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A FORMAL THEORY OF SORTAL QUANTIFICATION

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1 *Introduction* The standard quantification theory, first-order predicate calculus with identity, (called **QT** hereinafter; see [2], §§30, 40, 48, and [9], Ch. 2, §§1, 3, 8, for standard formulations) makes no distinctions between different kinds of one-place predicate. But many philosophical logicians have made a distinction between "sortal" predicates such as 'is a man' and other predicates such as 'is white'. Aristotle introduced the notion of "secondary substance"—the kind of substance a particular thing is, as opposed to the qualities it has ([1], Ch. 5, see especially 2^a11, 2^b29, 3^b10). Frege distinguished concepts which "isolate in a definite manner what falls under them" from those which do not ([3], §54, p. 66), although he did not represent the distinction in his formal system of quantification. In recent philosophical logic, Strawson has distinguished sortal universals from characterizing ones ([18], Ch. 5, §8, p. 168ff), Quine has distinguished terms with divided reference from mass terms ([10], Ch. 3, §19, p. 90ff), and Geach has distinguished substantival countable terms from those which are adjectival or non-countable ([4], Ch. 2, §31, p. 38ff).

We can distinguish between a sortal predicate, e.g., 'is a man', and the corresponding sortal term, 'man'. Grammatical marks of sortal terms are that they admit the definite and indefinite articles, they have plurals, they can appear in the singular after 'every', 'some', 'no', 'this', etc., and in the plural after 'all', 'some', 'most', 'at least two', 'those', etc., and in the singular in phrases of the form 'is the same . . . as'. But words like 'object', 'individual', 'thing', 'entity', etc., pass these grammatical tests. We shall say that a word is a sortal term iff it supplies a *criterion of numerical identity* for whatever it applies to, that is, iff it can occur in true or false sentences of the form 'There are n F's such that . . .', where n is any integer. The fact that there are no determinate truth-conditions for 'There are three red things in this room' implies that 'red thing' is not a sortal term (*cf.* [4], p. 38-9 and [3], p. 66). So 'man', 'tree', 'lump of coal', 'university', 'battle', 'real number', 'character in Shakespeare's plays' are sortal terms, but 'white', 'new', 'coal', 'six feet tall', 'interesting', 'came into existence in 1925', 'divisible by three', are not. Thus by the

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countability test used here, the notion of sortal is not restricted to words for material particulars, but applies wherever there is the possibility of counting; it may thus be said to be ontologically or categorially neutral. But no further analysis is offered here of the notion of countability or numerical identity, so we have no answer to the question whether terms such as 'material object', 'institution', 'event', 'number', 'fictional entity' should count as sortals. (Wiggins suggests the notion of "sortal-schema" for such cases, [21], appendix, 5.4., p. 63.) We do not discuss the use of sortals with demonstratives, nor with mass terms (e.g., forming 'lump of coal' from the mass term 'coal'), because such uses seem essentially peculiar to sortals for material particulars.

We concentrate here on three kinds of occurrences of sortal terms—in "sortal predications" of the form 'x is an S'; after "quantifier" words, e.g., in 'Every S is ϕ'; and in identity-statements of the form 'x is the same S as y'. **QT** systematically treats all sortal terms as one-place predicates, rendering 'Every S is ϕ' as '$(x)(Sx \supset \phi(x))$', and 'x is the same S as y' as '$x = y \;\&\; Sx$'. Such treatment is convenient, and no doubt legitimate for certain purposes, but the distinction thus slurred over may be of importance in other ways, so it is worth trying to construct a system of formal logic in which sortal terms are distinguished from one-place predicates, and the above three roles of sortals terms represented.

This is what is attempted here. We use ideas of Geach [4], Wallace [19], and Wiggins [21], although we depart from each in certain respects. We represent 'x is the same S as y' by '$x \underset{S}{=} y$' (following [21], p. 2), and 'x is an S' by 'xS' (following [19], p. 12), introducing the latter into the formal theory as an abbreviation for '$x \underset{S}{=} x$' (following [4], p. 191). 'Every S is ϕ' and 'Some S is ϕ' will be represented by formulas with sortally-restricted quantifiers: '$(\forall xS)\phi(x)$' and '$(\exists xS)\phi(x)$' respectively.

The theory will allow quantifiers with different sortal restrictions in one formula, e.g., 'Every boy loves some girl' will be represented by '$(\forall xB)(\exists yG)Lxy$', so in this respect it will be analogous to standard many-sorted theories ([13], [14], [20], [2] exercise 55.24, and [16]). But it will differ from these theories in that there will be only one syntactic category of individual variables; the range-restricting job is done by the sortal terms in the quantifiers, leaving the variables to do only the cross-referencing job of indicating which quantifier binds which position in the formula. The variables will therefore be theoretically eliminable by Schönfinkel's methods ([15], [11]). The theory will differ from Hailperin's theory of restricted quantification ([6]) in that it will have a syntactic category of sortal terms; and only sortal terms, not arbitrary formulas, will be allowed to appear in the range-restricting position.

What is truly distinctive of sortal terms is not their range-restricting role, for as Hailperin has shown, this can be done by any formula, but their role in identity-statements; this would be expected from our definition of sortals as terms which supply a criterion of identity. We shall follow Wiggins ([21] Part One) in accepting Leibniz's law of the indiscernability of identicals, in the form: if x is the same S as y, then anything true of x is true of y; we shall thus differ from Geach's views on "relative identity" ([4], p. 157, and [5]); see [17] for a defence of the position taken here. We must take into account certain relations which may hold between sortals, and hence between the criteria of identity they supply—e.g., 'fisherman' and 'man' can apply to one and the same individual man, so they may be said to *intersect*; and one sortal 'S' may be *subordinate* to (a restriction of) another 'T' in the sense that all S's are T's; in these cases the two sortals must give the same criterion of identity. We shall develop our formal theory on the following two assumptions: that if two sortals intersect then there is a sortal to which they are both subordinate (*cf.* [21], p. 33), and that every sortal is subordinate to some *ultimate* sortal, i.e., a sortal which is subordinate to no other sortal (*cf.* [21], p. 33 and note 40). An ultimate sortal may be said to give the criterion of identity of everything it applies to, and of all sortals subordinate to it. Accordingly we shall introduce a primitive logical constant $\cup$ and for any individual term t or any sortal term S we shall construct the corresponding ultimate sortal term $\cup t$ or $\cup S$.

We also make the following two simplifying assumptions, which could possibly be dropped by amendments to the theory: that every individual term and every sortal term is non-empty, and that every sortal term is syntactically simple, apart from those resulting from the applications of the $\cup$-function introduced above. These two assumptions are to some extent unrealistic, for 'dragon' is an empty sortal term, and 'man who habitually fishes' is presumably synonymous with the sortal term 'fisherman'. But if we count 'All dragons are ϕ' true just because there are no dragons, 'dragon' will turn out to be subordinate to every sortal term, even 'real number', which is counter-intuitive; and if we allow the formation of syntactically complex sortal terms from simple sortals plus predicates, then any such complex may turn out to be empty. In matters of logical style we generally follow Mendelson [9].

2 *Syntax*

2.1 *Symbols, Wffs, and Abbreviations*

Symbols

(i) Denumerably many individual variables $x, y, z, x_1, x_2, \ldots$
(ii) Denumerably many individual constants $a, b, c, a_1, a_2, \ldots$
(iii) Denumerably many function constants $f_1^1, f_2^1, \ldots, f_i^n, \ldots$
(iv) Denumerably many predicate constants $P_1^1, P_2^1, \ldots, P_i^n, \ldots$

(The superscript of a function or predicate constant indicates the number of arguments it requires.)

(v) Denumerably many sortal constants $A, B, C, A_1, A_2, \ldots$
(vi) Improper symbols (primitive) $\sim$, &, $\exists$, $=$, $\cup$, (,).
(vii) Improper symbols (defined) $\vee$, $\supset$, $\equiv$, $\forall$, $\subseteq$, υ.

Individual Terms

(i) Any individual variable is an individual term.
(ii) Any individual constant is an individual term.
(iii) Any n-place function constant followed by n individual terms (not necessarily all different) is an individual term.
(iv) A string of symbols is an individual term only if it can be shown to be one by (i)-(iii).

A *closed individual term* (cit) is an individual term which contains no individual variables.

Sortal Terms

(i) Any sortal constant is a sortal term.
(ii) If t is an individual term then $\mathsf{U}t$ is a sortal term.
(iii) If S is a sortal term then $\mathsf{U}S$ is a sortal term.
(iv) A string of symbols is a sortal term only if it can be shown to be one by (i)-(iii).

($\mathsf{U}t$ and $\mathsf{U}S$ will often be written U_t and U_S, and can be read as "the ultimate sortal of 't'" and "the ultimate sortal of 'S'" respectively.) A *closed* sortal term is one with no individual variables.

