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THE GENERAL DECISION PROBLEM FOR MARKOV ALGORITHMS WITH AXIOM

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*Introduction** Let $\mathcal{M}_{\mathcal{A}}$ denote the general decision problem for Markov algorithms with axiom. Of interest to us is whether or not this class of problems is as richly structured, with regard to degrees of unsolvability, as those classes studied in Hughes, Overbeek, and Singletary [2]. In this paper we shall present proofs which show this to be so. In particular we shall show that the general decision problem for the range of total recursive functions is many-one reducible to $\mathcal{M}_{\mathcal{A}}$ and consequently that every r.e. many-one degree of unsolvability is represented by $\mathcal{M}_{\mathcal{A}}$. Furthermore we shall show this result to be best possible, with regard to degree representation, in that every r.e. one-one degree is not represented by this family of decision problems. And finally we shall demonstrate a simple application of these results to the study of splinters.

Preliminaries A semi-Thue system S is a pair $\langle \Sigma, P\rangle$ where Σ is a finite alphabet and P is a finite set of rules each of which is of the form $\alpha \rightarrow \beta$, for α and β words over Σ. For any arbitrary pair of words W_1, W_2 over Σ, we say that W_2 is an immediate successor of W_1 in S, denoted $(W_1, W_2)_S$, if there exist a pair of words U, V over Σ and a rule $\alpha \rightarrow \beta$ in P such that $W_1 \equiv U\alpha V$ and $W_2 \equiv U\beta V$. W_2 is said to be derivable from W_1 in S, denoted $W_1 \vdash_S W_2$, if either

(i) $W_1 \equiv W_2$,

or

(ii) there exists a finite sequence $V_1, \ldots, V_k$, where $k > 1$, of words over Σ such that $W_1 \equiv V_1$, $W_2 \equiv V_k$, and $(V_i, V_{i+1})_S$, for $i = 1, \ldots, k - 1$.

A Markov alogorithm M is a pair $\langle \Sigma, P\rangle$ where Σ is a finite alphabet and $P = \{\alpha_i R_i \beta_i \mid 1 \leqslant i \leqslant m\}$ is a finite ordered set of rules where

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$m \geq 1$, $R_i \in (\rightarrow, \rightarrow.)$ and α_i and β_i are words over Σ. A rule of the form $\alpha \rightarrow. \beta$ is called a conclusive rule. Let W_1 and W_2 be arbitrary words over Σ. Then W_2 is the immediate successor of W_1 in M, denoted $(W_1, W_2)_M$, if there exists an i, $1 \leq i \leq m$, such that

(i) $W_1 \equiv U\alpha_i V$ and $W_2 \equiv U\beta_i V$ for some words U and V over Σ,
(ii) there exists no pair of words U', V' over Σ such that the length of U' is less than the length of U and $W_1 \equiv U'\alpha_i V'$,
(iii) there exists no j, $1 \leq j < i$, such that $W_1 \equiv U''\alpha_j V''$ for some words U'' and V'' over Σ.

W_2 is said to be derivable from W_1 in M, denoted $W_1 \vdash_M W_2$, if either

(i) $W_1 \equiv W_2$,

or

(ii) there exists a finite sequence $V_1, \ldots, V_k$, where $k > 1$, of words over Σ such that $W_1 \equiv V_1$, $W_2 \equiv V_k$, $(V_i, V_{i+1})_M$, for $i = 1, \ldots, k-1$, and no $(V_i, V_{i+1})_M$, for $1 \leq i \leq k-2$, is the result of the application of a conclusive rule.

Let G be a semi-Thue system or Markov algorithm and let A be a fixed word over the alphabet of G. Then G_A shall denote such a system with axiom. The decision problem for G_A is the problem to decide, for an arbitrary word W over the alphabet of G, whether or not $A \vdash_G W$ (written $\vdash_G W$ whenever A is understood from context). The general decision problem for semi-Thue systems (Markov algorithms) with axiom is then the family of decision problems for all such systems.

