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COMPUTABILITY ON FINITE LINEAR CONFIGURATIONS

THOMAS H. PAYNE

In [1] H. Friedman posed the following problem: "... to fill in the blank in: Turing's operations are to finite linear configurations as are ________ to arbitrary finite configurations." For these purposes we make the following assumptions:

(1) The arbitrary finite configurations of members of $\mathcal{S}$ are the elements of the closure $\mathcal{S}^{\#}$ of $\mathcal{S}$ under finite set formation, i.e., $\mathcal{S}^{\#} = \bigcap\{\mathcal{U}: \mathcal{S} \subset \mathcal{U}$ and $\{x_1, \ldots, x_n\} \in \mathcal{U}$ whenever $x_1, \ldots, x_n \in \mathcal{U}\}$.
(2) The finite linear configurations of members of $\mathcal{S}$ are the elements of the closure $\mathcal{S}^*$ of $\mathcal{S}$ under formation of ordered pairs, i.e., $\mathcal{S}^* = \bigcap\{\mathcal{U}: \mathcal{S} \subset \mathcal{U}$ and $\langle x, y\rangle \in \mathcal{U}$ whenever $x, y \in \mathcal{U}\}$.

Such mathematical renderings of abstract concepts, e.g., "configuration" and "computability," must remain theses until such time as one accepts as evident enough mathematical properties of the notion involved to prove some sort of a characterization theorem on the basis of those properties. However, experience at translating mathematics into set theory seems to indicate that whatever "arbitrary finite configurations" and "finite linear configurations" are, reasonable representations for them can be found in $\mathcal{S}^{\#}$ and $\mathcal{S}^*$ respectively.

In [2], on the basis of a number of evident mathematical properties of computability, the notion of computability on $\mathcal{S}^{\#}$ is characterized in terms of computability on the natural numbers. Since $\mathcal{S}^*$ is a computable subset of $\mathcal{S}^{\#}$, we automatically get a characterization of computability on finite linear configurations. It is the purpose of this note to show that computability on arbitrary finite configurations is a strict generalization of computability on finite linear configurations in the sense of the following theorem.

Henceforth, we assume that $\mathcal{S}$ is infinite and free in the sense that the members of $\mathcal{S}$ are ϵ-minimal in $\mathcal{S}^{\#}$ (i.e., $s \cap \mathcal{S}^{\#} = \emptyset$ for all $s \in \mathcal{S}$).

Theorem *There is no embedding θ of $\mathcal{S}^{\#}$ into $\mathcal{S}^*$ such that for every partial function F on the range of θ, F is computable on $\mathcal{S}^*$ iff $\theta^{-1} F \theta$ is computable on $\mathcal{S}^{\#}$.*

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To prove this theorem we need three lemmas and a number of definitions.

Definition If $\psi: \mathcal{S} \to \mathcal{S}$ then $\psi^{\#}: \mathcal{S}^{\#} \to \mathcal{S}^{\#}$ is the extension of ψ to an ϵ-homomorphism, i.e., $\psi^{\#}(\{y_1, \ldots, y_n\}) = \{\psi^{\#}(y_1), \ldots, \psi^{\#}(y_n)\}$ for all $y_1, \ldots, y_n \in \text{Dom } \psi^{\#}$.

Definition If $x \in \mathcal{S}^{\#}$, x^{b} denotes the smallest set contained in $\mathcal{S}$ such that $x \in (x^{b})^{\#}$ $\left(\text{i.e., } x^{b} = \bigcap \{\mathcal{U} \subset \mathcal{S} \mid x \in \mathcal{U}^{\#}\}\right)$.

Definition A set $\mathcal{U} \subset \mathcal{S}$ is said to *determine* a partial function F on $\mathcal{S}^{\#}$ iff $F\theta^{\#} = \theta^{\#}F$ for every permutation θ on $\mathcal{S}$ such that $\theta(u) = u$ for all $u \in \mathcal{U}$.

Note: In [2] it is shown that every computable partial function on $\mathcal{S}^{\#}$ is determined by a finite set.

Definitions Let $R \subset \mathcal{S}^{\#} \times \mathcal{S}^{\#}$ be a binary relation on $\mathcal{S}^{\#}$.

(1) R is *computable* on $\mathcal{S}^{\#}$ iff the identity function on R is computable on $\mathcal{S}^{\#}$.
(2) R is *left-finite* iff for every $y \in \mathcal{S}^{\#}$, $\{x\colon xRy\}$ is finite.
(3) R is *almost left-regular* iff there is a finite set $\mathcal{U} \subset \mathcal{S}$ such that $x \in (\mathcal{U} \cup y^{b})^{\#}$ whenever xRy.
(4) A partial function F is a selector for R iff $\text{Dom } F = \{y \in \mathcal{S}^{\#} \colon \text{for some } x \in \mathcal{S}^{\#}, xRy\}$ and $F(y)Ry$ for all $y \in \text{Dom } F$.