Well-formed formulas (*Wffs*)

(i) If F_i^n is an n-place predicate constant, and $t_1, \ldots, t_n$ are n individual terms (not necessarily all different) then $F_i^n t_1 \ldots t_n$ is an atomic wff.
(ii) If S is a sortal term, and t_1 and t_2 are two individual terms (not necessarily different) then $t_1 = St_2$ is an atomic wff.

($t_1 = St_2$ will usually be written as $t_1 \underset{S}{=} t_2$.)

(iii) If P and Q are wffs, then $(\sim P)$ and $(P \,\&\, Q)$ are wffs.
(iv) If P is a wff, x an individual variable, and S a sortal term, then $((\exists xS)P)$ is a wff.
(v) A string of symbols is a wff only if it can be shown to be one by (i)-(iv).

Definitions and Abbreviations An expression of the form $(\exists xS)$, where S is a sortal term, is called an *S-restricted existential quantifier,* and similarly $(\forall xS)$ is an *S-restricted universal quantifier,* and $\underset{S}{=}$ is *S-relative identity.* In an expression of the form $((\exists xS)P)$, P is called the *scope* of the quantifier. An occurrence of an individual variable in a wff is *bound* if it is in a quantifier in the wff or in the scope of a quantifier in the wff, otherwise the occurrence is *free.* A *closed* wff is one in which no variable occurs free. The outermost pair of brackets of a wff may be omitted, and the primitive and defined connectives and quantifiers are ordered as follows: $\equiv$, $\supset$, quantifiers, $\vee$, $\&$, $\sim$, so that brackets may be omitted according to the rule that the symbol latest in the list forms the shortest possible wff from the symbols surrounding it.

For any wffs P and Q:

$$P \vee Q \text{ is defined as } \sim(\sim P \,\&\, \sim Q),$$
$$P \supset Q \text{ is defined as } \sim(P \,\&\, \sim Q),$$
$$P \equiv Q \text{ is defined as } (P \supset Q) \,\&\, (Q \supset P).$$

For any wff P, any individual variable x, and any sortal term S:

$$(\forall xS)P \text{ is defined as } \sim(\exists xS) \sim P.$$

For any sortal terms S and T, and any individual term t: tS is defined as $t \underset{S}{=} t$, and can be read as "t is an S". $S \subseteq T$ is defined as $(\forall xS)xT$, and can be read as "All S's are T's", or as "'S' is subordinate to 'T'". $S = T$ is defined as $(S \subseteq T) \,\&\, (T \subseteq S)$, and can be read as "'$S$' and '$T$' are coextensional". (We could use a new symbol, e.g., $\equiv$, for this relation between sortals, but since there will be no danger of confusion with unrestricted individual identity, we do not need to.) $\upsilon(S)$ is defined as $\mathsf{U}S = S$, and can be read as "'S' is ultimate". $S\mathsf{I}T$ is defined as $(\exists xS)xT$, and can be read as "Some S's are T's", or as "'S' intersects 'T'".

2.2 *Axioms, Rules of Inference, and Proofs*

Logical Axioms

If P, Q, and R are any wffs, x and y any individual variables, t, t_1, and t_2 any individual terms, and S any sortal term, then the following are logical axioms:

(1) $P \supset (Q \supset P)$.
(2) $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$.
(3) $(\sim Q \supset \sim P) \supset ((\sim Q \supset P) \supset Q)$.
(4) $(\forall xS)(P \supset Q) \supset (P \supset (\forall xS)\, Q)$ *if x does not occur free in P.*
(5) $(\forall xS)\, \phi \supset (tS^x_t \supset \phi^x_t)$ *where ϕ is any wff and ϕ^x_t is the result of replacing all free occurrences of x in ϕ by t, and no variable occurring in t becomes bound in ϕ^x_t by such replacement, and S^x_t is the result of replacing any occurrences of x in S by t.*
(6) $t_1 \underset{S}{=} t_2 \supset t_1S \,\&\, t_2S$.
(7) $x \underset{S}{=} y \supset \left(\phi \supset \phi \dfrac{x}{y}\right)$ *where ϕ is any wff and $\phi \dfrac{x}{y}$ is the result of replacing some, but not necessarily all, free occurrences of x in ϕ by y, and no such replacement yields a bound occurrence of y.*
(8) $(\exists xS)xS$.
(9) $t\mathsf{U}_t$.
(10) $S \subseteq \mathsf{U}_S$.
(11) $tS \supset \mathsf{U}_t = \mathsf{U}_S$.

Rules of Inference

Modus Ponens (MP): Q follows from P and $P \supset Q$.
Generalization (Gen): $(\forall xS)P$ follows from $xS \supset P$.

A particular formal system of sortal quantification theory is determined by:

(a) a *vocabulary*, namely:

(i) any subset (possibly empty) of the individual constants,
(ii) any subset (possibly empty) of the function constants,
(iii) any non-empty subset of the predicate constants,
(iv) any non-empty subset of the sortal constants.

(b) a set of *proper axioms*, possibly empty.

Such a system will be called *a (first-order) sortal quantification theory*; if there are no proper axioms it will be called a *(first-order) sortal calculus*. The sortal calculus which has every constant in its vocabulary will be called **SQT**.

A wff ϕ is said to be a *consequence* in a sortal quantification theory K of a set Γ of wffs of K, and we write $\Gamma \vdash_K \phi$, iff there is a finite sequence $P_1, P_2, \ldots, P_n$ of wffs of K such that P_n is ϕ and for each i, either P_i is an axiom of K, or is in Γ, or follows from one or more previous wffs in the sequence by one of the rules of inference. Such a sequence is said to be a *proof* (or deduction) of ϕ from Γ in K. A wff ϕ of K is said to be a *theorem* of K, and we write $\vdash_K \phi$, iff there is a proof of ϕ from the empty set of wffs in K. In what follows we often abbreviate $\Gamma \vdash_{\mathbf{SQT}} \phi$ and $\vdash_{\mathbf{SQT}} \phi$ to $\Gamma \vdash \phi$ and $\vdash \phi$. If $\Gamma \cup \{P\} \vdash Q$ we write $\Gamma, P \vdash Q$.

3 *Elementary Metatheory*

3.1 *Consistency* We shall say that a sortal quantification theory K is *consistent* iff there is no wff P such that $\vdash_K P$ and $\vdash_K \sim P$. The consistency of any sortal calculus can be proved easily by interpreting it, in effect, in a domain consisting of only one element, to which every sortal applies:

Metatheorem 3.1 *Any first-order sortal calculus K is consistent.*

Proof: For each wff P of K we define a wff $c(P)$ of the propositional calculus as follows ($c(P)$ will be called the c-transform of P):

(i) If P is an atomic wff $F_i^n t_1 \ldots t_n$ then $c(P)$ is F_i^n.
(ii) If P is an atomic wff $t_1 =_S t_2$ then $c(P)$ is $F \supset F$.
(iii) If P is of the form $\sim Q$ then $c(P)$ is $\sim c(Q)$.
(iv) If P is of the form $Q \,\&\, R$ then $c(P)$ is $c(Q) \,\&\, c(R)$.
(v) If P is of the form $(\exists xS)Q$ then $c(P)$ is $c(Q)$.

If we regard the letters F and F_i^n as statement letters, then $c(P)$ is a wff of the propositional calculus. It is easily verified that if P is any axiom of a sortal calculus, then $c(P)$ is a tautology, and that the rules of inference, MP and Gen, preserve the property of having the c-transform a tautology. It follows that every theorem of a sortal calculus has a tautology as its c-transform. So if there were a wff P such that $\vdash_K P$ and $\vdash_K \sim P$, then $c(P)$ and $\sim c(P)$ would both be tautologies, but this is impossible, so there can be no such wff.

3.2 *The Deduction Theorem* If we have a deduction $P_1, \ldots, P_n$ in a sortal quantification theory K from a set Γ of wffs which includes the wff P, with a justification for each step in the deduction, we shall say that P_i *depends upon* P in this proof iff:

either (i) P_i in P and the justification for P_i is that it is in Γ;
or (ii) P_i is justified by its following by MP or Gen from previous wffs in the proof, at least one of which depends upon P.

It follows easily that if Q does not depend upon P in a deduction $\Gamma, P \vdash Q$, then $\Gamma \vdash Q$. We also note here that any substitution instance of a tautology is a theorem of **SQT**, since **SQT** includes axiom schema (1)-(3) and Modus Ponens, which are known to make every tautology provable (see, e.g., [9], Chapter 1, §4). Any use of propositional calculus in what follows is indicated by '**PC**'.

Metatheorem 3.2 *If* Γ, $P \vdash Q$ (*where* Γ *is a set of wffs of* **SQT**) *and in the deduction no application of* Gen *to a wff which depends upon* P *has as its quantified variable a free variable of* P, *then* $\Gamma \vdash P \supset Q$.

