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THE NUMERICAL EPSILON

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In this paper* Leśniewski's system of ontology extended by an axiom of infinity is used to derive Peano's arithmetic. Section 1 gives the main theses of this derivation which parallels the work of [6]. Using the numerical epsilon, defined in section 2, Peano's arithmetic is given a characteristically ontological model in section 3. Thus, the paper provides, for Peano's arithmetic, the two ways of treating logical concepts in ontology, the one, protothetical (section 1), the other, ontological (section 3).

1. *Numerals as predicates* The following proposition, in which the epsilon is primitive and is a proposition forming functor for two name arguments, is taken as the single axiom of ontology and is understood to be added to some given development of protothetic.

$$[Ab] \therefore A\varepsilon b .\equiv: [\exists C].C\varepsilon A:[C]:C\varepsilon A.\supset.C\varepsilon b:[CD]:C\varepsilon A.D\varepsilon A.\supset.C\varepsilon D$$

There is no rule which determines the style of letters to be used for variables, but throughout the paper capital Latin letters will be used for proper name variables and lower case Latin letters for general name variables; Greek letters will be employed for variables of higher semantical categories. Two types of definition, with the usual restrictions for bound and free variables, are allowable in ontology; ontological definitions which have the form:

$$[Aabc\ \ldots]:[\exists b].A\varepsilon b.\Phi(Aabc\ \ldots).\equiv.A\varepsilon\tau\langle abc\ \ldots\rangle$$

and protothetical definitions which are of the form:

$$[abc\ \ldots]:\Phi(abc\ \ldots).\equiv.\tau(abc\ \ldots)$$

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where τ is the constant being defined and is a name forming functor in the first case and a proposition forming functor in the second. The latter type of definition is called protothetical since, although the directive for ontology explicitly allows it as an acceptable form of definition, it is also allowable as a form of definition in Leśniewski's system of protothetic.

Once a constant of a given semantical category is introduced into the system, the principle of extensionality for that category is allowable as a thesis of the system. A more complete explanation of the directives for ontology and protothetic is found in [1], [4] and [5].

Two consequences of this axiom,

T1.2 $[Ab]:A\varepsilon b.\supset.A\varepsilon A$

T1.3 $[ABc]:A\varepsilon B.B\varepsilon c.\supset.A\varepsilon c$

serve to illustrate the difference between the primitive epsilon of ontology and other epsilons (of, say, set theory), since the former is seen to be semi-reflexive and transitive.

A list of well known basic definitions and theses follows, many of which can be found in [2].

D1.1 $[a]:[\exists A].A\varepsilon a.\equiv.\,!\{a\}$

D1.2 $[a]\therefore[BC]:B\varepsilon a.C\varepsilon a.\supset.B\varepsilon C:\equiv.\rightarrow\{a\}$

T1.4 $[A]:!\{A\}.\rightarrow\{A\}.\equiv.A\varepsilon A$

D1.3 $[A]:A\varepsilon A.\equiv.A\varepsilon\vee$

D1.4 $[A]:A\varepsilon A.\sim(A\varepsilon A).\equiv.A\varepsilon\wedge$

D1.5 $[AB]:A\varepsilon B.B\varepsilon A.\equiv.A=B$

D1.6 $[AB]:A\varepsilon A.B\varepsilon B.\sim(A=B).\equiv.A\neq B$

D1.7 $[ab]\therefore[A]:A\varepsilon a.\equiv.A\varepsilon b:\equiv.a\circ b$

T1.5 $[ab]\therefore[A]:A\varepsilon a.\equiv.A\varepsilon b:\equiv:[\varphi]:\varphi\{a\}.\equiv.\varphi\{b\}$

T1.6 $[ab]\therefore a\circ b.\equiv:[\varphi]:\varphi\{a\}.\equiv.\varphi\{b\}$

D1.8 $[A\varphi]:A\varepsilon A.\varphi\{A\}.\equiv.A\varepsilon\,\mathrm{stsf}<\varphi>$

T1.7 $[AB]\therefore A=B.\equiv:A\varepsilon A.B\varepsilon B:[\varphi]:\varphi\{A\}.\equiv.\varphi\{B\}$

D1.9 $[ab]\therefore[A]:A\varepsilon a.\supset.A\varepsilon b:\equiv.a\subset b$

D1.10 $[Aab]:A\varepsilon a.A\varepsilon b.\equiv.A\varepsilon a\cap b$

D1.11 $[Aab]\therefore A\varepsilon A:A\varepsilon a.\vee.A\varepsilon b:\equiv.A\varepsilon a\cup b$

T1.8 $[Aab]\therefore A\varepsilon a.\vee.A\varepsilon b:\equiv.A\varepsilon a\cup b$

T1.9 $[Aa]:A\varepsilon a.\supset.a\cup A\circ a$

D1.12 $[Aab]:A\varepsilon a.\sim(A\varepsilon b).\equiv.A\varepsilon a-b$

T1.10 $[Aa]:A\varepsilon A.\sim(A\varepsilon a).\supset.(a\cup A)-A\circ a$

D1.13 $[ab]:a\cap b\circ\wedge.\equiv.a\nabla b$

D1.14 $[\varphi]:[\exists a]\varphi\{a\}.\equiv.\,!(\varphi)$

D1.15 $[\varphi\psi\mu]:\psi(\varphi).\mu(\varphi).\equiv.\cap<\psi\mu>(\varphi)$

D1.16 $[\varphi\psi]\therefore[a]:\varphi\{a\}.\supset.\psi\{a\}:\equiv.\varphi\subset\psi$

D1.17 $[\varphi\psi]\therefore[a]:\varphi\{a\}.\equiv.\psi\{a\}:\equiv.\varphi\circ\psi$

T1.11 $[\varphi\psi]\therefore[a]:\varphi\{a\}.\equiv.\psi\{a\}:\equiv:[\mu]:\mu(\varphi).\equiv.\mu(\psi)$

T1.12 $[\varphi\psi]\therefore\varphi\circ\psi.\equiv:[\mu]:\mu(\varphi).\equiv.\mu(\psi)$

As can be seen from the above, there are many parallels between ontology and other systems, for instance [6], though in onotlogy one obtains two analogues of each logical concept, one ontological, the other protothetical, see for instance, *D1.1* with *D1.14* and *D1.9* with *D1.16*. Indeed, one obtains a definition of cardinal number from that of equinumerosity as in [6], and in section 3 an ontological, rather than protothetical, development of this theory is given.

D1.18 $[ab] .\therefore [\exists \varphi] : [ABC] : \varphi\{AC\} . \varphi\{BC\} .\supset. A = B : [ABC] : \varphi\{CA\} . \varphi\{CB\} .\supset. A = B : [A] : A \varepsilon a .\equiv. [\exists B] . \varphi\{BA\} : [B] : B \varepsilon b .\equiv. [\exists A] . \varphi\{BA\} :\equiv. a \infty b$
T1.13 $[a] . a \infty a$
T1.14 $[ab] : a \infty b .\supset. b \infty a$
T1.15 $[abc] : a \infty b . b \infty c .\supset. a \infty c$
D1.19 $[ab] : a \infty b .\equiv. \infty < a > \{b\}$
T1.16 $[a] . \infty < a > \{a\}$
D1.20 $[\varphi] .\therefore [ab] : \varphi\{a\} . a \infty b .\supset. \varphi\{b\} :\equiv. \mathbf{N}(\varphi)$
T1.17 $[a] . \mathbf{N}(\infty < a >)$
D1.21 $[\varphi] .\therefore [ab] : \varphi\{a\} . \varphi\{b\} .\supset. a \infty b :\equiv. \mathbf{Q}(\varphi)$
T1.18 $[a] . \mathbf{Q}(\infty < a >)$
D1.22 $[\varphi] : !(\varphi) . \mathbf{N}(\varphi) . \mathbf{Q}(\varphi) .\equiv. \mathbf{NC}(\varphi)$
T1.19 $[a] . \mathbf{NC}(\infty < a >)$

Definitions *D1.20* and *D1.21* provide the concepts of a proposition forming functor (for one name argument) being *numerical* and *quantitative*. These concepts, together with the condition that a given functor is unempty, provide the defining characteristics for a cardinal number—see *D1.22*.