Let C_1 and C_2 be two general decision problems. Then we say that C_1 is many-one (one-one) reducible to C_2 if there exists an effective mapping ψ of the decision problems p in C_1 into the decision problems $\psi(p)$ in C_2 such that p and $\psi(p)$ are of the same many-one (one-one) degree of unsolvability. C_2 is said to represent every r.e. many-one (one-one) degree of unsolvability if the general decision problem for the range of total recursive functions, denoted $\mathcal{R}$, is many-one (one-one) reducible to C_2.

Background Results In the next section we have need of the following theorem concerning semi-Thue systems with axiom.

Theorem 1 *There exists an effective procedure ψ_1 which, when applied to an arbitrary total recursive function f, produces a semi-Thue system S and a word A over the alphabet of S such that*

(i) *the decision problem for the range of f is of the same many-one degree as that for S_A;*
(ii) *there is no non-trivial derivation of A from A. That is, there exists no word W over the alphabet of S such that $(A, W)_S$ and $W \vdash_S A$;*
(iii) *if $\vdash_S W$ by a non-trival derivation and there exist W', W'' such that $(W', W)_S$ and $(W'', W)_S$ then $W' \equiv W''$ and $\vdash_S W'$. Stated differently this says*

that the semi-Thue system S^{-1}, *whose alphabet is that of S and which contains the rule* $\beta \to \alpha$ *if and only if* $\alpha \to \beta$ *is a rule of S, is deterministic*[1] *over words which are non-trivially derivable from A in S;*

(iv) *if* $\vdash_S W$ *and* $(W, W')_S$ *by some rule of S and* $(W, W'')_S$ *by the same rule then* $W' \equiv W''$;

(v) *the word of length zero is not derivable from A in S. In particular this means that A may not be the empty word;*

(vi) *no rule* $\alpha \to \beta$ *of S is such that either* α *or* β *is the word of length zero.*

Proof: In [5] Overbeek showed that $\mathcal{R}$ is many-one reducible to the general halting problem for Turing machines. Following this, Hughes and Singletary [3], Lemma 3, demonstrated an effective procedure which, when applied to an arbitrary Turing machine T, produces a semi-Thue system with axiom (denoted $\bar{S}_{hqHh}$ in their paper), which system satisfies properties (ii) through (vi) above and whose decision problem is of the same many-one degree as the halting problem for T. These two results may then be combined to provide a proof of the desired theorem. Q.E.D.

Reduction of $\mathcal{R}$ *to* $\mathcal{M}_{\mathcal{A}}$ In this section we shall demonstrate a uniform effective procedure ψ_2 which, when applied to an arbitrary semi-Thue system S with axiom A that satisfies properties (ii) through (vi) of Theorem 1, produces a Markov algorithm M with axiom B such that the decision problem for S_A is of the same many-one degree as that for M_B.

Let S_A be a semi-Thue system with axiom which satisfies properties (ii) through (vi) of Theorem 1. Further, let the alphabet of S be $\Sigma = \{a_1, \ldots, a_n\}$ and let the rule set of S be $P = \{\alpha_i \to \beta_i \mid 1 \leqslant i \leqslant \rho\}$. We define the Markov algorithm $M = \langle \Sigma', P' \rangle$ as follows:

$$\Sigma' = \Sigma \cup \{1, *\} \cup \{R_i, l_i, L_i, f_i, g_i, h_i \mid 1 \leqslant i \leqslant \rho\} \cup \{\$, Q, r, e_1, e_2\};$$

P' consists of the rules defined below, where a set of rules labelled (i) may have any internal order provided these rules follow all those in sets labelled (j), where $j < i$, and precede all rules in sets labelled (j), where $j > i$.

(1)	$\$ \to Q$	
(2)	$R_i\alpha_i \to \beta_i r$	$\forall 1 \leqslant i \leqslant \rho$
(3)	$R_i a_j \to a_j R_i$	$\forall 1 \leqslant i \leqslant \rho$ and $\forall 1 \leqslant j \leqslant n$
(4)	$R_i * \to l_i *$	$\forall 1 \leqslant i \leqslant \rho$
(5)	$ra_j \to a_j r$	$\forall 1 \leqslant j \leqslant n$
(6)	$r*11 \to e_2*1$	
(7)	$r*1 \to e_1*$	
(8)	$a_j e_2 \to e_2 a_j$	$\forall 1 \leqslant j \leqslant n$
(9)	$e_2 \to R_1$	
(10)	$a_j e_1 \to e_1 a_j$	$\forall 1 \leqslant j \leqslant n$