Proof: Let $R_1, R_2, \ldots, R_n$ (R_n being Q) be such a deduction of Q from Γ and P. We show by induction on i that $\Gamma \vdash P \supset R_i$ for each $i \leqslant n$. Suppose then, as induction hypothesis, that $\Gamma \vdash P \supset R_j$ for $1 \leqslant j < i$. We prove, as induction step, that $\Gamma \vdash P \supset R_i$. If R_i is an axiom or is in Γ, then $\Gamma \vdash P \supset R_i$, since $R_i \supset (P \supset R_i)$ is an axiom, by (1). If R_i is P, then $\Gamma \vdash P \supset R_i$, since $P \supset P$ is a theorem of **PC**. If R_i follows by MP from R_j and R_k, where $1 \leqslant j, k < i$ and R_k is $R_j \supset R_i$, then by induction hypothesis, $\Gamma \vdash P \supset R_j$ and $\Gamma \vdash P \supset (R_j \supset R_i)$, hence by (2) and MP, $\Gamma \vdash P \supset R_i$. Finally, suppose R_i follows by Gen from R_j, where $1 \leqslant j < i$, and R_i is $(\forall xA)R$ and R_j is $xA \supset R$. By hypothesis, $\Gamma \vdash P \supset (xA \supset R)$, and there is a proof $\Gamma, P \vdash xA \supset R$ such that either $xA \supset R$ does not depend upon P, or if it does, then x is not free in P. If $xA \supset R$ does not depend upon P, then $\Gamma \vdash xA \supset R$, hence by Gen $\Gamma \vdash (\forall xA)R$, hence $\Gamma \vdash P \supset (\forall xA)R$, i.e., $\Gamma \vdash P \supset R_i$. If x is not free in P, by hypothesis, $\Gamma \vdash P \supset (xA \supset R)$, hence $\Gamma \vdash xA \supset (P \supset R)$ by **PC**, hence by Gen $\Gamma \vdash (\forall xA)(P \supset R)$, hence by (4), since x is not free in P, $\Gamma \vdash P \supset (\forall xA)R$, i.e., $\Gamma \vdash P \supset R_i$. So in every possible case, $\Gamma \vdash P \supset R_i$, and thus the induction step is completed. The induction base is the case $i = 1$, in which case R_1 either is an axiom, or is in Γ, or is P, and we have seen that in all these three cases $\Gamma \vdash P \supset R_1$. So it follows by induction that $\Gamma \vdash P \supset R_n$, i.e., $\Gamma \vdash P \supset Q$.

The following corollaries are useful.

Metatheorem 3.2A *If a deduction* Γ, $P \vdash Q$ *involves no application of* Gen *of which the quantified variable is free in* P, *then* $\Gamma \vdash P \supset Q$.

Metatheorem 3.2B *If* P *is a closed wff and* Γ, $P \vdash Q$, *then* $\Gamma \vdash P \supset Q$.

We see also, in the proof of Metatheorem 3.2, that the new proof of $\Gamma \vdash P \supset Q$ involves an application of Gen to a wff depending on a wff S of Γ only if there is an application of Gen in the original proof of Γ, $P \vdash Q$ which involves the same quantified variable and is applied to a wff which depends on S. Therefore the Deduction Theorem can be applied repeatedly, e.g., to get $\Gamma \vdash P \supset (Q \supset R)$ from $\Gamma, P, Q \vdash R$.

3.3 *Sortally-Restricted Quantification*

Derived Rule 3.3.1 (Gen′) *If* $\vdash P$ *then* $\vdash (\forall xS)P$ *for any sortal term* S.

Proof: If $\vdash P$ then $\vdash xS \supset P$ by (1) and MP, hence $\vdash (\forall xS)P$ by Gen. (Gen′ is a slightly weaker rule than Gen; it is often sufficient, but there are certain places where Gen is needed.)

Theorem 3.3.2 $\vdash (\forall xS)P$ *iff* $\vdash xS \supset P$.

Proof: By (5) and MP, if $\vdash (\forall xS)P$ then $\vdash xS \supset P$. Conversely, if $\vdash xS \supset P$ then $(\forall xS)P$ by Gen.

Theorem 3.3.3 *If* x *does not occur free in* P, *then* $\vdash (\forall xS)P \supset P$.

Proof: $\vdash (\forall xS)P \supset (xS \supset P)$ by (5), hence $\vdash xS \supset ((\forall xS)P \supset P)$ by **PC**, hence $\vdash \sim ((\forall xS)P \supset P) \supset \sim xS$ by **PC**, hence $\vdash (\forall xS)(\sim ((\forall xS)P \supset P) \supset \sim xS)$ by Gen′, hence $\vdash \sim ((\forall xS)P \supset P) \supset (\forall xS) \sim xS$ by (4), since x is not free in P or in $(\forall xS)P$, hence $\vdash (\exists xS)xS \supset ((\forall xS)P \supset P)$ by **PC**, hence $\vdash (\forall xS)P \supset P$ by (8) and MP.

Corollaries 3.3.4 *If* x *does not occur free in* P, *then* $\vdash (\forall xS)P \equiv P$, $\vdash (\forall xS)P \equiv (\forall xT)P$, $\vdash P$ *iff* $\vdash xS \supset P$, *and* $\vdash xS \supset P$ *iff* $\vdash xT \supset P$.

It is interesting that Theorem 3.3.3 is actually equivalent to axiom schema (8), which states the non-emptiness of each sort. We have shown that (8) makes 3.3.3 provable. To prove the converse, notice that $\vdash (\forall xS) \sim xS \supset (xS \supset \sim xS)$ by (5), hence $\vdash (\forall xS) \sim xS \supset \sim xS$ by **PC**, hence $\vdash xS \supset (\exists xS)xS$ by **PC**, hence $\vdash (\forall xS)(\exists xS)xS$ by Gen; so if 3.3.3 is true, then since x is not free in $(\exists xS)xS$ we have $\vdash (\exists xS)xS$. The independence of (8) from (1)-(7) can easily be proved by interpreting in the empty domain (counting everything of the form $t_1 \underset{S}{=} t_2$ or $(\exists xS)P$ as false).

Theorem 3.3.5 $\vdash (\forall xS)(P \supset Q) \supset ((\forall xS)P \supset (\forall xS)Q)$.

Proof: From $(\forall xS)(P \supset Q)$ and $(\forall xS)P$ as hypotheses, we can deduce $xS \supset (P \supset Q)$ and $xS \supset P$ by (5), hence $xS \supset Q$ by **PC**, hence $(\forall xS)Q$ by Gen. Then the Deduction Theorem (DT) applies.

Theorem 3.3.6 $\vdash (\forall xS)(\forall yT)P \supset (\forall yT)(\forall xS)P$.

Proof:

1. $(\forall xS)(\forall yT)P$	hypothesis
2. $xS \supset (\forall yT)P$	(5) and MP
3. $(\forall yT)P \supset (yT \supset P)$	(5)
4. $xS \supset (yT \supset P)$	by **PC** from 2 and 3
5. $(\forall yS)(yT \supset P)$	Gen
6. $yT \supset (\forall xS)P$	by (4) and MP, since x is not free in yT
7. $(\forall yT)(\forall xS)P$	Gen

Then use DT.

We can also prove in **SQT** derived rules which are the natural amendments in sortal quantification of the usual rules of natural deduction versions of orthodox quantification theory.

Derived Rule 3.3.7 ($\forall$-elimination) *If* $\Gamma \vdash (\forall xS)\phi$ *and* $\Gamma \vdash t_1 S^x_{t_1}$ *then* $\Gamma \vdash \phi^x_{t_1}$, *if* $\phi^x_{t_1}$ *is as stated in* (5).

Proof: By (5), $\vdash (\forall xS)\phi \supset (t_1 S^x_{t_1} \supset \phi^x_{t_1})$.

Derived Rule 3.3.8 ($\exists$-introduction) *If* $\Gamma \vdash \phi^x_{t_1}$ *and* $\Gamma \vdash t_1 S^x_{t_1}$ *then* $\Gamma \vdash (\exists xS)\phi$, *if* $\phi^x_{t_1}$ *is as stated in* (5).

Proof: By (5), $\vdash (\forall xS) \sim \phi \supset (t_1 S^x_{t_1} \supset \sim \phi^x_{t_1})$, hence $\vdash t_1 S^x_{t_1} \supset ((\forall xS) \sim \phi \supset \sim \phi^x_{t_1})$ by **PC**, hence $\vdash t_1 S^x_{t_1} \supset (\phi^x_{t_1} \supset (\exists xS)\phi)$ by **PC**.

Derived Rule 3.3.9 ($\forall$-introduction) *If* $\Gamma \vdash xS \supset \phi$ *then* $\Gamma \vdash (\forall xS)\phi$.

Proof: By Gen.

Derived Rule 3.3.10 ($\exists$-elimination) *If* $\Gamma \vdash (\exists xS)\phi$ *and* $\Gamma \vdash cS^x_c \;\&\; \phi^x_c \supset P$, *where* c *is any individual constant which does not occur in* Γ, *then* $\Gamma \vdash P$.