Finite names, zero, and the successor function are also definable, though the definition of successor given here is that of Frege rather than that of [6]. The definition of successor in [6] amounts to a particular case of addition which is defined as the union of disjoint sets. However, this definition is ambiguous as to type (the successor of a number is not necessarily of the same type as the number) and is therefore not as good a candidate for use in ontology as the Fregean definition. Of course the Fregean definition is derivable as a theorem of [6]—see ***110.63.**

D1.23 $[a] :: [\varphi] .\therefore \varphi\{\Lambda\} : [Ab] : A \varepsilon a . \varphi\{b\} .\supset. \varphi\{b \cup A\} :\supset. \varphi\{a\} .\therefore \equiv. \mathrm{Fin}\{a\}$
T1.20 $\mathrm{Fin}\{\Lambda\}$
T1.21 $[Aa] : A \varepsilon A . \mathrm{Fin}\{a\} .\supset. \mathrm{Fin}\{a \cup A\}$
T1.22 $\mathbf{N}(\mathrm{Fin})$
T1.23 $[ab] : \mathrm{Fin}\{a\} . b \subset a . \sim(a \subset b) .\supset. \sim(a \infty b)$
D1.24 $[a] : a \circ \Lambda .\equiv. 0\{a\}$
T1.24 $0\{\Lambda\}$
T1.25 $[a\varphi] : 0\{a\} . \varphi\{a\} .\supset. \varphi\{\Lambda\}$
T1.26 $\mathbf{N}(0)$
T1.27 $\mathbf{Q}(0)$
D1.25 $[a\varphi] : [\exists A] . A \varepsilon a . \varphi\{a - A\} .\equiv. \mathbf{S} < \varphi > \{a\}$
T1.28 $[\varphi] . \sim([\exists a] . 0\{a\} . \mathbf{S} < \varphi > \{a\})$
T1.29 $[a\varphi\psi] : \mathbf{N}(\varphi) . \mathbf{S} < \varphi > \{a\} . \mathbf{S} < \psi > \{a\} .\supset. [\exists b] . \varphi\{b\} . \psi\{b\}$

T1.30 $[\varphi]:\mathbf{N}(\varphi).\supset.\mathbf{N}(\mathbf{S}<\varphi>)$
T1.31 $[\varphi]:\mathbf{Q}(\varphi).\supset.\mathbf{Q}(\mathbf{S}<\varphi>)$

The following proposition is taken as the axiom of infinity,

$$[a]:\mathrm{Fin}\{a\}.\supset.[\exists A].A\varepsilon A.\sim(A\varepsilon a)$$

and given this axiom it is now possible to derive Peano's axioms. First, one defines natural number,

D1.26 $[\varphi]::[\mu].\because.\mu(0):[\psi]:\mu(\psi).\supset.\mu(\mathbf{S}<\psi>):\supset.\mu(\varphi).\because.\equiv.\mathbf{Nn}(\varphi)$

and then the following are derivable.

T1.32 $\mathbf{Nn}(0)$
T1.33 $[\varphi]:\mathbf{Nn}(\varphi).\supset.\sim(0\,\mathrm{o}\,\mathbf{S}<\varphi>)$
T1.34 $[\varphi]:\mathbf{Nn}(\varphi).\supset.\mathbf{Nn}(\mathbf{S}<\varphi>)$
T1.35 $[\varphi\mu].\because.\mu(0):[\psi]:\mathbf{Nn}(\psi).\mu(\psi).\supset.\mu(\mathbf{S}<\psi>):\mathbf{Nn}(\varphi):\supset.\mu(\varphi)$
T1.36 $[\varphi]:\mathbf{Nn}(\varphi).\supset.\varphi\subset\mathrm{Fin}$
T1.37 $\mathbf{Nn}\subset\mathbf{NC}$
T1.38 $[\varphi\psi]:\mathbf{Nn}(\varphi).\mathbf{Nn}(\psi).\mathbf{S}<\varphi>\mathrm{o}\,\mathbf{S}<\psi>.\supset.\varphi\,\mathrm{o}\,\psi$

Of these theses, *T1.32—T1.35* and *T1.38* are Peano's axioms and *T1.34* follows in the system banally since its consequent is a thesis, while only *T1.38* requires the axiom of infinity for its proof.

Given these axioms, it is possible to obtain the rest of Peano's arithmetic by defining the various arithmetical operations and deriving their consequences. The approach here is to use Frege's method of reducing implicit definitions to explicit ones using "impredicative" definitions. For instance, guided by the recursive definition of addition, one defines,

D1.27 $[\tau].\because.[\varphi]:\mathbf{Nn}(\varphi).\supset.\tau(\varphi\varphi0):\equiv.\mathrm{sm}(\tau)$
D1.28 $[\tau].\because.[\varphi\psi\mu]:\mathbf{Nn}(\varphi).\mathbf{Nn}(\psi).\mathbf{Nn}(\mu).\tau(\varphi\psi\mu).\supset.$
$(\mathbf{S}<\varphi>\psi\,\mathbf{S}<\mu>):\equiv.\mathrm{smm}(\tau)$
D1.29 $[\varphi\psi\mu].\because.\mathbf{Nn}(\varphi).\mathbf{Nn}(\psi).\mathbf{Nn}(\mu):[\tau]:\mathrm{sm}(\tau).\mathrm{smm}(\tau).\supset.$
$\tau(\varphi\psi\mu):\equiv.\mathrm{Sm}(\varphi\psi\mu)$

and then obtains,

T1.39 $[\varphi]:\mathbf{Nn}(\varphi).\supset.\mathrm{Sm}(\varphi\varphi0)$
T1.40 $[\varphi\psi\mu]:\mathrm{Sm}(\varphi\psi\mu).\supset.\mathrm{Sm}(\mathbf{S}<\varphi>\psi\,\mathbf{S}<\mu>)$
T1.41 $[\varphi\psi]:\mathbf{Nn}(\varphi).\mathbf{Nn}(\psi).\supset.[\exists\mu].\mathbf{Nn}(\mu).\mathrm{Sm}(\mu\varphi\psi)$
T1.42 $[\varphi\psi\mu\tau]:\mathrm{Sm}(\varphi\mu\tau).\mathrm{Sm}(\psi\mu\tau).\supset.\varphi\,\mathrm{o}\,\psi$

Now, given the definition of addition,

D1.30 $[a\varphi\psi]:\mathrm{Sm}(\infty<a>\varphi\psi).\equiv.+<\varphi\psi>\{a\}$

the uniqueness, closure and recursive properties of the operation follow.

T1.43 $[\varphi\psi\mu].\because.\mathbf{Nn}(\varphi).\supset:\mathrm{Sm}(\varphi\psi\mu).\equiv.\varphi\,\mathrm{o}\,\psi+\mu$
T1.44 $[\varphi\psi]:\mathbf{Nn}(\varphi).\mathbf{Nn}(\psi).\supset.\mathbf{Nn}(\varphi+\psi)$

T1.45 $[\varphi]:\mathbf{Nn}(\varphi).\supset.\varphi+0\,\mathrm{o}\,\varphi$
T1.46 $[\varphi\psi]:\mathbf{Nn}(\varphi).\mathbf{Nn}(\psi).\supset.\varphi+\mathbf{S}<\psi>\mathrm{o}\,\mathbf{S}<\varphi+\psi>$

Clearly, all recursively definable concepts are obtainable in this manner. But it is important to note that this is probably the only way they are obtainable in this system. For instance, one cannot use the concept of ordered pairs in order to define multiplication as is done in [6]. For though ordered pairs are definable in ontology, an ordered pair of arguments is of a different semantical category than its arguments. This immediately leads to the problem of having non-homogeneous arguments for arithmetical operations—a problem which cannot be ignored in ontology because the system does not allow "typical ambiguity" (as does the system of [6]): any variable appearing in a thesis is definite as to type in ontology.

However, it is clear that Peano's arithmetic is derivable in this system in a way at least similar to [6]. Numerals are construed as predicates, that is, proposition forming functors for names, and numerals for finite numbers are characteristically inductive.

2. *The numerical epsilon* Consider the following definition of a higher epsilon.