1. A system S is deterministic over a word W if and only if there exists at most one W' such that $(W, W')_S$.

(11)	$e_1 \to \$$	
(12)	$a_j l_i \to l_i a_j$	$\forall 1 \leqslant i \leqslant \rho$ and $\forall 1 \leqslant j \leqslant n$
(13)	$l_i \to R_{i+1}$	$\forall 1 \leqslant i < \rho$
(14)	$l_\rho \to Q$	
(15)	$QA* \to R_1 A*1$	
(16)	$Q \to L_1$	
(17)	$L_i \beta_i \to \alpha_i f_i$	$\forall 1 \leqslant i \leqslant \rho$
(18)	$L_i a_j \to a_j L_i$	$\forall 1 \leqslant i \leqslant \rho$ and $\forall 1 \leqslant j \leqslant n$
(19)	$L_i * \to g_i *$	$\forall 1 \leqslant i \leqslant \rho$
(20)	$f_i a_j \to a_j f_i$	$\forall 1 \leqslant i \leqslant \rho$ and $\forall 1 \leqslant j \leqslant n$
(21)	$f_i * \to h_i *1$	$\forall 1 \leqslant i \leqslant \rho$
(22)	$a_j h_i \to h_i a_j$	$\forall 1 \leqslant i \leqslant \rho$ and $\forall 1 \leqslant j \leqslant n$
(23)	$h_i \to R_{i+1}$	$\forall 1 \leqslant i < \rho$
(24)	$h_\rho \to Q$	
(25)	$a_j g_i \to g_i a_j$	$\forall 1 \leqslant i \leqslant \rho$ and $\forall 1 \leqslant j \leqslant n$
(26)	$g_i \to L_{i+1}$	$\forall 1 \leqslant i < \rho$

We now wish to show that the decision problem for S_A is of the same many-one degree as that for M with axiom $\$A*$. Before doing this we shall present the algorithm of which M is an implementation. This, we believe, will help to make the subsequent proofs more understandable.

The Basic Algorithm

(I) Let the word we are currently working on be of the form $\$W*$ for W a word over Σ. Then W is a word derivable from A in S. Generate $QW*$.

(II) Let the word we are currently working on be of the form $R_i W*1^m$ for $1 \leqslant i \leqslant \rho$, $m \geqslant 1$, 1^m being a shorthand notation for the sequence of m 1's, and W a word over Σ.

case a) If the i'th rule of S applies to W and $m > 1$ then generate $R_1 W'*1^{m-1}$, where $(W, W')_S$ by the i'th rule.
case b) If the i'th rule of S applies to W and $m = 1$ then generate $\$W'*$, where $(W, W')_S$ by the i'th rule.
case c) If the i'th rule of S does not apply to W and $i < \rho$ then generate $R_{i+1} W*1^m$.
case d) If the i'th rule of S does not apply to W and $i = \rho$ then generate $QW*1^m$.

(III) Let the word we are currently working on be of the form $QW*1^m$ for $m \geqslant 0$, 1^m denoting a sequence of m 1's, and W a word over Σ.

case a) If $W \equiv A$ then generate $R_1 A*1^{m+1}$.
case b) If $W \not\equiv A$ and $(W', W)_S$ by some rule i, $i < \rho$, then generate $R_{i+1} W'*1^{m+1}$.
case c) If $W \not\equiv A$ and $(W', W)_S$ by rule ρ, then generate $QW'*1^{m+1}$.
case d) If $W \not\equiv A$ and it is not the case that there exists a W' such that $(W', W)_S$ then stop.

The intention of the above algorithm, when started on the word $\$A*$, is to generate words of the form $\$W*$, when and only when W is derivable from A in S. Essentially if we think of (1) as performing the additional task of outputting W whenever it is entered with a word of the form $\$W*$ then this algorithm would simply enumerate the set of words derivable from A in S. To see this, observe that if we input $\$A*$ to this procedure then we will immediately derive $QA*$ and then R_1A*1. From this we will, by successive applications of II, I, and III, generate $\$W*$ for every word W such that $(A, W)_S$. After generating every immediate successor of A, in the order determined by our previous ordering of the rules of S, the algorithm will start working over the word R_1A*11. In general, the basic algorithm will, when started over the word R_1A*1^m, generate $\$W*$ for every word W such that $\vdash_S W$ by a derivation of length m. After generating all such words it will start working over R_1A*1^{m+1}.