Proof: In the proof $\Gamma \vdash cS^x_c \;\&\; \phi^x_c \supset P$, replace every occurrence of c by a variable y which does not occur anywhere in the proof, then we have a proof $\Gamma \vdash yS^x_y \;\&\; \phi^x_y \supset P$, hence $\Gamma, \sim P \vdash yS^x_y \supset \sim \phi^x_y$ by **PC**, hence $\Gamma, \sim P \vdash (\forall yS^x_y) \sim \phi^x_y$ by Gen, hence $\Gamma \vdash \sim P \supset (\forall yS^x_y) \sim \phi^x_y$ by the deduction theorem (DT) since y is not in P, hence $\Gamma \vdash \sim P \supset (\forall xS) \sim \phi$ (by alphabetic change of bound variable from y to x), hence $\Gamma \vdash (\exists xS)\phi \supset P$ by **PC**, so if $\Gamma \vdash (\exists xS)\phi$ then $\Gamma \vdash P$.

Theorem 3.3.11 $\vdash (\forall xS)P \equiv (\forall xS)(xS \supset P)$.

Proof: If $(\forall xS)P$ then $xS \supset P$ by (5), hence $(\forall xS)(xS \supset P)$ by Gen′. If $(\forall xS)(xS \supset P)$ then $xS \supset (xS \supset P)$ by (5), hence $xS \supset P$ by **PC**, hence $(\forall xS)P$ by Gen. DT applies in both directions.

Theorem 3.3.12 $\vdash (\exists xS)P \equiv (\exists xS)(xS \;\&\; P)$.

Proof: Put $\sim P$ for P in 3.3.11.

Theorem 3.3.13 $\vdash (\forall xS)P \supset (\forall xT)(xS \supset P)$.

Proof: If $(\forall xS)P$ then $xS \supset P$ by (5), hence $(\forall xT)(xS \supset P)$ and DT applies.

Theorem 3.3.14 $\vdash (\exists xT)(xS \;\&\; P) \supset (\exists xS)P$.

Proof: Put $\sim P$ for P in 3.3.13.

Theorem 3.3.15 $\vdash S \subseteq T \supset ((\forall xT)(xS \supset P) \supset (\forall xS)P)$.

Proof: If $(\forall xS)xT$ and $(\forall xT)(xS \supset P)$ then $xS \supset xT$ and $xT \supset (xS \supset P)$ by (5), hence $xS \supset P$ by **PC**, hence $(\forall xS)P$ by Gen, and DT applies.

Theorem 3.3.16 $\vdash S \subseteq T \supset ((\exists xS)P \supset (\exists xT)(xS \;\&\; P))$.

Proof: Put $\sim P$ for P in 3.3.15.

Theorems 3.3.13 and 3.3.14 show that the sortally-restricted quantifiers of **SQT** have almost the effect of the unrestricted quantifiers of orthodox quantification theory, for the sortal term T appearing in them can be any arbitrary sortal term quite independent of the sortal term S. But the need for the hypothesis $S \subseteq T$ in the converses, 3.3.15 and 3.3.16, shows that the effect is not exactly the same as that of unrestricted quantifiers.

Theorem 3.3.17 $\vdash (\forall xS) \sim xT \equiv (\forall xT) \sim xS$.

Proof: If $(\forall xS) \sim xT$, then $xS \supset \sim xT$ by (5), hence $xT \supset \sim xS$ by **PC**, hence $(\forall xT) \sim xS$ by Gen, and DT applies.

Theorem 3.3.18 $\vdash (\exists xS)xT \equiv (\exists xT)xS$.

Proof: From 3.3.17. $\vdash \sim (\forall xS) \sim xT \equiv \sim (\forall xT) \sim xS$.

The four formulas $(\forall xS)xT$, $(\exists xS)xT$, $(\forall xS) \sim xT$, and $(\exists xS) \sim xT$ have all the logical relations of the A, I, E, and O forms in the traditional square of opposition, because of the non-emptiness of each sort ensured by axiom schema (8); *cf.* [16].

3.4 *Sortal-Relative Identity* The axiom schemas (6) and (7) are formalizations of the notion of sortal-relative identity—(6) states that only an S can be the same S as something, and (7) is a formulation of Leibniz's law of the indiscernability of identicals. But since we have defined tS (t is an S) as $t \underset{S}{=} t$ (t is the same S as itself), and since sortal predications of the form tS play a vital role in other axiom schemas such as (5), it is impossible to have a sortal quantification theory without having sortal-relative identity built into it. In this respect sortal quantification and identity is fundamentally different from orthodox quantification and identity. Of course, we *could* take sortal predications of the form tS as primitive, but they would then be not theoretically distinguishable from arbitrary one-place predications.

Theorem 3.4.1 $\vdash (\forall xS)(x \underset{S}{=} x)$. (Reflexivity of sortal identity)

Proof: $\vdash x \underset{S}{=} x \supset x \underset{S}{=} x$ by **PC**, hence $\vdash xS \supset x \underset{S}{=} x$ by definition of xS, hence $\vdash (\forall xS)(x \underset{S}{=} x)$ by Gen.

Theorem 3.4.2 $\vdash (t_1 \underset{S}{=} t_2) \supset (t_2 \underset{S}{=} t_1)$ *for any individual terms* t_1, t_2.
(Symmetry of sortal identity)

Proof: Putting $x \underset{S}{=} x$ for ϕ in (7), we have $\vdash x \underset{S}{=} y \supset (x \underset{S}{=} x \supset y \underset{S}{=} x)$; by (6) $\vdash x \underset{S}{=} y \supset xS$, and by 3.4.1. $\vdash xS \supset x \underset{S}{=} x$, hence $\vdash x \underset{S}{=} y \supset x \underset{S}{=} x$ by **PC**; so $\vdash x \underset{S}{=} y \supset y \underset{S}{=} x$. By Gen′ $\vdash (\forall xS)(\forall yS)(x \underset{S}{=} y \supset y \underset{S}{=} x)$, hence by (5) and MP and **PC** (provided x and y are not in S) $\vdash t_1S \;\&\; t_2S \supset (t_1 \underset{S}{=} t_2 \supset t_2 \underset{S}{=} t_1)$, but by (6) $\vdash t_1 \underset{S}{=} t_2 \supset t_1S \;\&\; t_2S$, hence by **PC**, $\vdash t_1 \underset{S}{=} t_2 \supset t_2 \underset{S}{=} t_1$. If S is not closed we can always find variables which do not occur in S—and this applies to all the following proofs.

Theorem 3.4.3 $\vdash (t_1 \underset{S}{=} t_2 \supset (t_2 \underset{S}{=} t_3 \supset t_3 \underset{S}{=} t_1))$ *for any individual terms* t_1, t_2, t_3.
(Transitivity of sortal identity)

Proof: Putting $x \underset{S}{=} z$ for ϕ in (7), we have $\vdash x \underset{S}{=} y \supset (x \underset{S}{=} z \supset y \underset{S}{=} z)$; $\vdash y \underset{S}{=} x \supset x \underset{S}{=} y$ by 3.4.2, hence $\vdash y \underset{S}{=} x \supset (x \underset{S}{=} z \supset y \underset{S}{=} z)$ by **PC**. By Gen′ $\vdash (\forall yS)(\forall xS)(\forall zS)(y \underset{S}{=} x \supset (x \underset{S}{=} z \supset y \underset{S}{=} z))$, hence by (5), MP and **PC** $\vdash t_1S \;\&\; t_2S \;\&\; t_3S \supset (t_1 \underset{S}{=} t_2 \supset (t_2 \underset{S}{=} t_3 \supset t_1 \underset{S}{=} t_3))$, but by (6) $t_1 \underset{S}{=} t_2 \supset t_1S \;\&\; t_2S$ and $t_2 \underset{S}{=} t_3 \supset t_2S \;\&\; t_3S$, hence by **PC**, $\vdash (t_1 \underset{S}{=} t_2 \supset (t_2 \underset{S}{=} t_3 \supset t_3 \underset{S}{=} t_1))$.

Theorem 3.4.4 $\vdash xS \equiv (\exists yS)(x \underset{S}{=} y)$ (*x is an S iff x is the same S as some S*).

Proof: By (5) $\vdash (\forall yS) \sim x \underset{S}{=} y \supset (xS \supset \sim x \underset{S}{=} x)$, hence by **PC** $\vdash xS \;\&\; x \underset{S}{=} x \supset (\exists yS) x \underset{S}{=} y$; but $\vdash xS \supset x \underset{S}{=} x$, therefore $\vdash xS \supset (\exists yS)(x \underset{S}{=} y)$. Conversely, $\vdash x \underset{S}{=} y \supset xS$ by (6), hence $\vdash \sim xS \supset \sim x \underset{S}{=} y$ by **PC**, hence $\vdash (\forall yS)(\sim xS \supset \sim x \underset{S}{=} y)$ by Gen′, hence $\vdash \sim xS \supset (\forall yS) \sim x \underset{S}{=} y$ by (4) since y is not free in xS, hence $\vdash (\exists yS)(x \underset{S}{=} y) \supset xS$ by **PC**.