D2.1 $[\Phi\varphi\psi]\therefore[\exists a].\Phi\{a\}.\varphi\{a\}:[ab]:\Phi\{a\}.\Phi\{b\}.\supset.\psi\{ab\}:[ab]:\Phi\{a\}.\psi\{ab\}.\supset.\Phi\{b\}:[ab]:\varphi\{a\}.\psi\{ab\}.\supset.\varphi\{b\}:\equiv.\varepsilon<\psi>(\Phi\varphi)$

In what follows interest will be limited to the particular case,

T2.1 $[\Phi\varphi]\therefore[\exists a].\Phi\{a\}.\varphi\{a\}:[ab]:\Phi\{a\}.\Phi\{b\}.\supset.a\infty b:[ab]:\Phi\{a\}.a\infty b.\supset.\Phi\{b\}:[ab]:\varphi\{a\}.a\infty b.\supset.\varphi\{b\}:\equiv.\varepsilon<\infty>(\Phi\varphi)$

and hereafter '$\Phi\varepsilon_\infty\varphi$' will be written for '$\varepsilon<\infty>(\Phi\varphi)$'. This higher epsilon will be called a *numerical epsilon* and could have been defined by the following thesis.

T2.2 $[\Phi\varphi]:[\exists a].\Phi\{a\}.\varphi\{a\}.\mathbf{Q}(\Phi).\mathbf{N}(\Phi).\mathbf{N}(\varphi).\equiv.\Phi\varepsilon_\infty\varphi$ [*T2.1, D1.20, D1.21*]

In [1] the author gave *T2.2* as the definition of his numerical epsilon, but it is more perspicuous to view the definition as merely one instance of a general form for defining higher epsilons. For instance, *D2.1* also yields

T $\varepsilon<\mathrm{o}>$ $[\Phi\varphi]\therefore[\exists a].\Phi\{a\}.\varphi\{a\}:[ab]:\Phi\{a\}.\Phi\{b\}.\supset.a\,\mathrm{o}\,b:[ab]:\Phi\{a\}.a\,\mathrm{o}\,b.\supset.\Phi\{b\}:[ab]:\varphi\{a\}.a\,\mathrm{o}\,b.\supset.\varphi\{b\}:\equiv.\varepsilon<\mathrm{o}>(\Phi\varphi)$

which is equivalent to the well known definition of a higher epsilon

D $\varepsilon<\mathrm{o}>$ $[\Phi\varphi]\therefore[\exists a].\Phi\{a\}.\varphi\{a\}:[ab]:\Phi\{a\}.\Phi\{b\}.\supset.a\,\mathrm{o}\,b:\equiv.\varepsilon<\mathrm{o}>(\Phi\varphi)$

since in this case *T* $\varepsilon<\mathrm{o}>$ contains subformulas, which by the principle of extensionality are theses (see *T1.6*). Similarly one would have

T $\varepsilon<=>$ $[\Phi\varphi]\therefore[\exists A].\Phi\{A\}.\varphi\{A\}:[AB]:\Phi\{A\}.\Phi\{B\}.\supset.A=B:[AB]:\Phi\{A\}.A=B.\supset.\Phi\{B\}:[AB]:\varphi\{A\}.A=B.\supset.\varphi\{B\}:\equiv.\varepsilon<=>(\Phi\varphi)$

and equivalently

$$D\ \varepsilon <=> [\Phi\varphi] .\because . [\exists A] . \Phi\{A\} . \varphi\{A\} : [AB] : \Phi\{A\} . \Phi\{B\} . \supset . A = B :\equiv . \varepsilon <=> (\Phi\varphi)$$

which is the usual definition given for the restricted higher epsilon. This generalization and particularly its use for defining a *numerical* epsilon as opposed to a higher general or restricted epsilon marks the author's contribution to the theory of higher epsilons in the field of ontology. As will be seen, the numerical epsilon is useful since it allows the construction of a characteristically ontological model for numerical concepts.

First of all, on the basis of *D2.1* it is possible to derive a thesis employing the numerical epsilon which is analogous to the axiom of ontology (which uses the primitive epsilon of ontology).

T2.3 $[\Phi\varphi] : \Phi\varepsilon_\infty\varphi . \supset . \Phi\varepsilon_\infty\Phi$ [*T2.2*]

T2.4 $[\Phi\Psi\varphi] : \Phi\varepsilon_\infty\varphi . \Psi\varepsilon_\infty\Phi . \supset . \Psi\varepsilon_\infty\varphi$

PF: $[\Phi\Psi\varphi] .\because . \text{Hyp}(2) . \supset :$

3) $\mathbf{Q}(\Phi) . \mathbf{N}(\varphi) .$ [*T2.2*, 1]

4) $\mathbf{Q}(\Psi) . \mathbf{N}(\Psi) :$ [*T2.2*, 2]

$[\exists a] :$

5) $\Phi\{a\} . \varphi\{a\} .$ [*T2.2*, 1]

$[\exists b] .$

6) $\Psi\{b\} . \Phi\{b\}$ [*T2.2*, 2]

7) $a \infty b$ [*D1.21*, 3, 5, 6]

8) $\varphi\{b\} :$ [*D1.20*, 3, 5, 7]

$\Psi\varepsilon_\infty\varphi$ [*T2.2*, 3, 4, 6, 8]

In the demonstration of this thesis the first line abbreviates listing the hypotheses of the conditional to be proved and the numeral indicates the number of components in the antecedent of the conditional.

T2.5 $[\Phi\Psi X\varphi] : \Phi\varepsilon_\infty\varphi . \Psi\varepsilon_\infty\Phi . X\varepsilon_\infty\Phi . \supset . \Psi\varepsilon_\infty X$

PF: $[\Phi\Psi X\varphi] .\because . \text{Hyp}(3) . \supset :$

4) $\mathbf{Q}(\Phi)$ [*T2.2*, 1]

5) $\mathbf{Q}(\Psi) . \mathbf{N}(\Psi) .$ [*T2.2*, 2]

6) $\mathbf{N}(X) :$ [*T2.2*, 3]

$[\exists a] :$

7) $\Psi\{a\} . \Phi\{a\} .$ [*T2.2*, 2]

$[\exists b] .$

8) $X\{b\} . \Phi\{b\} .$ [*T2.2*, 3]

9) $a \infty b .$ [*D1.21*, 4, 7, 8]

10) $\Psi\{b\} :$ [*D1.20*, 5, 7, 9]

$\Psi\varepsilon_\infty X$ [*T2.2*, 10, 8, 5, 6]

T2.6 $[a\varphi] : \varphi\{a\} . \mathbf{N}(\varphi) . \equiv . \infty<a>\varepsilon_\infty\varphi$

[*T2.2*, *D1.19*, *T1.16*, *T1.17*, *T1.18*, *T1.14*]

T2.7 $[ab\varphi] .\because . a \infty b . \supset : \infty<a>\varepsilon_\infty\varphi . \equiv . \infty<b>\varepsilon_\infty\varphi$ [*T1.14*, *T2.6*]

T2.8 $[\Phi\Psi ab] .\because . \Psi\varepsilon_\infty\Phi : [\Psi X] : \Psi\varepsilon_\infty\Phi . X\varepsilon_\infty\Phi . \supset . \Psi\varepsilon_\infty X : \Phi\{a\} . \Phi\{b\} : \supset . a \infty b$

PF: $[\Phi\Psi ab] .\therefore \text{Hyp}(4) :\supset .$

5) $\mathbf{N}(\Phi)$. [*T2.2*, 1]

6) $\infty\langle a\rangle \varepsilon_\infty \Phi.$ [*T2.6*, 3, 5]

7) $\infty\langle b\rangle \varepsilon_\infty \Phi.$ [*T2.6*, 4, 5]

8) $\infty\langle a\rangle \varepsilon_\infty \infty\langle b\rangle .$ [2, 6, 7]

$[\exists c].$

9) $\infty\langle a\rangle \{c\} . \infty\langle b\rangle \{c\}.$ [*T2.2*, 8]

$a \infty b$ [*D1.19*, 9, *T1.14*, *T1.15*]

T2.9 $[\Phi\Psi] .\therefore \Psi\varepsilon_\infty\Phi : [\Psi X] : \Psi\varepsilon_\infty\Phi . X\varepsilon_\infty\Phi .\supset . \Psi\varepsilon_\infty X :\supset . \mathbf{Q}(\Phi)$ [*T2.8*, *D1.21*]

T2.10 $[\Phi\Psi\varphi] .\therefore \Psi\varepsilon_\infty\Phi : [\Psi X] : \Psi\varepsilon_\infty\Phi . X\varepsilon_\infty\Phi .\supset . \Psi\varepsilon_\infty X : [\Psi] : \Psi\varepsilon_\infty\Phi .\supset .$
$\Psi\varepsilon_\infty\varphi :\supset . \Phi\varepsilon_\infty\varphi$