The reader should now be convinced that the basic algorithm is one which, in essence, "flattens" the rooted graph induced by S_A. The special properties of S_A are essential to this for they ensure that this rooted graph is a rooted tree and therefore devoid of nodes with indegree greater than one. We may also note that this algorithm never terminates when started over $\$A*$. This may be seen by observing that the only way halting may occur is by the application of III.d. But this would only arise if there existed some word W, $W \neq A$, such that $\vdash_S W$ and there is no W' such that $(W', W)_S$. Clearly this is impossible.

We shall now proceed with our proof that the decision problem for S_A is of the same many-one degree as that for $M_{\$A*}$.

Lemma 1 *The decision problem for S_A is many-one reducible to that for $M_{\$A*}$.*

Proof: This may be seen to be true if we can verify that, for an arbitrary W over the alphabet of S, $\vdash_S W$ if and only if $\vdash_M \$W*$. But this follows immediately since M implements our basic algorithm where the rules of M may be corresponded to the steps of the basic algorithm as follows:

rules	*step*
{1}	I
{2, 3, 5, 6, 8, 9}	II.a
{2, 3, 5, 7, 10, 11}	II.b
{3, 4, 12, 13}	II.c
{3, 4, 12, 14}	II.d
{15}	III.a
{16, 17, 18, 19, 20, 21, 22, 23, 25, 26}	III.b
{16, 17, 18, 19, 20, 21, 22, 24, 25, 26}	III.c
{16, 18, 19, 25, 26}	III.d

Q.E.D.

Lemma 2 *The decision problem for $M_{\$A*}$ is many-one reducible to that for S_A.*

Proof: Let W be an arbitrary word over Σ'. It should be clear that we may assume, without loss of generality, that W is of the form $W'*1^m$ for $m \geq 0$ and W' a word over $(\Sigma' - \{*, 1\})$ which contains exactly one occurrence of a letter from $((\Sigma' - \Sigma) - \{*, 1\})$. For if this were not so then $\not\vdash_M W$ (that is, W would not be derivable from $\$A*$ in M). Hence we shall hereafter assume W to be of this form. Now we may, with the aid of an oracle for deciding the decision problem for S_A, determine whether or not $\vdash_M W$ by the following case analysis:

(a) Assume W contains an occurrence of the letter \$. If W is not of the form $\$Y*$ for Y a word over Σ then $\not\vdash_M W$. If W is of this form then $\vdash_M W$ if and only if $\vdash_S Y$.
(b) Assume W contains an occurrence of the letter $R_i (1 \leq i \leq \rho)$. If W is not of the form YR_iZ*1^m for $m \geq 1$ and Y and Z words over Σ then $\not\vdash_M W$. If W is of this form then check to see if $R_iYZ*1^m \vdash_M W$ in exactly the length of Y steps. If it does not then $\not\vdash_M W$. If it does then $\vdash_M W$ if and only if $\vdash_S YZ$.
(c) Assume W contains an occurrence of l_i $(1 \leq i \leq \rho)$. If W is not of the form Yl_iZ*1^m for $m \geq 1$ and Y and Z words over Σ then $\not\vdash_M W$. If W is of this form then check to see if $R_iYZ*1^m \vdash_M W$ in exactly the length of Y plus twice the length of Z plus one steps. If it does not then $\not\vdash_M W$. If it does then $\vdash_M W$ if and only if $\vdash_S YZ$.
(d) Assume W contains an occurrence of the letter r. If W is not of the form $YrZ*1^m$ for $m \geq 1$ and Y and Z words over Σ then $\not\vdash_M W$. If W is of this form check to see if there exists an i, $1 \leq i \leq \rho$, and a pair of words Y', Z' over Σ such that $Y \equiv Y'\beta_iZ'$. If there does not exist exactly one triple i, Y', Z' satisfying the above requirement then $\not\vdash_M W$, else $\vdash_M W$ if and only if $\vdash_S Y'\alpha_iZ'Z$.
(e) Assume W contains an occurrence of the letter e_2. If W is not of the form Ye_2Z*1^m for $m \geq 1$ and Y and Z words over Σ then $\not\vdash_M W$. If W is of this form then $\vdash_M W$ if and only if $\vdash_M YZr*1^{m+1}$ and our analysis returns to case (d).
(f) Assume W contains an occurrence of the letter e_1. If W is not of the form Ye_1Z* for Y and Z words over Σ then $\not\vdash_M W$. If W is of this form then $\vdash_M W$ if and only if $\vdash_M YZr*1$ and our analysis returns to case (d).
(g) Assume W contains an occurrence of the letter Q. If W is not of the form $QY*1^m$ for $m \geq 0$ and Y a word over Σ then $\not\vdash_M W$. If W is of this form then $\vdash_M W$ if and only if $\vdash_S Y$.
(h) Assume W contains an occurrence of the letter L_i $(1 \leq i \leq \rho)$. If W is not of the form YL_iZ*1^m for $m \geq 0$ and Y and Z words over Σ then $\not\vdash_M W$. If W is of this form and $YZ \equiv A$ then $\not\vdash_M W$. If W is of this form and $YZ \not\equiv A$ then check to see if $L_1YZ*1^m \vdash_M W$ in exactly the length of Y plus $(i - 1)$ times (length of YZ plus 2) steps. If it does not then $\not\vdash_M W$ else $\vdash_M W$ if and only if $\vdash_S YZ$.
(i) Assume W contains an occurrence of the letter $g_i (1 \leq i < \rho)$. If W is not of the form Yg_iZ*1^m for $m \geq 0$ and Y and Z words over Σ then $\not\vdash_M W$. If W is of this form then $\vdash_M W$ if and only if $\vdash_M L_{i+1}YZ*1^m$ and our analysis returns to case (h).
(j) Assume W contains an occurrence of the letter g_ρ then $\not\vdash_M W$.

(k) Assume W contains an occurrence of the letter $f_i (1 \leqslant i \leqslant \rho)$. If W is not of the form Yf_iZ*1^m for $m \geqslant 0$ and Y and Z words over Σ then $\nvdash_M W$. If W is of this form then check to see if there exists a pair of word Y', Z' over Σ such that $Y \equiv Y'\alpha_i Z'$. If there does not exist exactly one pair Y', Z' satisfying the above requirement then $\nvdash_M W$, else $\vdash_M W$ if and only if $\vdash_M L_i Y'\beta_i Z'Z*1^m$ and our analysis returns to case (h).
(l) Assume W contains an occurrence of the letter $h_i (1 \leqslant i \leqslant \rho)$. If W is not of the form Yh_iZ*1^m for $m \geqslant 1$ and Y and Z words over Σ then $\nvdash_M W$. If W is of this form then $\vdash_M W$ if and only if $\vdash_M YZf_i*1^{m-1}$ and our analysis returns to case (k).

The proof may now be seen to be complete by simply observing that, for any given word W, the above decision procedure asks at most one question of the oracle for S_A and, if it chooses to ask a question, reports the oracle's answer faithfully.[2] Q.E.D.

Theorem 2 *Each of the following holds*

(I) $\mathcal{R}$ *is many-one reducible to* $\mathcal{M}_{\mathcal{A}}$,
(II) *Every* r.e. *many-one degree is represented by* $\mathcal{M}_{\mathcal{A}}$.

Proof: (I) is an immediate consequence of Lemmas 1 and 2 and Theorem 1. (II) follows from (I) by the definition of what it means for a general decision problem to represent every r.e. many-one degree. Q.E.D.

Corollary 1 *Define a restricted Markov algorithm to be one which does not contain any conclusive rules. Then* Theorem 2 *holds with* $\mathcal{M}_{\mathcal{A}}$ *replaced by* $\mathcal{M}'_{\mathcal{A}}$ *where* $\mathcal{M}'_{\mathcal{A}}$ *is the general decision problem for restricted Markov algorithms with axiom.*

Proof: Immediate from the fact that the Markov algorithms defined by our procedure ψ_2 are always restricted. Q.E.D.