Theorem 3.4.5 $\vdash t_1 \underset{S}{=} t_2 \;\&\; t_1T \supset t_1 \underset{T}{=} t_2$, *for any individual terms t_1 and t_2, any sortal terms S and T.*

Proof: Putting $x \underset{T}{=} x$ for ϕ in (7), we have $\vdash x \underset{S}{=} y \supset (x \underset{T}{=} x \supset x \underset{T}{=} y)$, but by 3.4.1 $\vdash xT \supset x \underset{T}{=} x$, so by **PC** $\vdash x \underset{S}{=} y \;\&\; xT \supset x \underset{T}{=} y$. By Gen′ $\vdash (\forall xS)(\forall yS)(x \underset{S}{=} y \;\&\; xT \supset x \underset{T}{=} y)$, hence by (5), MP, & **PC**, $t_1S \;\&\; t_2S \supset (t_1 \underset{S}{=} t_2 \;\&\; t_1T \supset t_1 \underset{T}{=} t_2)$, but by (6), $\vdash t_1 \underset{S}{=} t_2 \supset t_1S \;\&\; t_2S$, hence by **PC** $\vdash t_1 \underset{S}{=} t_2 \;\&\; t_1T \supset t_1 \underset{S}{=} t_2$. This is a formal proof, from Leibniz's Law, that it is not possible for t_1 to be the same A as t_2 and yet not the same B as t_2, where 'B' is another sortal which applies to t_1. Thus on this question we agree with [21], §1.2, and differ from [4], p. 157; see [17]. There are several different ways in which $t_1 = t_2$ may be false (*cf.* [21], §1.3): t_1 and t_2 may both be S's and yet not the same S; or one of them may be an S and the other not an S, in which case they cannot of course be the same S; or they may both not be S's, in which case they may or may not be the same T (for some other sortal term 'T'). We may want to pick out the first case from the others, and to do this we can use the following abbreviation:

$$t_1 \underset{S}{\neq} t_2 \text{ for } t_1S \;\&\; t_2S \;\&\; \sim(t_1 \underset{S}{=} t_2).$$

$t_1 \underset{S}{\neq} t_2$ should be read '*t_1 is a different S from t_2*'; it implies, but is not implied by, $\sim(t_1 \underset{S}{=} t_2)$, which means only that it is not the case that t_1 is the same S as t_2.

3.5 *Ultimate Sortals* Our definition of the notion of sortal term, together with our axiom schemas (9) and (10), formalize the principles that every individual falls under an ultimate sortal and that every sortal is subordinate to some ultimate sortal. In the first case the ultimate sortal may be said to give the criterion of identity of the individual, since $t \underset{Ut}{=} t$, and in the second it may be said to give the criterion of identity given by the subordinate sortal, since $(\forall xS)(x \underset{US}{=} x)$. Axiom schema (11) adds the principle that if t is an S, then the criterion of identity of t is that given by 'S'. These intuitively plausible principles are enough to generate all the properties we expect of ultimate sortals.

Theorem 3.5.1 $\vdash S = T$ *is an equivalence relation between sortal terms.*

Proof: $\vdash S = T$ iff $\vdash xS \equiv xT$, from its definition, and the latter is an equivalence relation, by **PC**.

Theorem 3.5.2 $\vdash t_1 \underset{S}{=} t_2 \supset \mathsf{U}_{t_1} = \mathsf{U}_{t_2}$ *for any individual terms* t_1, t_2.
(If t_1 is the same S as t_2, then t_1 is of the same ultimate sort as t_2.)

Proof: By (6), if $t_1 \underset{S}{=} t_2$ then t_1S and t_2S, hence by (11) $\mathsf{U}_{t_1} = \mathsf{U}_S$ and $\mathsf{U}_{t_2} = \mathsf{U}_S$. hence by 3.5.1. $\mathsf{U}_{t_1} = \mathsf{U}_{t_2}$.

Theorem 3.5.3 $\vdash S \mathsf{I} T \supset \mathsf{U}_S = \mathsf{U}_T$.
(Intersecting sortals have coextensional ultimate sortals)

Proof: If $(\exists xS)xT$, let c be an individual constant and suppose cS & cT, then by (11), $\mathsf{U}_c = \mathsf{U}_S$ and $\mathsf{U}_c = \mathsf{U}_T$, hence by 3.5.1. $\mathsf{U}_S = \mathsf{U}_T$. So by $\exists$-elimination (3.3.10), $\vdash (\exists xS)xT \supset \mathsf{U}_S = \mathsf{U}_T$.

Theorem 3.5.4 $\vdash S = T \supset \mathsf{U}_S = \mathsf{U}_T$.
(Coextensional sortals have coextensional ultimate sortals)

Proof: If $S = T$, then $S\mathsf{I}T$ since $(\exists xS)xS$ by (8). Hence by 3.5.3.

Theorem 3.5.5 $\vdash \mathsf{u}(S)$ & $\mathsf{u}(T)$ & $\sim S = T \supset \sim S\mathsf{I}T$.
(Any two ultimate sortals which are not coextensional are disjoint)

Proof: Suppose $\mathsf{u}(S)$ & $\mathsf{u}(T)$ & $S\mathsf{I}T$, then $\mathsf{U}_S = \mathsf{U}_T$ by 3.5.3., but since $\mathsf{u}(S)$ and $\mathsf{u}(T)$, $\mathsf{U}_S = S$ and $\mathsf{U}_T = T$, hence $S = T$ by 3.5.1.

Theorem 3.5.6 $\vdash \mathsf{u}(\mathsf{U}_t)$ *and* $\vdash \mathsf{u}(\mathsf{U}_S)$.
(For any individual term t, and any sortal term S)

Proof: $\vdash t\mathsf{U}_t$ by (9), hence $\vdash \mathsf{U}_t = \mathsf{U}_{\mathsf{U}_t}$ by (11), i.e., $\mathsf{u}(\mathsf{U}_t)$. $\vdash (\exists xS)xS$ by (8); suppose cS for some constant c, then $c\mathsf{U}_S$ since $S \subseteq \mathsf{U}_S$ by (9), so $\mathsf{U}_c = \mathsf{U}_S$ and $\mathsf{U}_c = \mathsf{U}_{\mathsf{U}_S}$ by 3.5.1, i.e., $\mathsf{u}(\mathsf{U}_S)$. Hence $\vdash \mathsf{u}(\mathsf{U}_S)$ by $\exists$-elimination (3.3.10).

Theorem 3.5.7 *If S is any sortal term, then either S is a sortal constant, or* $\vdash S = \mathsf{U}A$ *for some sortal constant, or* $\vdash S = \mathsf{U}t$ *for some individual term t.*

Proof: This follows immediately from the definition of sortal term and the facts that $\vdash \mathsf{UU}A = \mathsf{U}A$ and $\vdash \mathsf{UU}t = \mathsf{U}t$ by 3.5.6.

4 *Semantics*

4.1 *S-Sets* The essential part of any semantics for **SQT** must be a representation of the distinctive properties of *sorts*—those sets which consist of all the individuals to which a given sortal applies—as opposed to the arbitrary sets which correspond to one-place predicates in the Tarskian semantics for orthodox quantification theory. To represent sorts in a set-theoretic semantics for **SQT** we introduce the purely set-theoretic notion of an S-system. A set S of sets will be called an S-*system* iff the following conditions are satisfied:

(1) Every set in S is non-empty;
(2) If two sets in S have a non-empty intersection, then there is a set of S of which they are both subsets;
(3) Every set in S is a subset of some set in S which is not itself a proper subset of any set in S.

We shall call the sets in an S-system *S-sets* (intuitively, they will play the role of srots). An S-set will be said to be a *US-set* (intuitively, an ultimate sort) iff it is not a proper subset of any S-set in the system. Two S-sets will be said to be *in the same family* iff there is a US-set of which they are both subsets. The union of all the sets in an S-system will be called the *domain* of the system.

For the philosophical motivation behind these definitions see [21], Part Two, but note that the notion of ultimate sort used there is slightly different.

Theorem 4.1.1 *Any two US-sets in an S-system are disjoint.*

Proof: If A and B are two US-sets with a non-empty intersection, then by (2) there is an S-set C such that $A \subseteq C$ and $B \subseteq C$; but if A and B are different sets then one of them, say A, has an element, say a, which the other lacks, so $a \in A$, $a \notin B$, so $a \in C$, so $B \neq C$, so B is a proper subset of C, contradicting the supposition that B is a US-set.

Theorem 4.1.2 *Each S-set in an S-system is included in one and only one US-set, and each element of the domain is a member of one and only one US-set.*

Proof: By (3) each S-set is included in one US-set, and by 4.1.1 it cannot be included in more than one, since each S-set is non-empty by (1). Any element of the domain is in at least one S-set, and hence in at least one US-set, and by 4.1.1 it cannot be in more than one US-set.