PF: $[\Phi\Psi\varphi] .\therefore \text{Hyp}(3) :\supset .$

4) $\mathbf{N}(\Phi)$. [*T2.2*, 1]

5) $\mathbf{Q}(\Phi)$. [*T2.9*, 1, 2]

$[\exists a].$

6) $\Phi\{a\}$ [*T2.2*, 1]

7) $\Phi\varepsilon_\infty\Phi .$ [*T2.2*, 6, 5, 4]

$\Phi\varepsilon_\infty\varphi$ [3, 7]

T2.11 $[\Phi\varphi] .\therefore \Phi\varepsilon_\infty\varphi .\equiv: [\exists\Psi] . \Psi\varepsilon_\infty\Phi : [\Psi X] : \Psi\varepsilon_\infty\Phi . X\varepsilon_\infty\Phi .\supset .$
$\Psi\varepsilon_\infty X : [\Psi] : \Psi\varepsilon_\infty\Phi .\supset . \Psi\varepsilon_\infty\varphi$ [*T2.10*, *T2.3*, *T2.4*, *T2.5*]

This thesis is the analogue of the axiom of ontology since it can be expressed as a proposition identical in shape with the axiom except for the parentheses which are generally associated with the primitive epsilon on the one hand and the numerical epsilon on the other.

Under certain conditions, the principle of extensionality holds for the numerical epsilon. The limitation imposed on employing extensionality for the numerical epsilon amounts to restricting the principles's application to numerical predicates. This is clear when one considers *T2.14*. On the basis of the auxiliary definition,

D2.2 $[\Phi\varphi] : \Phi\varepsilon_\infty\varphi .\equiv . \varepsilon_\infty\langle\Phi\rangle(\varphi)$

there follows,

T2.12 $[\varphi\psi] .\therefore [\mu] : \mu(\varphi) .\equiv . \mu(\psi) :\supset: [\Phi] : \Phi\varepsilon_\infty\varphi .\equiv . \Phi\varepsilon_\infty\psi$ [*D2.2*]

T2.13 $[\varphi\psi] .\therefore \mathbf{N}(\varphi) . \mathbf{N}(\psi) : [\Phi] : \Phi\varepsilon_\infty\varphi .\equiv . \Phi\varepsilon_\infty\psi :\supset: [\mu] : \mu(\varphi) .\equiv . \mu(\psi)$

PF: $[\varphi\psi] .\therefore \text{Hyp}(3) :\supset:$

4) $[a] : \infty\langle a\rangle \varepsilon_\infty\varphi .\equiv . \infty\langle a\rangle \varepsilon_\infty\psi :$ [3]

5) $[a] : \varphi\{a\} . \mathbf{N}(\varphi) .\equiv . \psi\{a\} . \mathbf{N}(\psi) :$ [*T2.6*, 4]

6) $[a] : \varphi\{a\} .\equiv . \psi\{a\} :$ [5, 1, 2]

$[\mu] : \mu(\varphi) .\equiv . \mu(\psi)$ [*T1.11*, 6]

T2.14 $[\varphi\psi] :: \mathbf{N}(\varphi) . \mathbf{N}(\psi) .\supset .\therefore [\Phi] : \Phi\varepsilon_\infty\varphi .\equiv . \Phi\varepsilon_\infty\psi :\equiv: [\mu] : \mu(\varphi) .\equiv . \mu(\psi)$
[*T2.12*, *T2.13*]

The hypothesis of this thesis guarantees that for numerical predicates, there is derivable a thesis analogous to the principle of ontological extensionality for the primitive epsilon: compare the consequent of this thesis with *T1.5*. Next, identity for numerical names is defined and a thesis analogous to the principle of extensionality for identical names is derived.

D2.3 $[\Phi\Psi] : \Phi \varepsilon_{\infty} \Psi . \Psi \varepsilon_{\infty} \Phi . \equiv . \Phi =_{\infty} \Psi$

T2.15 $[\Phi\Psi] \therefore \Phi =_{\infty} \Psi . \supset : [\mu] : \mu(\Phi) . \equiv . \mu(\Psi)$

PF: $[\Phi\Psi] \therefore \mathrm{Hyp}(1) . \supset :$

2) $\Phi \varepsilon_{\infty} \Psi . \Psi \varepsilon_{\infty} \Phi .$ \quad [*D2.3*, 1]

3) $\mathbf{N}(\Phi) . \mathbf{N}(\Psi) :$ \quad [*T2.2*, 2]

4) $[X] : X \varepsilon_{\infty} \Phi . \equiv . X \varepsilon_{\infty} \Psi :$ \quad [*T2.4*, 2]

$[\mu] : \mu(\Phi) . \equiv . \mu(\Psi)$ \quad [*T2.14*, 3, 4]

T2.16 $[\Phi\Psi] \therefore [\mu] : \mu(\Phi) . \equiv . \mu(\Psi) : \Phi \varepsilon_{\infty} \Phi . \Psi \varepsilon_{\infty} \Psi : \supset . \Phi =_{\infty} \Psi$

PF: $[\Phi\Psi] \therefore \mathrm{Hyp}(3) : \supset :$

4) $\mathbf{N}(\Phi)$ \quad [*T2.2*, 2]

5) $\mathbf{N}(\Psi) :$ \quad [*T2.2*, 3]

6) $[X] : X \varepsilon_{\infty} \Phi . = . X \varepsilon_{\infty} \Psi :$ \quad [*T2.14*, 1, 4, 5]

7) $\Phi \varepsilon_{\infty} \Psi .$ \quad [6, 2]

8) $\Psi \varepsilon_{\infty} \Phi .$ \quad [6, 3]

$\Phi =_{\infty} \Psi$ \quad [*D2.3*, 7, 8]

T2.17 $[\Phi\Psi] \therefore \Phi =_{\infty} \Psi . \equiv : \Phi \varepsilon_{\infty} \Phi . \Psi \varepsilon_{\infty} \Psi : [\mu] : \mu(\Phi) . \equiv . \mu(\Psi)$

[*T2.15*, *T2.16*, *D2.3*]

It should be noted that in dealing with identical numerical names, the condition imposed on the principle of numerical extensionality is always fulfilled. Thus, this exact analogue of the principle of extensionality for identical names is derivable, compare *T1.7*.

This section is concluded by establishing that under certain conditions a thesis analogous to an ontological definition is always forthcoming for any definable numerical name.

T2.18 $[\Phi\varphi] \therefore \Phi \varepsilon_{\infty} \Phi . \mathbf{N}(\varphi) : [a] : \Phi\{a\} . \supset . \varphi\{a\} : \supset . \Phi \varepsilon_{\infty} \varphi$ \quad [*T2.2*]

T2.19 $[\Phi\varphi] \therefore \Phi \varepsilon_{\infty} \varphi . \supset : \Phi \varepsilon_{\infty} \Phi . \mathbf{N}(\varphi) : [a] : \Phi\{a\} . \supset . \varphi\{a\}$

PF: $[\Phi\varphi] :: \mathrm{Hyp}(1) . \supset \therefore$

2) $\Phi \varepsilon_{\infty} \Phi . \mathbf{Q}(\Phi) . \mathbf{N}(\varphi) \therefore$ \quad [*T2.2*, 1]

$[\exists b] \therefore$

3) $\Phi\{b\} . \varphi\{b\} :$ \quad [*T2.2*, 1]

4) $[a] : \Phi\{a\} . \supset . b \infty a :$ \quad [*D1.21*, 2, 3]

5) $[a] : b \infty a . \supset . \varphi\{a\} \therefore$ \quad [*D1.20*, 2, 3]

6) $[a] : \Phi\{a\} . \supset . \varphi\{a\} :$ \quad [4, 5]

$\Phi \varepsilon_{\infty} \Phi . \mathbf{N}(\varphi) : [a] : \Phi\{a\} . \supset . \varphi\{a\}$ \quad [2, 6]

T2.20 $[\Phi\varphi] \therefore \Phi \varepsilon_{\infty} \varphi . \equiv : \Phi \varepsilon_{\infty} \Phi . \mathbf{N}(\varphi) : [a] : \Phi\{a\} . \supset . \varphi\{a\}$ \quad [*T2.18*, *T2.19*]

T2.21 $[\mu\Phi\varphi\psi] \therefore [\varphi a] : \mu < \varphi > (\infty < a >) . \equiv . \psi < \varphi > \{a\} : \Phi \varepsilon_{\infty} \psi < \varphi > : \supset .$
$\mu < \varphi > (\Phi)$

