Notre Dame Journal of Formal Logic
Volume XVI, Number 3, July 1975
NDJFAM

A HENKIN-STYLE COMPLETENESS PROOF FOR THE PURE IMPLICATIONAL CALCULUS

GEORGE F. SCHUMM

Pollock has shown in [1] that Henkin-style completeness proofs can be obtained for deductive theories lacking negation, provided that disjunction is available. In this note, I show how to construct such proofs for implicational calculi without recourse to the special properties of disjunction exploited by Pollock. I shall run the argument through only for $\mathbf{PC_I}$, the pure implicational calculus, but the proof is easily adapted for richer theories as well.

For the sake of definiteness, we suppose $\mathbf{PC_I}$ to have

A1. $A \supset (B \supset A)$
A2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3. $((A \supset B) \supset A) \supset A$

as axiom-schemes and *modus ponens* as its only rule of inference. The relation '$\vdash$' of deducibility for $\mathbf{PC_I}$ is defined in the usual fashion.

Definition 1 A set Γ of formulas is *consistent* if $\Gamma \nvdash A$ for some formula A.

Definition 2 A set Γ of formulas is *maximal consistent* if

(1) Γ is consistent,
(2) $\Gamma \cup \{A\}$ is consistent, then $A \in \Gamma$.

We can now establish a familiar batch of theorems, the proofs of the first seven being straightforward and left to the reader.

Theorem 1 *If $A \in \Gamma$ or A is an axiom, then $\Gamma \vdash A$.*

Theorem 2 *If $\Gamma \vdash A \supset B$ and $\Gamma \vdash A$, then $\Gamma \vdash B$.*

Theorem 3 *If $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash A \supset B$.*

Proof: As usual, using A1 and A2.

Theorem 4 *If $\Gamma \vdash A$, then $\Gamma \cup \Delta \vdash A$.*

Theorem 5 *If $\Gamma \vdash A$, then $\Delta \vdash A$ for some finite subset Δ of Γ.*

Received June 28, 1973

Theorem 6 *If Γ is maximal consistent, then $A \supset B \in \Gamma$ iff $A \notin \Gamma$ or $B \in \Gamma$.*

Theorem 7 *If Γ is maximal consistent, then there is a valuation V such that for each formula A, $V(A) = T$ iff $A \in \Gamma$.*

Proof: As usual, using Theorem 6.

Theorem 8 *If $\Gamma \nvdash A$, then there is a maximal consistent extension Θ of Γ which does not contain A.*

Proof: Suppose $\Gamma \nvdash A$ and let $A_1, A_2, \ldots$ be some fixed enumeration of all formulas. Define Θ inductively as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{i+1} = \Gamma_i \cup \{A \supset A_i\}$$

$$\Delta_0 = \bigcup_{i \in \omega} \Gamma_i$$

$$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{A_i\} \text{ if } \Delta_i \cup \{A_i\} \text{ is consistent} \\ \Delta_i \text{ otherwise} \end{cases}$$

$$\Theta = \bigcup_{i \in \omega} \Delta_i$$

With an eye to showing that Θ has the desired properties, we record the following two lemmas.

Lemma 1 *$\Gamma_i \nvdash A$ for all $i \in \omega$.*

Proof: By assumption, $\Gamma_0 \nvdash A$. Now suppose $\Gamma_{i+1} \vdash A$. Then $\Gamma_i \cup \{A \supset A_i\} \vdash A$, and hence $\Gamma_i \vdash (A \supset A_i) \supset A$. But $\Gamma_i \vdash ((A \supset A_i) \supset A) \supset A$ since $((A \supset A_i) \supset A) \supset A$ is an axiom, and so $\Gamma_i \vdash A$. Hence, if $\Gamma_i \nvdash A$, then $\Gamma_{i+1} \nvdash A$. Therefore, by induction, $\Gamma_i \nvdash A$ for all $i \in \omega$.

Lemma 2 *Δ_i is consistent for all $i \in \omega$.*

Proof: Suppose Δ_0 were inconsistent. Then $\Delta_0 \vdash A$, whence it follows from Theorems 4 and 5 that $\Gamma_i \vdash A$ for some $i \in \omega$ contrary to Lemma 1. Hence, Δ_0 is consistent. But if Δ_i is consistent, then by definition Δ_{i+1} is consistent. Therefore, by induction, Δ_i is consistent for all $i \in \omega$.

Returning now to the proof of Theorem 8, we suppose for *reductio* that Θ is inconsistent or $A \in \Theta$. In either case, we know that $\Theta \vdash A$, whence it follows from Theorems 4 and 5 that $\Delta_i \vdash A$ for some $i \in \omega$. But $A \supset B \in \Delta_i$ for every formula B, and so $\Delta_i \vdash A \supset B$. Hence, $\Delta_i \vdash B$. But then Δ_i is inconsistent contrary to Lemma 2. Hence, Θ is consistent and $A \notin \Theta$. Finally, if $\Theta \cup \{A_i\}$ is consistent, then $\Delta_i \cup \{A_i\}$ is consistent, whence it follows that $A_i \in \Delta_{i+1}$ and hence that $A_i \in \Theta$. We have thus determined that Θ is a maximal consistent extension of Γ which does not contain A, and this completes the proof of the theorem.

Strong semantical completeness for **PC$_\mathbf{I}$** follows by the familiar Henkin-style argument: Suppose $\Gamma \nvdash A$. Then, by Theorems 7 and 8, there is a valuation which simultaneously satisfies Γ but does not satisfy A, and so Γ does not semantically imply A.

REFERENCE

[1] Pollock, John L., "Henkin style completeness proofs in theories lacking negation," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 509-511.

Ohio State University
Columbus, Ohio