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ON SYMBOLIZING SINGULARY S5 FUNCTIONS

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A normal form representation for S5 functions has been given by Massey in [4]. Consider the following schema

$$T(p_1, \ldots, p_n, F_1(p_1, \ldots, p_n), \ldots, F_k(p_1, \ldots, p_n))$$

where $k = m - n$ and $m = 2^n + n - 1$. Massey showed that there exists a k-tuple $\langle F_1, \ldots, F_k \rangle$ of n-ary S5 functions (in fact, several such k-tuples) such that as T runs through the m-ary truth functions, the above expression generates all the n-ary S5 functions.

This result suggests an interesting symbolism for the singulary S5 functions. According to the above there are 16 such functions and so, symbols for binary truth functions are obvious candidates for symbolizing these singularies. In what follows Leśniewski's symbols for binary truth functions will be employed for singulary S5 functions and Łukasiewicz's symbols for truth functions will be retained in their usual role. In particular, C, E, and N are used for conditionals, biconditionals and negations (see [5]).

Selecting ♀- as the symbol for $\otimes_1$ of [3] and using this functor as the F_1 in the normal form schema given above, we have the following 16 singulary S5 functions (Massey's notation and terminology of [3] is given in the last column).

Vp♀-(p)	=	C♀-$(p)Cpp$	=	-⌽-(p)	=	($?p$, constant truth)
Ap♀-(p)	=	CNp♀-(p)	=	-♀-(p)	=	($\Diamond p$, possibility)
Lp♀-(p)	=	C♀-$(p)p$	=	-⌽(p)	=	($\otimes_3 p$, hybrid)
Dp♀-(p)	=	CpN♀-(p)	=	-♁-(p)	=	($\boxminus p$, non-necessity)
Cp♀-(p)	=	Cp♀-(p)	=	⌽-(p)	=	($\otimes_5 p$, hybrid)
Gp♀-(p)	=	p	=	-♀(p)	=	($?p$, affirmation)
Bp♀-(p)	=	♀-(p)	=	♀-(p)	=	($\otimes_1 p$, hybrid)
Ep♀-(p)	=	Ep♀-(p)	=	⌽(p)	=	($\otimes_7 p$, determinateness)
Rp♀-(p)	=	NEp♀-(p)	=	-o-(p)	=	($\otimes_8 p$, contingency)
Pp♀-(p)	=	N♀-(p)	=	-♁(p)	=	($\otimes_2 p$, hybrid)
Jp♀-(p)	=	Np	=	♁-(p)	=	($\sim p$, negation)

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Hp ϙ-(p) = NCp ϙ-(p) = -o (p) = ($\otimes_6 p$, hybrid)
Kp ϙ-(p) = NDp ϙ-(p) = ϙ̲ (p) = ($\Box p$, necessity)
Tp ϙ-(p) = NLp ϙ-(p) = o- (p) = ($\otimes_4 p$, hybrid)
Sp ϙ-(p) = NAp ϙ-(p) = ϭ (p) = (⌽ p, impossibility)
Fp ϙ-(p) = NVp ϙ-(p) = o (p) = (? p, constant falsehood)

It should be noted that as Massey points out in [4] there are four singulary S5 functions which can serve as the F_1 in the normal form representation of singulary S5 functions. Thus, the selection of Leśniewskian symbols for the S5 functions is dependent upon which of the four is used in the representation. The selection of symbols given here is guided by the following consideration. Recalling the truth tables for binary truth functions

p q	Kpq	Apq	Bpq	Epq
t t	Ktt = t	Att = t	Btt = t	Ett = t
f f	Kff = f	Aff = f	Bff = f	Eff = t
t f	Ktf = f	Atf = t	Btf = f	Etf = f
f t	Kft = f	Aft = t	Bft = t	Eft = f

consider the following complete truth tabular analyses.

p	Kp ϙ-(p)	Ap ϙ-(p)	Bp ϙ-(p)	Ep ϙ-(p)
t	Ktt = t	Att = t	Btt = t	Ett = t
f	Kff = f	Aff = f	Bff = f	Eff = t
t	Ktf = f	Atf = t	Btf = f	Etf = f
f	Kft = f	Aft = t	Bft = t	Eft = f

Thus, since the Leśniewskian symbolism for K, A, B and E is ϙ, -ϙ-, ϙ-, and ⌽ respectively, these symbols are selected for the singulary S5 functions whose normal forms are Kp ϙ-(p), Ap ϙ-(p), Bp ϙ-(p) and Ep ϙ-(p) respectively. But regardless of which of the four possible ways is used to symbolize the singulary S5 functions, certain advantages of using Leśniewskian symbolism can be noted.

First of all, the symbol for the functor indicates the intended interpretation of the functor. The 16 symbols are obtained by placing (or not placing) a stroke in one of four positions from a point of origin. Relative to the above selection of symbols, if one reads the top position as "always false", the bottom as "always true", and the left and right as "true, but sometimes false" and "false, but sometimes true", then the placement of the stroke indicates the value of the function as "true" under the given conditions for its argument. For instance, ϙ(p) is only true when its argument is always true (necessity), while -ϙ-(p) is only false when its argument is always false (possibility).

Secondly, Leśniewski's iconic symbolism is designed to indicate syntactic relations among the functions symbolized. For example, necessity and non-necessity are symbolized by ϙ and -ϭ- respectively, that is, by signs which are complementary in respect to the positioning of strokes.

Further, the conjunction and disjunction of these functions are the constantly false and constantly true functions respectively while the symbols of these latter functions are the intersection and union of the positioning of strokes in them, that is, ∘ and -ϕ- (see Luschei, p. 292 ff. of [2] for a fuller discussion on combining this symbolism).

Symbolism for n-ary S5 functions for $n > 1$ could be developed in an analogous way, but it hardly seems worthwhile: for the binaries alone one needs a symbolism adequate for the quintinary truth functions. However, since there is a correspondence between the fully modal n-ary S5 functions and the proper quantifiers of n arguments investigated by Borkowski in [1] it might be interesting to apply the ideas of symbolization given here to Borkowski's proper quantifiers.

REFERENCES

[1] Borkowski, L., "On proper quantifiers, I and II," *Studia Logica*, v. 8 (1958), pp. 65-130 and v. 10 (1960), pp. 7-28 (Correction to the above given in v. 15 (1964), p. 272).

[2] Luschei, E. C., *The Logical Systems of Leśniewski*, North-Holland Publishing Co., Amsterdam, 1962.

[3] Massey, G. J., "The theory of truth tabular connectives, both truth functional and modal," *The Journal of Symbolic Logic*, v. 31 (1966), pp. 593-408.

[4] Massey, G. J., "Normal form generation of S5 functions via truth functions," *Notre Dame Journal of Formal Logic*, v. 9 (1968), pp. 81-85.

[5] Scharle, T. W., "A diagram for functors of two-valued propositional calculus," *Notre Dame Journal of Formal Logic*, v. 3 (1962), pp. 243-255.

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