Theorem 4.1.3 *The relation 'in the same family' is an equivalence relation over S-sets.*

Proof: It is reflexive, for every S-set is a subset of a US-set by 4.1.2. Clearly it is symmetric. And it is transitive, since each S-set is a subset of only one US-set by 4.1.2, so if A and B have the same ultimate sort, and B and C have, then A and C have the same ultimate sort.

The equivalence classes under this relation may be called *families*; each such family is the set of all subsets of some US-set.

4.2 *S-Interpretations* Given the notion of an S-system, we can now define that of S-interpretation.

An S-interpretation $\mathcal{I}$ of a first-order sortal quantification theory K consists of:

(i) An S-system (call the domain of the S-system D);
(ii) For each sortal constant A in K, an S-set $\mathcal{I}(A)$ of the S-system;

(iii) For each individual constant a in K, an element $\mathscr{I}(a)$ of D;
(iv) For each n-place function constant f_i^n in K, an n-place operation $\mathscr{I}(f_i^n)$ on D, i.e., a function from D^n, the set of all ordered n-tuples of elements of D, into D);
(v) For each n-place predicate constant P_i^n in K, a subset $\mathscr{I}(P_i^n)$ of D^n.

An *evaluation,* in a given S-interpretation, is an assignment of an element of D to each individual variable. (The evaluation can therefore be identified with an infinite sequence of elements of D, not necessarily all different, arranged in the order of the individual variables to which they are assigned.) For a given evaluation **e** in a given S-interpretation $\mathscr{I}$, we recursively define:

(a) a function $\mathbf{e}(t)$ which takes individual terms t as arguments and has values in the domain D of $\mathscr{I}$, as follows:

(i) If t is an individual variable x, then $\mathbf{e}(t)$ is the element of D assigned to x by the evaluation **e**;
(ii) If t is an individual constant a, then $\mathbf{e}(t)$ is $\mathscr{I}(a)$;
(iii) If t is of the form $f_i^n t_1 \ldots t_n$, then $\mathbf{e}(t)$ is the result of applying the n-place operation $\mathscr{I}(f_i^n)$ to the n-tuple $\langle \mathbf{e}(t_1), \ldots, \mathbf{e}(t_n)\rangle$.

(b) a function $\mathbf{e}(S)$ which takes sortal terms S as arguments, and has S-sets in the S-system of $\mathscr{I}$ as values, as follows:

(i) If S is a sortal constant A, then $\mathbf{e}(S)$ is $\mathscr{I}(A)$;
(ii) If S is a sortal term of the form $\mathsf{U}t$, where t is an individual term, then $\mathbf{e}(S)$ is the unique $\mathsf{U}S$-set (in the S-system of $\mathscr{I}$) which contains $\mathbf{e}(t)$;
(iii) If S is a sortal term of the form $\mathsf{U}S'$, where S' is a sortal term, then $\mathbf{e}(S)$ is the unique $\mathsf{U}S$-set (in the S-system of $\mathscr{I}$) which includes $\mathbf{e}(S')$.

Note that for a *closed* individual term t, the value of $\mathbf{e}(t)$ is independent of the evaluation **e**, and depends only on $\mathscr{I}$, so it can be written $\mathscr{I}(t)$. Similarly, for a *closed* sortal term S, $\mathbf{e}(S)$ can be written $\mathscr{I}(S)$.

We can now define what it is for an evaluation **e** in a given S-interpretation $\mathscr{I}$ to *satisfy* a wff P of **SQT**:

(i) If P is an atomic wff of the form $P_i^n t_1 \ldots t_n$, then **e** satisfies P iff the n-tuple $\langle \mathbf{e}(t_1), \ldots, \mathbf{e}(t_n)\rangle$ is in the set $\mathscr{I}(P_i^n)$;
(ii) If P is an atomic wff of the form $t_1 =_S t_2$, then **e** satisfies P iff $\mathbf{e}(t_1)$ and $\mathbf{e}(t_2)$ are the same element of the S-set $\mathbf{e}(S)$;
(iii) If P is of the form $\sim Q$, then **e** satisfies P iff **e** does not satisfy Q;
(iv) If P is of the form $Q \,\&\, R$, then **e** satisfies P iff **e** satisfies Q and **e** satisfies R;
(v) If P is of the form $(\exists xS)Q$, then **e** satisfies P iff some evaluation which assigns an element of $\mathbf{e}(S)$ to the individual variable x, and is the same as **e** for all other individual variables, satisfies Q.

A wff of **SQT** is said to be *true* in a given S-interpretation iff every evaluation in that S-interpretation satisfies it. An S-interpretation is said to be an *S-model* for a given set of wffs of **SQT** iff every wff in the set is true in that S-interpretation, and for a given sortal quantification theory K iff every axiom of K is true in that S-interpretation. A wff of **SQT** is said to be *S-valid* iff it is true in every S-interpretation.

5 *Completeness Proof*

Metatheorem 5.1 *Every theorem of a sortal calculus is S-valid.*

Proof: We show that every logical axiom is S-valid, and that the rules of inference preserve S-validity. Any instance of axiom schemas (1)-(3) is an instance of tautology, and hence S-valid; for (4), suppose an evaluation $\mathbf{e}$ satisfies $(\forall xS)(P \supset Q)$ and P, and let $\mathbf{f}$ be any evaluation which assigns a member of $\mathbf{e}(S)$ to x and is otherwise like $\mathbf{e}$, then $\mathbf{f}$ satisfies $P \supset Q$ and also satisfies P if x is not free in P, therefore $\mathbf{f}$ satisfies Q, so $\mathbf{e}$ satisfies $(\forall xS)Q$; for (5), suppose $\mathbf{e}$ satisfies $(\forall xS)\phi$ and tS_t^x, then $\mathbf{e}(t)$ is in the set $\mathbf{e}(S_t^x)$, so $\mathbf{e}$ must satisfy ϕ_t^x; for (6), suppose $\mathbf{e}$ satisfies $t_1 \underset{S}{=} t_2$, then $\mathbf{e}(t_1)$ and $\mathbf{e}(t_2)$ are the same element of $\mathbf{e}(S)$, so $\mathbf{e}$ satisfies t_1S and t_2S; for (7), suppose $\mathbf{e}$ satisfies $x \underset{S}{=} y$, then $\mathbf{e}(x)$ is the same as $\mathbf{e}(y)$, so if $\mathbf{e}$ satisfies ϕ then $\mathbf{e}$ satisfies $\phi\,\frac{x}{y}$; for (8), every S-set is non-empty, so for any evaluation $\mathbf{e}$ and for any sortal term S, $\mathbf{e}(S)$ is non-empty, so $\mathbf{e}$ satisfies $(\exists xS)xS$; for (9), $\mathbf{e}(\mathsf{U}_t)$ is by definition the $\mathsf{U}S$-set which contains $\mathbf{e}(t)$, so any evaluation $\mathbf{e}$ satisfies $t\mathsf{U}_t$; for (10), $\mathbf{e}(\mathsf{U}_S)$ is by definition the $\mathsf{U}S$-set which includes $\mathbf{e}(S)$, so any evaluation $\mathbf{e}$ satisfies $(\forall xS)x\mathsf{U}_S$; for (11), if $\mathbf{e}$ satisfies tS then $\mathbf{e}(t)$ is in $\mathbf{e}(S)$, so $\mathbf{e}(\mathsf{U}_t)$ is the same $\mathsf{U}S$-set as $\mathbf{e}(\mathsf{U}_S)$, so $\mathbf{e}$ satisfies $(\forall x\mathsf{U}_t)x\mathsf{U}_S$ and $(\forall x\mathsf{U}_S)x\mathsf{U}_t$. Modus Ponens preserves S-validity, for if an evaluation $\mathbf{e}$ satisfies P and $P \supset Q$ it satisfies Q; and Generalization also preserves S-validity, for if every evaluation satisfies $xS \supset P$, then every evaluation $\mathbf{e}$ which assigns an element of $\mathbf{e}(S)$ to x satisfies P, so every evaluation satisfies $(\forall xS)P$.

The other half of the completeness proof—that all S-valid wffs are theorems—takes more effort to prove, as it does in most systems. But we shall see that Henkin's method (see [8], [7], [9], Proposition 2.12) can be adapted to sortal quantification theory. First some definitions:

a first-order sortal quantification theory K will be said to be *negation-complete* iff for any closed wff P of K, either $\vdash_K P$ or $\vdash_K \sim P$. A theory K' with the same vocabulary as K will be said to be an *extension* of K iff every theorem of K is a theorem of K'. We need the following lemmas:

Lemma 5. 2 *The set of wffs of any sortal quantification is denumerable.*

Proof: The set of symbols is denumerable, and each wff is a finite string of symbols, so a Gödel numbering can be used to enumerate them, (*cf.* [9], Lemma 2.10).