PF: $[\mu\Phi\varphi\psi] :: \mathrm{Hyp}(2) : \supset \therefore$

3) $[ab] : \Phi\{a\} . \Phi\{b\} . \supset . a \infty b :$ \quad [*T2.2*, 2, *D1.21*]

4) $[ab]: \Phi\{a\} . a \infty b .\supset. \Phi\{b\}:$ [*T2.2*, 2, *D1.20*]
$[\exists a] .\therefore$
5) $\Phi\{a\} . \psi<\varphi>\{a\}:$ [*T2.2*, 2]
6) $[b]: \Phi\{b\} .\supset. a \infty b:$ [3, 5]
7) $[b]: a \infty b .\supset. \Phi\{b\}:$ [4, 5]
8) $[b]: \Phi\{b\} .\equiv. \infty<a>\{b\}:$ [*D1.19*, 7, 6]
9) $\mu<\varphi>(\Phi) .\equiv. \mu<\varphi>(\infty<a>) .\therefore$ [*T1.11*, 8]
$\mu<\varphi>(\Phi)$ [1, 5, 9]

T2.22 $[\mu\psi] .\therefore [\varphi a]: \mu<\varphi>(\infty<a>) .\equiv. \psi<\varphi>\{a\} :\supset: [\Phi\varphi]: \Phi\varepsilon_{\infty}\psi<\varphi> .\supset.$
$\mu<\varphi>(\Phi)$ [*T2.21*]

T2.23 $[\mu\Phi\varphi\psi] .\therefore [\varphi a]: \mu<\varphi>(\infty<a>) .\equiv. \psi<\varphi>\{a\}:$
$\mu<\varphi>(\Phi) . \Phi\varepsilon_{\infty}\Phi . \Phi\{a\} :\supset. \psi<\varphi>\{a\}$

PF: $[\mu\Phi\varphi\psi] .\therefore$ Hyp(4) $:\supset:$
5) $\mathbf{Q}(\Phi) . \mathbf{N}(\Phi):$ [*T2.2*, 3]
6) $[b]: \Phi\{b\} .\supset. a \infty b:$ [*D1.21*, 5, 4]
7) $[b]: a \infty b .\supset. \Phi\{b\}:$ [*D1.20*, 5, 4]
8) $[b]: \Phi\{b\} .\equiv. \infty<a>\{b\}:$ [*D1.19*, 7, 6]
9) $\mu<\varphi>(\Phi) .\equiv. \mu<\varphi>(\infty<a>):$ [*T1.11*, 8]
$\psi<\varphi>\{a\}$ [9, 2, 1]

T2.24 $[\mu\Phi\varphi\tau] .\therefore [\psi a]: \mu<\psi>(\infty<a>) .\equiv. \tau<\psi>\{a\} : \Phi\varepsilon_{\infty}\Phi . \mu<\varphi>(\Phi) :\supset:$
$[a]: \Phi\{a\} .\supset. \tau<\varphi>\{a\}$ [*T2.23*]

T2.25 $[\mu\Phi\varphi\tau] .\therefore [\psi a]: \mu<\psi>(\infty<a>) .\equiv. \tau<\psi>\{a\} : \mathbf{N}(\tau<\varphi>) .$
$\mu<\varphi>(\Phi) . \Phi\varepsilon_{\infty}\Phi :\supset. \Phi\varepsilon_{\infty}\tau<\varphi>$ [*T2.24*, *T2.20*]

T2.26 $[\mu\tau] :: [\psi a]: \mu<\psi>(\infty<a>) .\equiv. \tau<\psi>\{a\} :\supset.\because [\Phi\varphi] .\therefore \mathbf{N}(\tau<\varphi>) .\supset:$
$\Phi\varepsilon_{\infty}\tau<\varphi> .\equiv. \Phi\varepsilon_{\infty}\Phi . \mu<\varphi>(\Phi)$ [*T2.22*, *T2.3*, *T2.25*]

This thesis guarantees that given (a) a protothetical definition of a functor and (b) that the functor is numerical, then a thesis is derivable analogous to an ontological definition of the functor—analogous in the sense that the thesis is expressible as a proposition which has the form of a proper ontological definition, compare the consequent of *T2.26* with the form of ontological definition given in section 1.

Actually, *T2.26* considers as the functor to be defined only one which is a single link numerical name forming functor (whose link is a proposition forming functor for a single name argument). However, an inspection of the above proof shows that similar theses can be established for numerical names with any finite number of links, for instance,

T2.27 $[\mu\tau] :: [\varphi\psi a]: \mu<\varphi\psi>(\infty<a>) .\equiv. \tau<\varphi\psi>\{a\} :\supset.\because$
$[\Phi\varphi\psi] .\therefore \mathbf{N}(\tau<\varphi\psi>) .\supset: \Phi\varepsilon_{\infty}\tau<\varphi\psi> .\equiv. \Phi\varepsilon_{\infty}\Phi . \mu<\varphi\psi>(\Phi)$
[similar to *T2.26*]

T2.28 $[\mu\tau] :: [a]: \mu(\infty<a>) .\equiv. \tau\{a\} :\supset.\because [\Phi] .\therefore \mathbf{N}(\tau) .\supset: \Phi\varepsilon_{\infty}\tau .\equiv. \Phi\varepsilon_{\infty}\Phi . \mu(\Phi)$
[similar to *T2.26*]

It should be noted that theses *T2.26*, *T2.27*, etc., provide an effective means for obtaining theses which have the form of ontological definitions.

In general, once an ontological definition of a numerical name forming functor is decided upon, it is obtainable from an effectively given protothetical definition. An application of this process is given in the next section.

With the availability of ontological definitions for numerical names, as well as extensionality and an ontological axiom, it is clear that the numerical epsilon is completely analogous to the primitive epsilon. So, as long as one restricts attention to numerical functors, all theses which involve the primitive epsilon are also derivable for the numerical epsilon. And given this basis, the means is available for providing an ontological development of Peano's arithmetic in the system.

3. *Numerals as names* Peano's axioms expressed with the numerical epsilon are given as *T3.1*, *T3.3*, *T3.5*, *T3.6* and *T3.15*. The proof of *T3.15* is of particular interest since it requires the use of an ontological definition for numerical names. In this development *T3.5* does not follow banally in the system as does its analogue *T1.34*: the antecedent of *T3.5* is required to establish its consequent.

T3.1 $0\varepsilon_{\infty}\mathrm{Fin}$ [*T2.2*, *T1.24*, *T1.26*, *T1.27*, *T1.20*, *T1.22*]

T3.2 $[\Phi].\because[a]:\mathrm{Fin}\{a\}.\supset.[\exists A].A\varepsilon A.\sim(A\varepsilon a):\Phi\varepsilon_{\infty}\mathrm{Fin}:\supset.\mathbf{S}<\Phi>\varepsilon_{\infty}\mathrm{Fin}$

PF: $[\Phi].\because\mathrm{Hyp}(2):\supset:$

3) $\mathbf{Q}(\Phi).\mathbf{N}(\Phi)$ [*T2.2*, 2]

$[\exists a]:$

4) $\Phi\{a\}.\mathrm{Fin}\{a\}.$ [*T2.2*, 2]

$[\exists A].$

5) $A\varepsilon A.\sim(A\varepsilon a).$ [1, 6]

6) $(a\cup A)-A\,\mathrm{o}\,a.$ [*T1.10*, 5]

7) $A\varepsilon a\cup A.$ [*T1.8*, 5]

8) $\Phi\{(a\cup A)-A\}.$ [*T1.6*, 6, 5]

9) $\mathbf{S}<\Phi>\{a\cup A\}.$ [*D1.25*, 7, 8]

10) $\mathrm{Fin}\{a\cup A\}:$ [*T1.21*, 5, 4]

$\mathbf{S}<\Phi>\varepsilon_{\infty}\mathrm{Fin}$ [*T2.2*, 9, 10, *T1.30*, 3, *T1.31*, *T1.22*]

T3.3 $[\Phi]:\Phi\varepsilon_{\infty}\mathrm{Fin}.\supset.\mathbf{S}<\Phi>\varepsilon_{\infty}\mathrm{Fin}$ [*T3.2*, Ax Infin]

T3.4 $[\varphi].\sim(0=_{\infty}\mathbf{S}<\varphi>)$ [*T2.2*, *D2.3*, *T1.28*]

On the basis of,

D3.1 $[\Phi\Psi]:\Phi\varepsilon_{\infty}\Phi.\Psi\varepsilon_{\infty}\Psi.\sim(\Phi\varepsilon_{\infty}\Psi).\equiv.\Phi\neq_{\infty}\Psi$

we have the following.