The One-One Degrees of $\mathcal{M}_{\mathcal{A}}$ We shall now prove that our result for $\mathcal{M}_{\mathcal{A}}$ is best possible with regard to degree representation. In order to do this we shall prove that no instance of $\mathcal{M}_{\mathcal{A}}$ is of the same one-one degree as a simple set [6], p. 298. This theorem and its proof are a particularization of a more general theorem of W. E. Singletary. Other papers which have employed his result include [1], [2], and [3].

Theorem 3 *It is not the case that every* r.e. *one-one degree is represented by* $\mathcal{M}_{\mathcal{A}}$.

Proof: Let M_A be a Markov algorithm with axiom whose decision problem is recursively unsolvable. Then we may deduce the existence of some word W_o over the alphabet of M such that $\nvdash_M W_o$, $D = \{W \mid W_o \vdash_M W\}$ is infinite and no

2. By reporting the answer faithfully we mean that the procedure may not continue to compute after asking a question of the oracle nor may it apply any Boolean function to the oracle's answer.

word contained in D is derivable from A. For assume no such W_0 exists then the decision problem for M_A may be decided as follows: Let W be an arbitrary word over the alphabet of M. Build two list of words in the following manner: Stage O, put W in list 1 and A in list 2. Stage $n+1$, put the word derivable from W in $n+1$ steps into list 1, if any such word exists. Do the same for list 2 with respect to A. Continue this process until either (1) list 1 contains all words derivable from W in M; or (2) list 1 and list 2 contain some word in common. By our assumption one of these cases must arise. Now if (1) occurs then $\nvdash_M W$ since A must have an infinite number of descendants in order for the decision problem for M_A to be unsolvable. If (2) first occurs due to W being placed in list 2 then $\vdash_M W$. Otherwise (2) first occurs because there exists some W' such that $W \vdash_M W'$, non-trivially, and $A \vdash_M W'$ in a derivation in which W does not arise (note: this includes the possibility that $W \vdash_M A$, non-trivially). But then $\nvdash_M W$. For, if $\vdash_M W$, then M would loop when started on A and hence M_A would be solvable. Thus any r.e. set of the same one-one degree as the decision problem for M_A must be non-simple and hence every r.e. one-one degree is not represented by $\mathcal{M}_{\mathcal{A}}$. Q.E.D.

Degrees of Splinters Let f be a total recursive function and let a_o be a natural number. Then $\{x \mid \exists n[f^n(a_o) = x]\}$, where $f^0(a_o) = a_o$ and $f^{m+1}(a_o) = f(f^m(a_o))$, is called a splinter. Splinters were first defined by Ullian [7] and were subsequently studied by Myhill [4]. One of the results of their research was the proof that every r.e. many-one degree is represented by the general decision problem for splinters. We shall now show that our results for $\mathcal{M}'_{\mathcal{A}}$ provide us with an independent proof of this.

Theorem 4 *Every* r.e. *many-one degree is represented by the general decision problem for splinters.*

Proof: Let M be an arbitrary restricted Markov algorithm, let A be a word over the alphabet of M, and let g_M be a Gödel numbering of the words of M onto the natural numbers. Define the total recursive function f_M as follows:

$$f_M(x) = \begin{cases} x & \text{if } g_M^{-1}(x) \text{ has no immediate successor in } M, \\ y & \text{if } g_M^{-1}(y) \text{ is the immediate successor of } g_M^{-1}(x) \text{ in } M. \end{cases}$$

Let $a_o = g_M(A)$. Then clearly $\{x \mid \exists n[f_M^n(a_o) = x]\}$ is just the set of Gödel numbers of all words derivable from A in M. Hence the decision problem for M_A is of the same many-one (in fact one-one) degree as that for the splinter arising from f_M and a_o. But then, since M and A were chosen arbitrarily, we have that $\mathcal{M}'_{\mathcal{A}}$ is many-one reducible to the general decision problem for splinters and this theorem is proved in light of Theorem 2, Corollary 1. Q.E.D.

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