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ON EXPLANATION OF NUMBER PROGRESSION

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In a recent article[1] Benacerraf asserts that the less-than relation R must be recursive in order for the set of elements on which R is defined to constitute a set of numbers. Benacerraf thinks that this is an essential and independent requirement for a set of elements to be a set of numbers. He rejects Quine's view that there is only one condition upon all acceptable explications of numbers, namely, a given set of elements must be a progression in the sense that it is an infinite series each of whose members has only finitely many precursors.[2] Now the question is whether the condition that R must be recursive is an independent one for the characterization of natural numbers. In this paper I shall show that a little closer examination of Quine's view should dispel the doubt that the condition in question is not an independent one.

For our purpose, it suffices to show that the less-than relation R can be defined in terms of Quine's characterization of natural numbers as a progression, and that the recursiveness of the less-than relation is an inherent feature of the progression by definition. Quine defines[3] the class of natural numbers N as follows:

$$\mathrm{N} =_{df} \{x\colon (z)\ (x \epsilon z . \breve{\mathrm{S}}``z \subseteq z . \supset . 0 \epsilon z)\}$$

In this sense to be a natural number is to be a member of all classes z fulfilling the initial condition "$0 \epsilon z$" and the closure condition "$\breve{\mathrm{S}}``z \epsilon z$" (precursors of $z$ are in $z$). A class $z$ which fulfills the initial condition "$0 \epsilon z$" and the closure condition "$\breve{\mathrm{S}}``z \epsilon z$" is in fact a progression each of whose members has only finitely many precursors. Of course, the definition does not restrict the size of a natural number, for a natural number can be infinitely large if there is a z which is infinitely large, even though an axiom of infinity for z is not needed to make sense of the definition.

To see that a progression is a class z, let the progression be:

$$A = a_1, a_2, a_3, \ldots, a_n, \ldots$$

Since each a_i has only finitely many precursors, each a_i must have 0 as its precursor, and has all the elements which are the precursors of each of its precursors as precursors. Thus A is indeed a class z.

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Now Quine defines the less-than relation on a number progression in terms of the equal to or less-than relation between members of a class z.

$$y \leqq x \equiv_{df} (z)\,(x \in z . \breve{\mathrm{S}}``z \subseteq z . \supset . y \in z)$$

$$y < x \equiv_{df} \mathrm{S}`y \leqq x$$

Thus to say that y is equal to or less than x is to say that both y and x belong to a progression and that y is either x or a precursor of x. It follows too that if the successor of y is either x or a precursor of x, y must be less than x. Thus it follows the definition of $y < x$ in terms of $\mathrm{S}`y \leqq x$.

Two things are clear from the above account. First, since the class N of natural numbers forms a progression, it can be defined as the class of x such that $0 \leqq x$ and 0 is the least element of the progression. Second, the less-than relation $<$ can be seen to be an inherent feature or property of the progression of natural numbers. Now $<$ is recursive in the sense that, given any two elements x and y in the progression, we can know which element of the two is greater by finite calculation. This follows from the definition of the progression and the definition of $<$. Since x and y belong to the progression, each of them has finitely many precursors, we are therefore able to tell how many precursors each has by a finite calculation. Since we can further tell whether x is a precursor, or a precursor of a precursor, or a precursor of a precursor of a precursor and so on, of y by the definition of $<$, thus, to conclude, to say that the natural numbers form a progression P is to implicitly claim that $<$ is definable on P and $<$ is recursive on P.[4] In fact a system of objects which forms a progression, with the successor or precursor functions suitably defined on the progression always have the property that $<$ is definable on the progression and $<$ is recursive on it.

Benacerraf attempts to construct a progression which is not recursive in order to show that recursiveness of progression with respect to $<$ is an independent requirement for an adequate account of natural numbers. But his attempt seems to end in failure. He asks us to choose a progression:

$$A = a_1, a_2, a_3, \ldots, a_n, \ldots$$

obtained as follows. Divide the positive integers into two sequences B and C, within each sequence letting the elements come in order of magnitude. Let B be the sequence of Gödel numbers of valid formulas of quantification, and let C be the sequence of positive integers which are not numbers of valid formulas of quantification. Then A is formed from B and C according to the specification $a_{2n-1} = b_n$ and $a_{2n} = c_n$. Benacerraf concludes that A is a progression but a non-recursive one.