T3.5 $[\Phi]:\Phi\varepsilon_{\infty}\mathrm{Fin}.\supset.0\neq_{\infty}\mathbf{S}<\Phi>$ [*T2.3*, *T3.3*, *T3.1*, *T3.4*, *D3.1*, *D2.3*]

T3.6 $[\Phi\Psi]:\Phi\varepsilon_{\infty}\mathrm{Fin}.\Psi\varepsilon_{\infty}\mathrm{Fin}.\mathbf{S}<\Phi>=_{\infty}\mathbf{S}<\Psi>.\supset.\Phi=_{\infty}\Psi$

PF: $[\Phi\Psi]:\mathrm{Hyp}(3).\supset.$

4) $\mathbf{Q}(\Phi).\mathbf{N}(\Phi).$ [*T2.2*, 1]

5) $\mathbf{Q}(\Psi).\mathbf{N}(\Psi).$ [*T2.2*, 2]

6) $\mathbf{S}<\Phi>\varepsilon_{\infty}\mathbf{S}<\Psi>.$ [*D2.3*, 3]

7) $[\exists a] . \mathbf{S}<\Phi>\{a\} . \mathbf{S}<\Psi>\{a\}$. [*T2.2*, 6]
8) $[\exists b] . \Phi\{b\} . \Psi\{b\}$. [*T1.29*, 5, 7]
9) $\Phi \varepsilon_{\infty} \Psi$. [*T2.2*, 8, 4, 5]
10) $\Psi \varepsilon_{\infty} \Phi$. [*T2.2*, 8, 4, 5]
$\Phi =_{\infty} \Psi$ [*D2.3*, 9, 10]

In order to obtain the last of Peano's axioms, an ontological definition of the intersection of numerical names is required. On the basis of the following definitions,

D3.2 $[\Phi\varphi\psi]: \Phi \varepsilon_{\infty} \varphi . \Phi \varepsilon_{\infty} \psi . \equiv . \cap_{\infty}<\varphi\psi>(\Phi)$
D3.3 $[a\varphi\psi]: \infty<a> \varepsilon_{\infty} \varphi . \infty<a> \varepsilon_{\infty} \psi . \equiv . \cap_{\infty}<\varphi\psi>\{a\}$

the hypotheses of *T2.27* can be obtained.

T3.7 $[a\varphi\psi]: \cap_{\infty}<\varphi\psi>(\infty<a>) . \equiv . \cap_{\infty}<\varphi\psi>\{a\}$ [*D3.2, D3.3*]
T3.8 $[ab\varphi\psi]: \cap_{\infty}<\varphi\psi>\{a\} . a \infty b . \supset . \cap_{\infty}<\varphi\psi>\{b\}$
PF: $[ab\varphi\psi]$: Hyp(2) $. \supset .$
3) $\infty<a> \varepsilon_{\infty} \varphi . \infty<a> \varepsilon_{\infty} \psi$. [*D3.3*, 1]
4) $\infty<b> \varepsilon_{\infty} \varphi . \infty<b> \varepsilon_{\infty} \psi$. [*T2.7*, 2, 3]
$\cap_{\infty}<\varphi\psi>\{b\}$ [*D3.3*, 4]

T3.9 $[\varphi\psi] . \mathbf{N}(\cap_{\infty}<\varphi\psi>)$ [*D1.20, T3.8*]

Now, on the basis of *T2.27*, the ontological definition of the intersection of numerical names is derivable.

T3.10 $[\Phi\varphi\psi]: \Phi \varepsilon_{\infty} \cap_{\infty}<\varphi\psi> . \equiv . \Phi \varepsilon_{\infty} \varphi . \Phi \varepsilon_{\infty} \psi . \Phi \varepsilon_{\infty} \Phi$ [*T2.27, T3.7, T3.9*]
T3.11 $[\Phi\varphi\psi]: \Phi \varepsilon_{\infty} \cap_{\infty}<\varphi\psi> . \equiv . \Phi \varepsilon_{\infty} \varphi . \Phi \varepsilon_{\infty} \psi$ [*T3.10, T2.3*]

And given this thesis, Peano's last axiom follows.

T3.12 $[Ab\varphi] \therefore 0 \varepsilon_{\infty} \varphi : [\Psi]: \Psi \varepsilon_{\infty} \varphi . \supset . \mathbf{S}<\Psi> \varepsilon_{\infty} \varphi : A \varepsilon A . \varphi\{b\} . \sim(A \varepsilon b) : \supset .$ $\varphi\{b \cup A\}$
PF: $[Ab\varphi] \therefore$ Hyp(5) $: \supset :$
6) $\mathbf{N}(\varphi)$. [*T2.2*, 1]
7) $\infty<b> \varepsilon_{\infty} \varphi$. [*T2.6*, 4, 6]
8) $\mathbf{S}<\infty<b>> \varepsilon_{\infty} \varphi$: [2, 7]
9) $[a]: \mathbf{S}<\infty<b>>\{a\} . \supset . \varphi\{a\}$: [*T2.20*, 8]
10) $A \varepsilon b \cup A$. [*T1.8*, 3]
11) $(b \cup A) - A \circ b$. [*T1.10*, 3, 5]
12) $\infty<b>\{(b \cup A) - A\}$. [*T1.6*, 11, *T1.16*]
13) $\mathbf{S}<\infty<b>>\{b \cup A\}$. [*D1.25*, 10, 12]
$\varphi\{b \cup A\}$ [9, 13]

T3.13 $[\varphi] \therefore 0 \varepsilon_{\infty} \varphi : [\Psi]: \Psi \varepsilon_{\infty} \varphi . \supset . \mathbf{S}<\Psi> \varepsilon_{\infty} \varphi : \supset :$
$[Ab]: A \varepsilon A . \varphi\{b\} . \supset . \varphi\{b \cup A\}$ [*T1.6, T1.9, T3.12*]
T3.14 $[\Phi\varphi] \therefore 0 \varepsilon_{\infty} \varphi : [\Psi]: \Psi \varepsilon_{\infty} \varphi . \supset . \mathbf{S}<\Psi> \varepsilon_{\infty} \varphi : \Phi \varepsilon_{\infty} \mathrm{Fin} : \supset . \Phi \varepsilon_{\infty} \varphi$
PF: $[\Psi\varphi] ::$ Hyp(3) $: \supset \therefore$
4) $[Ab]: A \varepsilon A . \varphi\{b\} . \supset . \varphi\{b \cup A\}$: [*T3.13*, 1, 2]
5) $\mathbf{N}(\varphi)$. [*T2.2*, 1]

6) $\mathbf{N}(\Phi).\mathbf{Q}(\Phi).$ [T2.2, 3]
7) $[\exists b].0\{b\}.\varphi\{b\}.$ [T2.2, 1]
8) $\varphi\{\Lambda\}.\therefore$ [T1.25, 7]
$[\exists a].\therefore$
9) $\Phi\{a\}.\mathrm{Fin}\{a\}:$ [T2.2, 3]
10) $[Ab]:A\varepsilon b.\varphi\{b\}.\supset.\varphi\{b\cup A\}:$ [T1.2, 4]
11) $\varphi\{a\}.\therefore$ [D1.23, 8, 9, 10]
$\Phi\varepsilon_{\infty}\varphi$ [T2.2, 9, 11, 5, 6]

T3.15 $[\Phi\varphi].\therefore 0\varepsilon_{\infty}\varphi:[\Psi]:\Psi\varepsilon_{\infty}\mathrm{Fin}.\Psi\varepsilon_{\infty}\varphi.\supset.\mathbf{S}\langle\Psi\rangle\varepsilon_{\infty}\varphi:\Phi\varepsilon_{\infty}\mathrm{Fin}:\supset.\Phi\varepsilon_{\infty}\varphi$
PF: $[\Phi\varphi].\therefore\mathrm{Hyp}(3):\supset:$
4) $\mathbf{N}(\varphi).$ [T2.2, 1]
5) $0\varepsilon_{\infty}\cap_{\infty}\langle\varphi\mathrm{Fin}\rangle:$ [T3.11, 1, T3.1]
6) $[\Psi]:\Psi\varepsilon_{\infty}\cap_{\infty}\langle\varphi\mathrm{Fin}\rangle.\supset.\mathbf{S}\langle\Psi\rangle\varepsilon_{\infty}\cap_{\infty}\langle\varphi\mathrm{Fin}\rangle:$ [T3.11, 2, T3.3]
7) $\Phi\varepsilon_{\infty}\cap_{\infty}\langle\varphi\mathrm{Fin}\rangle.$ [T3.14, 5, 6, 3]
$\Phi\varepsilon_{\infty}\varphi$ [T3.11, 7]

This thesis gives the principle of mathematical induction for finite numerical names.

