine, *Word and Object*, p. 186.

6. W. V. Quine, "On What There Is," in *From a Logical Point of View*, p. 13.

7. S. Leśniewski, "O podstawach matematyki," *Przegląd Filosoficzny*, XXX (1927), p. 187.

8. W. V. Quine, *Word and Object*, p. 181.

9. If all the variables were bound, the axioms would be true. In Leśniewski's systems there are not free variables.

10. I have found that a predicate for distinguishing what exists from what does not can be added to predicate calculus with identity. However, I wish to consider this in another paper.

11. C. Lejewski, "Logic and Existence," p. 115.

12. Parentheses are abbreviated according to the convention of A. Church, *Introduction to Mathematical Logic*.

13. W. V. Quine, *Word and Object*, p. 181.

14. The two systems will not be equivalent, because Ontology is a very elaborate system, much more so than predicate calculus.

15. In Leśniewski's systems, all quantified variables are enclosed in one quantifier.

16. The symbol 'ε,' as defined for predicate calculus, is so defined that it is possible to prove

$$\varepsilon(fg) \supset \cdot (\exists h)\ \varepsilon(hf)\ \&\ (h_1)(h_2)[\varepsilon(h_1 f)\ \&\ \varepsilon(h_2 f) \supset \varepsilon(h_1 h_2)]\ \&\ (h) \cdot \varepsilon(hf) \supset \varepsilon(hg)\ .$$

But the implication cannot be proved in the other direction without the addition of an axiom.

17. Andrzej Mostowski, "On the Rules of Proof in the Pure Functional Calculus of the First Order," *The Journal of Symbolic Logic*, vol. 16 (1951), no. 2, pp. 107-111. Theodore Hailperin, "Quantificational Theory and Empty Individual Domains," *The Journal of Symbolic Logic*, vol. 18 (1953), no 3, pp. 197-200.

18. The rules of substitution for propositional and predicate variables must be changed so that when expressions containing free individual variables are substituted, the free variables are then bound by an initial universal quantifier. There will be no rule of substitution for free individual variables, but there will be a corresponding rule for those variables bound by an initial universal quantifier. The rule of modus ponens is unchanged. The proof that this reformulated system is universally valid follows the essential lines of the proof of Mostowski, in the paper cited in note 17.

19. One who starts with Ontology rather than first order predicates calculus is not losing anything (except the possibility of proving that at least one individual exists). For everything that can be said by means of individual variables and predicates can be said without individual variables.

20. In discussing the connection between predicate calculus and Ontology, I have pointed out how it is possible to begin with second order predicate calculus and obtain an analogue to Ontology. It is also possible to begin with Ontology and obtain an analogue to first order predicate calculus.

However, if one starts with Ontology, it is not possible to make much use of definitions in formulating an analogue to first order predicate calculus If the names of Ontology are regarded as predicates, then a new category of expressions is needed to name the individuals of which the predicates are predicated. In addition to this category, a new constant is needed to symbolize the application of a predicate to an individual term. Consider the predicate calculus formula

$$f(x)$$

Let 'a' be the general term of Ontology corresponding to 'f.' Then the analogue in Ontology to '$f(x)$' will be

$$\alpha(ax) \; ,$$

where 'α' is the constant for applying the predicate 'a' to 'z.' In predicate calculus, application is primitive, and is indicated by applying a predicate to its argument. In Ontology application is not primitive; it must be made explicit by means of a constant.

The reason why definitions are not too useful in formulating an analogue to first order predicate calculus is that the notation for the application of a predicate to its argument involves two semantical categories not found in Ontology. To introduce two categories at once, axioms are required rather than definitions.

21. W. V. Quine, "Reification of Universals," *From a Logical Point of View*, p. 109.

22. W. V. Quine, "Reification of Universals," p. 113.

23. W. V. Quine, "Reification of Universals," p. 113.

24. C. Lejewski, *op. cit.*, pp. 113-114.

25. Of course, in dealing with languages, one wants some way of reading the formulas being considered. For such a purpose, I think Lejewski's is as good as any. But it is necessary to keep in mind the limitations of the English reading of a formula from a formalized language.

26. See, for example, p. 129 ff, *An Inquiry into Meaning and Truth*.

27. The development of Combinatory Logic indicates that other devices can sometimes be used to do the work of variables. Questions similar to those we are considering can then be considered with respect to these devices.

28. Leśniewski's formalized languages include rules for definition among their rules of procedure. A defined expression is regarded as a genuine expression of the formalized language, and not merely as an abbreviation. This practice receives additional justification in terms of the F-L view of variables. The formula

$$(\exists x)[\phi(x) \;\&\; (y).\,\phi(y) \supset x = y]$$

can be true by virtue of a defined expression as well as an undefined term.

29. *Op. cit.*, p. 118, Of course, Dr. Lejewski is considering two interpretations of quantifiers rather than two views of variables, but I think it is clear that the different interpretations depend on the different ways of regarding variables.

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