Lemma 5.3 *If K is a consistent sortal quantification theory, then there is a consistent, negation-complete extension of K.* (Lindenbaum's Lemma for sortal quantification theories.)

Proof: The standard form of proof works without alteration (see [9], Proposition 2.11). Enumerate all the closed wffs of K in a sequence $P_1, P_2, \ldots, P_k, \ldots$, and add in turn (as a proper axiom) each one of them which is not already provable.

Lemma 5.4 *If* $\vdash_K P(c_1, \ldots, c_n)$ *where* $c_1, \ldots, c_n$ *are n individual constants which do not occur among the proper axioms of K, then* $\vdash_K P(x_1, \ldots, x_n)$ *where* $x_1, \ldots, x_n$ *are any n individual variables which do not occur free in* $P(c_1, \ldots, c_n)$*, and* $P(x_1, \ldots, x_n)$ *is the result of substituting* x_i *for* c_i *everywhere in* $P(c_1, \ldots, c_n)$ *for* $1 \leqslant i \leqslant n$.

Proof: Let $P_1, \ldots, P_m$ be a proof of $P(c_1, \ldots, c_n)$ in K, in which the individual variables $x_1, \ldots, x_n$ do not occur. Replace every occurrence of c_i in the proof by x_i, for $1 \leqslant i \leqslant n$. This transforms logical axioms into logical axioms, for the axiom schemas (1)-(11) make no distinction between constants and variables, it leaves proper axioms of K unchanged, and it preserves the correctness of the applications of the rules of inference. So we thus produce a proof of $P(x_1, \ldots, x_n)$ in K.

Metatheorem 5.5 *Every consistent sortal quantification theory K has an S-model.*

Proof: (I) Add to the symbols of K a denumerable set $\{b_1, b_2, \ldots\}$ of new individual constants. The resulting theory K_0 has as its axioms all the axioms of K plus all logical axioms which involve the new constants. K_0 is consistent. For if not, $\vdash_{K_0} P \;\&\; \sim P$ for some wff P, and then by Lemma 5.4 we could replace every b_i in the proof by an individual variable and thus produce a proof of a contradiction in K, but by hypothesis K is consistent.

(II) We now construct an extension J of K_0 which will be *instantiated*, in the sense that for every closed wff of the form $(\exists xS)\phi(x)$, if $\vdash_J (\exists xS)\phi(x)$ then there is some constant c such that $\vdash_J \phi(c)$. By Lemma 5.2, let $P_1, P_2, \ldots, P_k, \ldots$ be an enumeration of all the closed wffs of K_0 of the form $(\exists xS)\phi$, and let P_k be $(\exists x_k S_k)\phi_k(x_k)$. We define a sequence $b_{j_1}, b_{j_2}, \ldots, b_{j_k}, \ldots$ of the new constants as follows: b_{j_1} is the first one which does not occur in P_1, and b_{j_k} is the first one which does not occur in $P_1, \ldots, P_k$ and is different from $b_{j_1}, \ldots, b_{j_{k-1}}$. For each k, we define the wff S_k to be $(\exists x_k S_k)\phi_k(x_k) \supset \phi_k(b_{j_k})$. Let K_n be the theory obtained from K_0 by adding $S_1, \ldots, S_n$ to its proper axioms, and let K_∞ be that obtained by thus adding all the S_k. To prove that K_∞ is consistent it is sufficient to prove that every K_n is consistent, for any proof of a contradiction in K_∞ would use only a finite number of axioms, and hence would be a proof of a contradiction in some K_n. We proceed by induction. K_0 is consistent, by (I). Suppose that K_n is inconsistent. Then by **PC** any wff is provable in K_n, so in particular $\vdash_{K_n} \sim S_n$, hence $S_n \vdash_{K_{n-1}} \sim S_n$ by definition of K_n, hence $\vdash_{K_{n-1}} S_n \supset \sim S_n$

by the Deduction Theorem, since S_n is closed, hence by **PC** $\vdash_{\overline{K}_{n-1}} \sim S_n$, hence by **PC** $\vdash_{\overline{K}_{n-1}} (\exists x_n S_n)\phi_n(x_n)$ and $\vdash_{\overline{K}_{n-1}} \sim \phi_n(b_{j_n})$. But by definition of b_{j_n} and K_{n-1}, b_{j_n} does not occur among the proper axioms of K_{n-1}, so by Lemma 5.4, $\vdash_{\overline{K}_{n-1}} \sim \phi_n(x_n)$, since x_n is not free in $\sim \phi_n(b_{j_n})$, hence by Gen′ $\vdash_{\overline{K}_{n-1}} (\forall x_n S_n) \sim \phi_n(x_n)$, hence $\vdash_{\overline{K}_{n-1}} \sim (\exists x_n S_n)\phi_n(x_n)$. But $\vdash_{\overline{K}_{n-1}} (\exists x_n S_n)\phi_n(x_n)$, so K_{n-1} is inconsistent. So if K_{n-1} is consistent, then K_n is. So by induction every K_n is consistent, hence K is.

By Lemma 5.3, let J be a consistent negation-complete extension of K. Then J is consistent, negation-complete, and instantiated, for if P is a closed wff of the form $(\exists xS)\phi$, then for some k, P is P_k, so $P_k \supset \phi_k(b_{j_k})$ is a proper axiom of J, so if $\vdash_J P$ then $\vdash_J \phi_k(b_{j_k})$.

(III) We now proceed to construct an S-system. Consider the denumerable set of closed individual terms (cits) of K_0, and consider the relation E which holds between two of these terms t_1 and t_2 iff, for some sortal term S, $\vdash_J t_1 \underset{S}{=} t_2$. By 3.4.2 and 3.4.3 E is symmetric and transitive, and by axiom schema (9) $\vdash_J t \underset{\mathsf{U}t}{=} t$ for any cit t, so E is reflexive; thus E is an equivalence relation. The set D of equivalence classes of cits of K_0 under relation E is to be the domain of our S-system. (We need to deal with these equivalence classes of cits rather than the cits themselves, because in an S-interpretation any sortal-relative identity $\underset{S}{=}$ must be interpreted as identity in that sort.) Let $[t]$ denote the equivalence class to which the cit t belongs; we note that as far as provability in J is concerned, it does not matter which member of each an equivalence class we choose to represent it; i.e., if $[t_1] = [t_2]$, then $\vdash_J \phi(t_1)$ iff $\vdash_J \phi(t_2)$; this follows easily from Leibniz's law as formulated in axiom schema (7).

For each closed sortal term (cst) S let $[S]$ be the set of $[t]$ such that $\vdash_J tS$. We show that the set of such sets $[S]$ is an S-system with domain D. By (9), $\vdash_J t\mathsf{U}_t$ for any cit t, so every member of D is in some $[S]$. Each $[S]$ is non-empty, for by (8) $\vdash_J (\exists xS)xS$, hence for some constant c $\vdash_J cS$ since J is instantiated, so $[c]$ is in $[S]$. Suppose $[S_1]$ and $[S_2]$ have a non-empty intersection, then for some cits t_1 and t_2, $\vdash_J t_1S_1$ and $\vdash_J t_2S_2$ and $[t_1] = [t_2]$, so for some S, $\vdash_J t_1 \underset{S}{=} t_2$; by (11) $\vdash_J \mathsf{U}t_1 = \mathsf{U}S_1$ and $\vdash_J \mathsf{U}t_2 = \mathsf{U}S_2$, and by 3.5.2 $\vdash_J \mathsf{U}t_1 = \mathsf{U}t_2$, so by 3.5.1 $\vdash_J \mathsf{U}S_1 = \mathsf{U}S_2$, so $[\mathsf{U}S_1] = [\mathsf{U}S_2]$, but from (10) $[S_1] \subseteq [\mathsf{U}S_1]$ and $[S_2] \subseteq [\mathsf{U}S_2]$, so we have a set, namely $[\mathsf{U}S_1]$, of which $[S_1]$ and $[S_2]$ are both subsets. To demonstrate the existence of a $\mathsf{U}S$-set including any S-set, by (10) we have $[S_1] \subseteq [\mathsf{U}S_1]$; suppose $[\mathsf{U}S_1]$ is a proper subset of a set $[S_2]$, then $[\mathsf{U}S_1]$ is non-empty, so $[\mathsf{U}S_1]$ and $[S_2]$ have a non-empty intersection, so by the above $[\mathsf{U}\mathsf{U}S_1] = [\mathsf{U}S_2]$, but by 3.5.6 $[\mathsf{U}\mathsf{U}S_1] = [\mathsf{U}S_1]$, so $[\mathsf{U}S_1] = [\mathsf{U}S_2]$, but $[S_2] \subseteq [\mathsf{U}S_2]$, so $[S_2] \subseteq [\mathsf{U}S_1]$ contrary to the hypothesis that $[\mathsf{U}S_1]$ is a proper subset of $[S_2]$; so for any $[S_1]$, $[\mathsf{U}S_1]$ is a $\mathsf{U}S$-set including $[S_1]$. This completes our proof that the $[S]$'s form an S-system.