From an inspection of Peano's axioms as given in terms of the numerical epsilon, the following equivalence could be expected.

T3.16 $[\Phi]:\Phi\varepsilon_{\infty}\Phi.\equiv.\mathbf{NC}(\Phi)$ [T2.2, D1.22, D1.14]

That is, cardinal numbers are just numerical individuals. Moreover, a similar correspondence obtains for inductive cardinals (*T3.28*). As preliminary theses, there are the following.

T3.17 $[a].\infty\langle a\rangle\varepsilon_{\infty}\infty\langle a\rangle$ [T2.2, D1.19, T1.16, D1.20, D1.21]
T3.18 $[a]:\mathrm{Fin}\{a\}.\equiv.\infty\langle a\rangle\varepsilon_{\infty}\mathrm{Fin}$
[T2.2, D1.19, T1.16, T1.17, T1.18, T1.22]
T3.19 $[\Phi]:\mathbf{Nn}(\Phi).\supset.\Phi\varepsilon_{\infty}\mathrm{Fin}$ [T1.37, T1.36, T3.16, T1.22, T2.20, D1.16]

Now, the protothetical definition of a numerical satisfier,

D3.4 $[\mu a]:[\exists\Phi].\Phi\varepsilon_{\infty}\Phi.\Phi\{a\}.\mu(\Phi).\equiv.\mathrm{stsf}_{\infty}\langle\mu\rangle\{a\}$

is used in conjunction with *T1.28* in order to obtain the ontological definition of that functor (*T3.25*).

T3.20 $[\mu a]:\mathrm{stsf}_{\infty}\langle\mu\rangle\{a\}.\supset.\mu(\infty\langle a\rangle)$
PF: $[\mu a]::\mathrm{Hyp}(1).\supset.\therefore$
$[\exists\Phi].\therefore$
2) $\Phi\varepsilon_{\infty}\Phi.\Phi\{a\}.\mu(\Phi):$ [D3.4, 1]
3) $[b]:\Phi\{b\}.\equiv.\infty\langle a\rangle\{b\}:$ [T2.2, 2, D1.20, D1.21, D1.19]
4) $[\mu]:\mu(\Phi).\equiv.\mu(\infty\langle a\rangle).\therefore$ [T1.11, 3]
$\mu(\infty\langle a\rangle)$ [2, 4]

T3.21 $[\mu a]:\mu(\infty\langle a\rangle).\supset.\mathrm{stsf}_{\infty}\langle\mu\rangle\{a\}$ [D3.4, T3.17, T1.16]
T3.22 $[\mu a]:\mu(\infty\langle a\rangle).\equiv.\mathrm{stsf}_{\infty}\langle\mu\rangle\{a\}$ [T3.20, T3.21]
T3.23 $[\mu ab]:\mathrm{stsf}_{\infty}\langle\mu\rangle\{a\}.a\infty b.\supset.\mathrm{stsf}_{\infty}\langle\mu\rangle\{b\}$

PF: $[\mu a b]$: Hyp(2) .⊃.
$[\exists \Phi]$.
3) $\Phi \varepsilon_{\infty} \Phi . \Phi\{a\} . \mu(\Phi)$. [*D3.4*, 1]
4) $\Phi\{b\}$. [*T2.2*, 3, *D1.20*, 2]
$\text{stsf}_{\infty}<\mu>\{b\}$ [*D3.4*, 3, 4]

T3.24 $[\mu] . \mathbf{N}(\text{stsf}_{\infty}<\mu>)$ [*D1.20*, *T3.23*]
T3.25 $[\Phi \mu]: \Phi \varepsilon_{\infty} \text{stsf}_{\infty}<\mu> .\equiv. \Phi \varepsilon_{\infty} \Phi . \mu(\Phi)$ [*T2.28*, *T3.24*, *T3.22*]

Given this ontological definition of a numerical satisfier of a predicate (compare *D1.8*), the desired equivalence is derivable.

T3.26 $[\Phi \mu]: \Phi \varepsilon_{\infty} \text{Fin} . \mu(0) : [\Psi]: \mu(\Psi) .\supset. \mu(\mathbf{S}<\Psi>) :\supset. \mu(\Phi)$
PF: $[\Phi \mu]$.˙. Hyp(3) :⊃:
4) $0 \varepsilon_{\infty} \text{stsf}_{\infty} <\mu>$: [*T3.25*, 2, *T3.1*, *T2.3*]
5) $[\Psi]: \Psi \varepsilon_{\infty} \text{Fin} . \Psi \varepsilon_{\infty} \text{stsf}_{\infty} <\mu> .\supset. \mathbf{S}<\Psi> \varepsilon_{\infty} \text{stsf}_{\infty}<\mu>$:
[*T3.25*, *T3.21*, *T2.3*, 3]
$\mu(\Phi)$ [*T3.15*, 1, 4, 5, *T3.25*]

T3.27 $[\Phi]: \Phi \varepsilon_{\infty} \text{Fin} .\supset. \mathbf{Nn}(\Phi)$ [*T3.26*, *D1.26*]
T3.28 $[\Phi]: \Phi \varepsilon_{\infty} \text{Fin} .\equiv. \mathbf{Nn}(\Phi)$ [*T3.27*, *T3.19*]

That is, natural numbers are just finite numerical individuals. The equivalence of the axiom of infinity and one of Peano's axioms is now shown.

T3.29 $[a]$.˙. $[\Phi]: \Phi \varepsilon_{\infty} \text{Fin} .\supset. \mathbf{S}<\Phi> \varepsilon_{\infty} \text{Fin} : \text{Fin}\{a\} : [A]: A \varepsilon A .\supset. A \varepsilon a :\supset.$
$[\exists A] . A \varepsilon A . \sim(A \varepsilon a)$
PF: $[a]$.˙. Hyp(3) :⊃:
4) $\infty<a> \varepsilon_{\infty} \text{Fin}$. [*T3.18*, 2]
5) $\mathbf{S}<\infty<a>> \varepsilon_{\infty} \text{Fin}$: [1, 4]
$[\exists b]$:
6) $\mathbf{S}<\infty<a>>\{b\} . \text{Fin}\{b\}$. [*T2.2*, 5]
$[\exists B]$.
7) $B \varepsilon b . \infty<a>\{b - B\}$. [*D1.25*, 6]
8) $b - B \subset a$. [*D1.9*, *D1.12*, *T1.2*, 3]
9) $a \circ b - B$. [*T1.23*, 8, 2, *D1.19*, 7]
10) $B \varepsilon B$. [*T1.2*, 7]
11) $B \varepsilon a$. [10, 3]
12) $\sim(B \varepsilon B)$: [*T1.5*, 9, *D1.12*, 11]
$[\exists A] . A \varepsilon A . \sim(A \varepsilon a)$ [12, 10]

T3.30 $[\Phi]: \Phi \varepsilon_{\infty} \text{Fin} .\supset. \mathbf{S}<\Phi> \varepsilon_{\infty} \text{Fin} :\supset: [a]: \text{Fin}\{a\} .\supset. [\exists A] . A \varepsilon A . \sim(A \varepsilon a)$
[*T3.29*]
T3.31 $[\Phi]: \Phi \varepsilon_{\infty} \text{Fin} .\supset. \mathbf{S}<\Phi> \varepsilon_{\infty} \text{Fin} :\equiv. \text{Ax Infin}$ [*T3.30*, *T3.3*, Ax Infin]

Naturally, theses analogous to the ontological definitions of arithmetical operations could be given and the properties of the operations are all derivable on the above basis: the requisite definitions for the operations having already been given in the introductory section. For example, in respect to addition, one has uniqueness, closure, and the recursive properties of the operation,

T3.32 $[\Phi\varphi\psi] \therefore \Phi\varepsilon_{\infty}\mathrm{Fin} .\supset: \Phi =_{\infty} \varphi + \psi .\equiv. \mathrm{Sm}(\Phi\varphi\psi)$
T3.33 $[\Phi\Psi]: \Phi\varepsilon_{\infty}\mathrm{Fin} . \Psi\varepsilon_{\infty}\mathrm{Fin} .\supset. [\exists X]. X\varepsilon_{\infty}\mathrm{Fin} . X =_{\infty} \Phi + \Psi$
T3.34 $[\Phi]: \Phi\varepsilon_{\infty}\mathrm{Fin} .\supset. \Phi =_{\infty} \Phi + 0$
T3.35 $[\Phi\Psi]: \Phi\varepsilon_{\infty}\mathrm{Fin} . \Psi\varepsilon_{\infty}\mathrm{Fin} .\supset. \Phi + \mathbf{S}\langle\Psi\rangle =_{\infty} \mathbf{S}\langle\Phi + \Psi\rangle$

and similarly for any other arithmetical operation which can be defined recursively.