Now I think that this way of producing a non-recursive progression is mistaken for the following reason. It can be shown that the progression in question is either not a progression or a recursive progression. A is clearly in fact not a progression because it is not guaranteed that each of its members has finitely many precursors. Since some elements in C, by definition, may have infinitely many precursors, C cannot be a progression, and therefore A cannot be a progression. On the other hand, Benacerraf apparently thinks that C is a progression, that is, elements can be generated each of which will be known to have finitely many precursors. If that were the case, one should not conclude that A is a progression but not a recursive one.

Finally, we come to the question whether in explicating the concept of natural number it is necessary to explain the notion of cardinality or cardinal number. Again Benacerraf criticizes Quine for rejecting the explanation of the notion of cardinality as a part of an acceptable explication of the notion of number. Now the notion of cardinality of a class of objects involves the notion of one-to-one correspondence between the class of objects in question and the elements in the progression of natural numbers 1, 2, 3, ... In other words, one might say that to know the cardinality of a class of objects is to know how to *count* the objects. In fact, this know-how is an ability to do the counting in an intuitive sense. What Benacerraf maintains is that an account of numbers must include an account of the use of number words for "transitive counting"--correlating members of a given class of objects with elements of the number progression--for the following reason. "One cannot tell what number a particular expression represents without being given the sequence of which it forms a part. It will then be from its place in that sequence--that is, from its relation to other members of the sequence, *and from the rule governing the use of the sequence in counting*--that it will derive its individuality. It is for this last reason that I urged, contra Quine, that the account of cardinality must explicitly be included in the account of number."[5] That is, to learn what numbers are is to learn how to use number words in counting and in measuring multiplicity. Therefore, in order to explicate the notion of number, it is necessary to explain the notion of counting.

Now there are at least three objections to the above argument. First, it might be quite true that to learn what numbers are is to learn how to use number words in counting or in measuring multiplicity. In fact one might say that our intuitive notion of numbers just consists in that. But the question of explicating the notion of numbers is to explain how counting is possible and how the use of number words in measuring multiplicity is possible. This question is to be answered by showing what structure a number progression exhibits independently of the use of number words for measuring multiplicity. But to say this is not to say that the account of numbers by way of exhibiting the structure of the number progression will not enable us to do counting. On the contrary, an explanation of numbers should meet the condition that the use of the number words for counting is guaranteed on the given account.[6]

Second, Benacerraf has assumed that number words do not refer to anything and therefore their meanings depend upon rules governing the use of the number sequence (indeed sequence of *number words*) in counting. But this is not a justified assumption. Because from the premises that numbers do not have their individuality independently of their relations in the number progression, it need not follow that number words do not refer to anything. Instead, in so far as a set of sets exhibits the relations of a number progression, there is no *prima facie* implausibility that a number progression can be identified with the set of sets and individual numbers with individual sets in that set on the basis of some criteria of identity formulated in an overall theory.[7] Thus a theoretical account of numbers need not explicitly include as a part a prior understanding of the rules governing the use of the number words in counting, for these rules must be already presupposed and therefore should be perfectly constructible from the explication of the notion of numbers. Finally, the notion of cardinality itself involves and presupposes the notion of natural numbers. Therefore an explanation of cardinality need not be said to completely explain the notion of natural numbers since it would be question begging.

In order to dispel any doubts that Quine's condition of being a progression constitutes an adequate account of natural numbers from which the use of number words in counting or measuring multiplicity can be explained, we might consider how the statement "There are n objects x such that Fx" is to be explained. Now there are two ways of explaining this. One way does not involve reference to numbers as such, but only in terms of the *numerically definite* quantifiers[8]. Thus we can explain the statement "There are n objects x such that Fx" in the following fashion:

$$\text{`}(\exists_0 x)Fx\text{' for: `}\sim(\exists x)Fx\text{'}$$

Then we can explain $(\exists_1 x)Fx$ in terms of $(\exists_0 x)Fx$ by defining

$$\text{`}(\exists_1 x)Fx\text{' for: `}(\exists x)\ (Fx\cdot(\exists_0 y)\ (Fy\cdot y\neq x))\text{'}$$

and then define:

$$\text{`}(\exists_n x)Fx\text{' for: `}(\exists x)\ (Fx\cdot(\exists_{n-1}y)(Fy\cdot y\neq x))\text{'}$$

To expand this, we have:

$$(\exists x_1)(\exists x_2)\ldots(\exists x_n)(Fx_1\cdot Fx_2\ldots Fx_n\cdot x_1\neq x_2\cdot x_1\neq x_3\ldots$$
$$x_{n-1}\neq x_n\cdot(y)(Fy\supset . y=x_1 \vee y=x_2 \vee\ldots\vee y=x_n)$$

This shows that the question as to how many x are such that Fx can be answered by exhibiting all the distinguishable elements in the class of x's. This exhibition is representable in terms of the theory of quantification and the theory of identity.