Lemma 1 *$\{\langle x, y\rangle \in \mathcal{S}^{\#}\colon x \in y\}$ has no computable selector on $\mathcal{S}^{\#}$.*

Proof: Suppose F were such a selector. Let $\mathcal{U} \subset \mathcal{S}$ be a finite set that determines F. Let s and t be distinct members of $\mathcal{S} - \mathcal{U}$. Let $\psi = \lambda x[x$ if $x \notin \{s, t\}$; t if $x = s$; s if $x = t]$. Notice that $\psi^{\#}(\{s, t\}) = \{s, t\}$. Suppose $F(\{s, t\}) = s$. Then since $\mathcal{U}$ determines F and $\psi(u) = u$ for all $u \in \mathcal{U}$, $s = F(\{s, t\}) = \psi^{\#^{-1}}F\psi^{\#}(\{s, t\}) = \psi^{\#^{-1}}F(\{s, t\}) = \psi^{\#^{-1}}(s) = t$. This contradicts our assumption that $s \neq t$ and, similarly, so does the supposition that $F(\{s, t\}) = t$. Q.E.D.

Lemma 2 *Let R be a relation on $\mathcal{S}^{\#}$ that is left-finite and computable on $\mathcal{S}^{\#}$. Then R is almost left-regular on $\mathcal{S}^{\#}$.*

Proof: Since R is computable, then id_R is computable and hence id_R is determined by a finite set $\mathcal{U} \subset \mathcal{S}$. Suppose there exist $x, y \in \mathcal{S}^{\#}$ such that xRy and $x \notin (y^{b} \cup \mathcal{U})^{\#}$. Then there exists $t \in x^{b} - (y^{b} \cup \mathcal{U})$. Let $s_1, s_2, \ldots$ be distinct members of $\mathcal{S} - (y^{b} \cup \mathcal{U})$ and let $\psi_i = \lambda z\, [s_i$ if $z = t$; t if $z = s_i$; z otherwise$]$ so that

(1) $\psi_1^{\#}(x), \psi_2^{\#}(x), \ldots$ are distinct.
(2) $\psi_i^{\#}(y) = y$ for $i = 1, 2, \ldots$.
(3) $\psi_i(u) = u$ for all $u \in \mathcal{U}$ and $i = 1, 2, \ldots$.

It follows that $\psi_i^{\#}(x)Ry$ for $i = 1, 2, \ldots$ contrary to our assumption that R is left-finite, for $\langle\psi_i^{\#}(x), y\rangle = \langle\psi_i^{\#}(x), \psi_i^{\#}(y)\rangle = \psi_i^{\#}\langle x, y\rangle = \psi_i^{\#}\text{id}_R\langle x, y\rangle = \text{id}_R\psi_i^{\#}\langle x, y\rangle = \text{id}_R\langle\psi_i^{\#}(x), \psi_i^{\#}(y)\rangle = \text{id}_R\langle\psi_i^{\#}(x), y\rangle$. Q.E.D.

Lemma 3 *Let R be an almost left-regular computable relation on $\mathcal{S}^{\#}$. Then $R \cap (\mathcal{S}^{\#} \times \mathcal{S}^{*})$ has a computable selector F.*

Proof: For every member x of $\mathcal{S}^*$ we define an enumeration x^Δ of $x^\flat$ via the recursion formula:

$$x^\Delta = (s_1, \ldots, s_m, t_1, \ldots, t_n) \text{ if } x = \langle y, z\rangle$$
$$\text{and } y^\Delta = (s_1, \ldots, s_m) \text{ and } z^\Delta = (t_1, \ldots, t_n).$$

We now define $E: \mathcal{S}^* \times \mathbf{N} \to \mathcal{S}^\#$ so that $\lambda n[E(x, n)]$ is an enumeration of $(x^\flat)^\#$ as follows:

$E(x, 2n)$ is the n'th member of x^Δ if n is less than or equal to the length of x^Δ; $\emptyset$ otherwise.

$E(x, 2(2^{y_1} + \ldots + 2^{y_m}) + 1)$ is $\{E(x, y_1), \ldots, E(x, y_m)\}$.

Clearly E is computable on $\mathcal{S}^\#$. We complete the proof by defining F so that

$F(y) = E((y, u_1, \ldots, u_k), m)$ where m denotes $\mu n[E((y, u_1, \ldots, u_k), n)Ry]$ and $\{u_1, \ldots, u_k\}$ is such that $x \in (y^\flat \cup \{u_1, \ldots, u_k\})^\#$ whenever xRy. Q.E.D.

Proof of main theorem: Suppose to the contrary that such a θ exists. Let ϵ_θ denote $\{\langle\theta(x), \theta(y)\rangle: x \in y\}$. Then ϵ_θ is computable and left-finite on $\mathcal{S}^\#$. Hence ϵ_θ is almost left-regular on $\mathcal{S}^\#$. So ϵ_θ has a computable selector F. But then $\theta^{-1}F\theta$ would be a computable selector for ϵ on $\mathcal{S}^\#$ contrary to Lemma 1. Q.E.D.

REFERENCES

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University of California, Riverside
Riverside, California