4. *Concluding remarks* In section 1, as in section 3, an analysis of the cardinality of (finite) names is given. These sections differ in style: the former treats numerals as predicates, the latter as names. Consider the following.

T1.32 **Nn**(0)
T3.1 $0\varepsilon_{\infty}\mathrm{Fin}$

Under the intended interpretation both of these propositions state that zero is a natural number. Naturally the numeral in these propositions belongs to but one semantical category, for the numeral is a unique constant (of the category of proposition forming functors for one name argument). But the first proposition corresponds in a simple way to,

***120.12.** $\vdash . 0 \epsilon \mathrm{NC}$ induct

of [6], while the second proposition does not.

For the authors of [6], classes are but "fictitious objects" and their use of an epsilon and class abstractor is but a notational device for speaking extensionally about predicates. But the logic of section 1 is completely extensional, and this is achieved without the introduction of "fictitious objects" via notational devices. Section 1 gives, then, a development of cardinality closely akin to [6], but without the fictions of that work.

On the other hand, the numerals of section 3 can clearly be considered to be names. Section 2, in providing an internal ontological model for the numerical epsilon, justifies this claim. Recall that in section 2 analogues of the ontological axiom, extensional theses and definitional theses of the primitive epsilon were derived for the numerical epsilon. This guarantees that an analogue of each thesis derivable for the primitive epsilon is derivable for the numerical epsilon. In this exposition, unlike [6], there is no question of numerals being "really" predicative and only "conventionally" nominative—since there is an internal ontological model for the numerical epsilon, its arguments are by analogy nominal.

Thus, if one wishes to construe numerals as predicates there is the exposition of section 1 without the disadvantages of [3], while if one wishes to construe numerals as names, there is the exposition of section 3. In the latter case there is, as has been seen, formal justification for considering numerals as names, while in the former case, one can at best consider them fictitious names—they are indeed only predicates.

Though an analysis of the cardinality of names has been given, what is lacking in this exposition is an analysis of cardinality in general. There is, for instance, a definition of a constant number (zero) for names, but nowhere has the definition of the constant number zero for proposition forming functors of one name argument (or two, or three name arguments, etc.) been given. Only the cardinality for the primitive semantical category of names has been investigated, while the cardinality of the non-primitive categories remains to be discussed.

Consider the following definition.

$$[\Phi\varphi] .\therefore [\exists a] . \Phi\{a\} . \varphi\{a\} : [ab] : \Phi\{a\} . \Phi\{b\} . \supset . a \circ b :\equiv . \Phi \varepsilon \varphi$$

Here a higher (ontological) epsilon is defined. With this definition slight adjustments in the exposition of section 2 would establish the existence of an internal ontological model for this epsilon. Higher epsilons of this type are well known, and it can be shown, in fact that a higher epsilon exists for each and every semantical category definable in ontology.

Since there is a higher epsilon for each category of ontology, the exposition for the cardinality of names which has been given serves as an exposition for the cardinality of any particular category of ontology. The cardinality of any given category is to be developed in exactly the manner in which the cardinality of names has been given, employing the higher epsilon peculiar to the category in question in place of the primitive epsilon.

It is important to note that the axiom of infinity is reproducible at higher levels. It is already clear that the ontological directives and axiom are available for any category, but the axiom of infinity must also be available. However, it can be shown that the axiom of infinity for proposition forming functors of one name argument is implied by the axiom of infinity for names, and in general, that the axiom of infinity for any non-primitive semantical category ultimately definable in terms of the primitive category of names is implied by the axiom of infinity for names.

At this point a precise exposition of arithmetic for each category of ontology has been attained. No doubt this is a far cry from the standpoint of ordinary arithmetic which ignores differences of type in numbers, but on the basis of these different types of arithmetic one can safely reach the standpoint of ordinary arithmetic.

The only remaining requirement in the quest for arithmetic is the ability to handle arithmetical operations whose arguments are non-homogeneous. The possibility already exists of producing, for instance, the sum of homogeneous cardinals. But consider the problem of adding the cardinality of a name to the cardinality of some other semantical category. Since the operation of addition has homogeneous arguments, it is necessary to secure elements of the same category in order to perform the operation. But this can always be done in a satisfactory manner by producing an element of the higher category equinumerous with the element of the lower category.

Of course, this requires expanding the notion of equinumerosity so that it countenances non-homogeneous arguments. But the higher epsilons afford a precise basis for such an extension of the notion. It can be seen that it is possible in the presence of the higher epsilons, to perform arithmetical operations, now, for arguments from any semantical category whatsoever. One simply finds a representative from the higher category for the element of the lower category and performs the operation in the arithmetic of the higher level.

And it is now clear in what sense the standpoint of ordinary arithmetic which ignores the difference in type among numbers can be attained in ontology. A sufficiently high semantical category is selected and all work is carried on in the arithmetic of that level. It is not that numbers are ambiguous as to type, as the authors of [6] would have it, but that any given arithmetical problem can be handled in a single sufficiently high semantical category.

At this point, the work of deriving arithmetic is finished. Indeed, the above discussion closely parallels section ***126** of [6] which discusses "typically indefinite inductive cardinals". The advantage to this exposition lies in the fact that the standpoint of ordinary arithmetic has been achieved without any of the ambiguities latent in the development of [6]. By relying on the theory of higher epsilons, the effect of employing typically ambiguous symbolism as in [6] can be obtained in ontology without however exhibiting theorems which are ambiguous.

Finally, having supplied an ontological model for Peano's arithmetic in ontology (extended by an axiom of infinity), the incompleteness of this system will follow, if its directives are recursive. In this respect, the numerical epsilon proves very useful. The directives for ontology were given by Leśniewski [3] in a list of terminological explanations which are developed by employing his mereological concepts. Now, the numerical epsilon provides an efficient means of modeling Leśniewski's original terminological explanations in Peano's arithmetic as given in section 3, thus showing the applicability of Gödel's incompleteness result to ontology. The details of this work are left for another paper.

REFERENCES

[1] Canty, John Thomas, *Leśniewski's ontology and Gödel's incompleteness theorem*, Ph.D. Thesis, June, 1967, University of Notre Dame, Notre Dame, Indiana.

[2] Lejewski, Czesław, "On Leśniewski's ontology," *Ratio*, v. 1 (1957), pp. 150-176.

[3] Leśniewski, Stanisław, "Grundzüge eines neuen Systems der Grundlagen der Mathematik," *Fundamenta Mathematicae*, v. 14 (1929), pp. 1-81. "Über die Grundlagen der Ontologie," *Comptes rendus des seances de la Société des sciences et des lettres de Varsovie, Classe III*, v. 23 (1930), pp. 111-132.

[4] Słupecki, Jerzy, "St. Leśniewski's protothetics," *Studia Logica*, v. 1 (1953), pp. 44-112 and "St. Leśniewski's calculus of names," v. 3 (1955), pp. 7-71.

[5] Sobociński, Bolesław, "On the single axioms of protothetic," *Notre Dame Journal of Formal Logic,* v. 1 (1960), pp. 52-73 and v. 2 (1961), pp. 111-126, 129-148 and "On successive simplifications of the axiomatizations of Leśniewski's ontology," (in translation) *Polish Logic,* Storrs McCall (Editor), Oxford University Press, 1967, pp. 188-200.

[6] Whitehead, Alfred N., and Bertrand Russell, *Principia mathematica,* Vol. I-III, (second edition), The University Press, Cambridge, 1963.

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