Now if one still presses to know how many x are such that Fx in terms of *numbers*, we must turn to the second way of explaining the statement "There are n objects x such that Fx". This is to explain this statement as if it is equivalent to the statement that "The number of objects x such that Fx is n." Since numbers form a progression, this makes it possible for us to correlate elements of the number progression with members in the class of objects x such that Fx. A correlation is a function. Thus to say that the number of x such that Fx is n can be explained as saying that there exists a function which assigns exhaustively members of class of objects x to those of the number progression, no two to the same. Following Quine, we may define "α has no more members than β"

$$\text{`}\alpha \leq \beta\text{'} \text{ for: } \text{`}(\exists x)(\text{Func}x . \alpha \subseteq x\text{``}\beta)\text{'}$$

In terms of this terminology, we may construe "The number of the class of objects x such that Fx is n" as saying the same thing: "The class of objects x such that Fx and the number n are alike in size."[9] This can be expressed as follows:[10]

$$\{x : Fx\} \simeq \{y : y < n\}$$

Now clearly $y{:}y < n$ can be explained in terms of a member in the number progression which has n *precursors* and this member is uniquely determined in the number progression because the less-than relations $<$ is uniquely defined in the progression. From this explanation of the meaning of the statement "There are n objects x such that Fx", we see that the account of numbers as a progression in no logical sense depends upon the use of number words for counting or measuring multiplicity, but rather provides a basis for the possibility of such use when an one-to-one correspondence is introduced into the theory of relations.

NOTES

1. Paul Benacerraf, "What Number Could Not Be", *Philosophical Review*, LXXIV, 1, January, 1965, 47-73.

2. Quine maintains this in his *Word and Object*, MIT and John Wiley, 1960. Benacerraf has quoted Quine from this source in his article "What Numbers Could Not Be." Quine has also reaffirmed this view in his *Set Theory and its Logic*, Harvard University Press, 1963. In that book Quine says, "Any objects will serve as numbers so long as the arithmetical operations are defined for them and the laws of arithmetic are preserved. It has sometimes been urged that we account for pure arithmetic, we must also account for the application of number in the measurement of multiplicity. But this position, in so far as it is thought of as contrary to the other, is wrong. We have seen how to define not only the arithmetical operations but also the *Anzahlbegriff*, "α has x members," without having yet decided what numbers are." (p. 81).

3. Cf. Quine, *Set Theory and its Logic*, p. 75. Note Quine's definition of natural numbers cited above differs from earlier definitions of Frege and Russell in this: whereas those earlier definitions take N to be $\{x{:}(z)\ (0 \in z \cdot \text{S``}z \subseteq z \cdot \supset \cdot x \in z)\}$ where S is the successor function such that for each number x, S'x is $x + 1$. This definition requires infinite classes for characterizing the class of natural numbers. The cited definition however does not depend on an axiom of infinity for z. For any x to be a natural number, z need not contain more than x positive members and 0. Furthermore, the law of mathematical induction can be justified on the basis of this definition.

4. An intuitive notion of successor or precursor of course is presupposed in this account.

5. Op. Cit., 72. Italics Benacerraf's.

6. It is curious that Benacerraf considers intransitive counting, that is, counting without referring to objects, as prior to transitive counting, that is, counting in reference to objects other than numbers. An explication of the notion of numbers is acceptable if it can explain how intransitive counting is possible, for it will necessarily explain how transitive counting is possible. Quine's condition seems quite apt to satisfy this condition. See above.

7. See discussions in my paper "Referential Involvements of Numbers", forthcoming.

8. See Quine's, *Methods of Logic,* Henry Holt and Co., New York, 1955, 231.

9. Cf. Charles Parson's "Frege's Theory of Numbers", *Philosophy in America,* edited by Max Black, Cornell University Press, 1965, 183, regarding Frege's definition of *Gleichzahligkeit* or "numerical equivalence."

10. Cf. Quine's *Set Theory and its Logic,* 78.

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