(IV) We now give an S-interpretation $\mathcal{I}$ for the wffs of J. The S-system of I is that defined in (III) above. For a sortal constant $A, \mathcal{I}(A)$ is the set of $[t]$ such that $\vdash_J tA$; for an individual constant a, $\mathcal{I}(a)$ is $[a]$; for an n-place function constant $f_i^n, \mathcal{I}(f_i^n)$ is the n-place operation which has for arguments

$[t_1] \dots [t_n]$ the value $[f_i^n t_1 \dots t_n]$; for an n-place predicate constant P_i^n, $\mathcal{J}(P_i^n)$ is the set of n-tuples $\langle [t_1] \dots [t_n] \rangle$ such that $\vdash_J P_i^n t_1 \dots t_n$.

(V) We now show that for each closed wff P of J, P is true in this S-interpretation $\mathcal{J}$ iff $\vdash_J P$. The proof is by induction on the number of connectives and quantifiers in P. The induction base is therefore the case in which P is an atomic wff. There are two subcases of this. Subcase 1: P is of the form $P_i^n t_1 \dots t_n$, where $t_1, \dots, t_n$ are cits. Then P is true in $\mathcal{J}$ iff $\vdash_J P_i^n t_1 \dots t_n$, by the definition of $\mathcal{J}$. Subcase 2: P is of the form $t_1 \underset{S}{=} t_2$, where t_1 and t_2 are cits and S is a cst. Then P is true in $\mathcal{J}$ iff $[t_1]$ and $[t_2]$ are the same element of $[S]$, i.e., iff $\vdash_J t_1 S$ and $\vdash_J t_2 S$ and for some sortal term T $\vdash_J t_1 \underset{T}{=} t_2$, i.e., iff $\vdash t_1 \underset{S}{=} t_2$, by 3.4.5 and (6).

For the induction step, suppose as induction hypothesis that for all closed wffs Q with fewer than n connectives and quantifiers, $\vdash_J Q$ iff Q is true in $\mathcal{J}$. Let P be a closed wff with n connectives and quantifiers. There are three cases.

Case 1: P is of the form $\sim Q$. Then by induction hypothesis Q is false iff it is not the case that $\vdash_J Q$, hence iff $\vdash_J \sim Q$, since J is negation-complete, so P is true iff $\vdash_J \sim Q$, i.e., iff $\vdash_J P$.

Case 2: P is of the form Q & R. Then P is true iff Q and R are true, hence iff $\vdash_K Q$ and $\vdash_K R$, by induction hypothesis, hence iff $\vdash_K Q \,\&\, R$, by **PC**, i.e., iff $\vdash_K P$.

Case 3: P is of the form $(\exists xS)Q$, where x is the only variable, if any, which occurs free in Q (since P is closed). Suppose first that $\vdash_J P$, then by 3.3.12 $\vdash_J (\exists xS)(xS \,\&\, Q)$, so since J is instantiated there is some constant c such that $\vdash_J cS \,\&\, Q_c^x$, then by induction hypothesis $\vdash_J cS$ iff cS is true and $\vdash Q_c^x$ iff Q_c^x is true (for Q_c^x is closed), so cS is true and Q_c^x is true, so $(\exists xS)Q$ is true, since any evaluation which assigns $[c]$ to x satisfies Q_c^x, thus P is true. Conversely, suppose that it is not the case that $\vdash_J P$, then since J is negation-complete, $\vdash_J \sim P$, hence $\vdash (\forall xS) \sim Q$, hence by (5) $\vdash_J tS_t^x \supset \sim Q_t^x$ for every cit t, hence for every cit t it is not the case that $\vdash_J tS_t^x$ and $\vdash_J Q_t^x$, since J is consistent; now by induction hypothesis $\vdash_J tS_t^x$ iff tS_t^x is true (for S_t^x is a cst) and $\vdash_J Q_t^x$ iff Q_t^x is true (for Q_t^x is closed), so for every cit t it is not the case that tS_t^x and Q_t^x are true, so $(\exists xS)Q$ is false. So in Case 3, $\vdash_J P$ iff P is true, and our induction is completed.

(VI) We now show that every axiom of K is true in the S-interpretation $\mathcal{J}$, i.e., that $\mathcal{J}$ is an S-model for K. Let P be any axiom of K, then $\vdash_K P$, therefore $\vdash_J P$ since J is an extension of K by (I) and (II). Therefore if P is closed then P is true in $\mathcal{J}$, by (V). If P is not closed, let $x_1 \dots x_n$ be all the individual variables occurring in it, then for any n individual constants $c_1 \dots c_n$, $P_{c_1 \dots c_n}^{x_1 \dots x_n}$ is closed, and $\vdash_J P_{c_1 \dots c_n}^{x_1 \dots x_n}$, since by Gen$'$ from $\vdash_J P$ $\vdash_J (\forall x_1 \mathsf{U}_{c_1}) \dots (\forall x_n \mathsf{U}_{c_n})P$, hence by (5) $\vdash_J c_1 \mathsf{U}_{c_1} \,\&\, \dots \,\&\, c_n \mathsf{U}_{c_n} \supset P_{c_1 \dots c_n}^{x_1 \dots x_n}$, hence by (9) $\vdash_J P_{c_1 \dots c_n}^{x_1 \dots x_n}$. So $\vdash_J P_{c_1 \dots c_n}^{x_1 \dots x_n}$ is true in $\mathcal{J}$ for *any* individual constants $c_1 \dots c_n$, so P is true in $\mathcal{J}$, since it will be satisfied by any evaluation in $\mathcal{J}$.

Metatheorem 5.6 *Any S-valid wff of a first-order sortal quantification theory K is a theorem of K.*

Proof: Suppose first that P is a *closed* S-valid wff of K. Then if P is not a theorem of K, then the theory K' which is obtained from K by adding $\sim P$ as an extra axiom is consistent, hence by 5.5 it has an S-model, so $\sim P$ is true in this S-model, so P is false in it. But since P is S-valid this is impossible, so P must be a theorem of K. If P is not closed, let $x_1 \ldots x_n$ be all the individual variables occurring free in P, and let $c_1 \ldots c_n$ be any n individual constants which do not occur in P or in the proper axioms of K then $P^{x_1 \ldots x_n}_{c_1 \ldots c_n}$ is closed and S-valid, so $\vdash_K P^{x_1 \ldots x_n}_{c_1 \ldots c_n}$ by the above argument. Hence $\vdash_K P$ by Lemma 5.4.

Metatheorem 5.7 *In any first-order sortal calculus, a wff is a theorem iff it is S-valid.*

Proof: From 5.1 and 5.6.

6 Derivation of Unrestricted Quantification The bound individual variables of sortal quantification theory are restricted in their range by the sortal term which appears in the relevant quantifier, when that sortal term is a closed one. But any individual variables which occur free are in effect unrestricted, as are the individual constants, although for any individual variable or constant t there is an ultimate sortal $\mathsf{U}t$ which gives the corresponding criterion of identity, by axiom schema (9). Consider a wff of the form $(\forall x \mathsf{U}_x)\phi(x)$; it says, in effect, that any individual x, with its corresponding criterion of identity given by its ultimate sortal U_x, satisfies the condition $\phi(x)$. So the variable here is really not restricted to any particular sort, and we have a version of unrestricted quantification. If we define $(\forall x)\phi$ as $(\forall x \mathsf{U}_x)\phi$, we then have every wff of ordinary unrestricted quantification theory **QT** definable in **SQT**, and we shall now show that all the theorems of **QT** are theorems of **SQT**, taking as our standard formulation of **QT** that in [9], Chapter 1, section 3.

Metatheorem 6.1 *If* $\vdash_{\mathbf{QT}} P$ *then* $\vdash_{\mathbf{SQT}} P$.

Proof: We show that the axioms and rules of **QT** ([9], p. 57) are theorems and rules of **SQT**. The three schema for propositional calculus are common to both. The schema $(\forall x)(P \supset Q) \supset (P \supset (\forall x)Q)$ if P is not free in Q, is just the schema $(\forall x \mathsf{U}_x)(P \supset Q) \supset (P \supset (\forall x \mathsf{U}_x)Q)$ in **SQT**. The schema $(\forall x)\phi \supset \phi^x_t$ is derivable from (5) in **SQT**, since $(\forall x \mathsf{U}_x)\phi \supset (t\mathsf{U}_t \supset \phi^x_t)$ is a special case of the latter, and $t\mathsf{U}_t$ is axiom schema (9) of **SQT**, and the restriction on t is the same in both cases. The rule of Modus Ponens is common to **QT** and **SQT**, and the rule of Generalization (from P to $(\forall x)P$) in **QT** is just a special case of Gen′ in **SQT** (from P to $(\forall x \mathsf{U}_x)P$).

A similar derivation of unrestricted identity can be made, if we define $t_1 = t_2$ as $t_1 \underset{\mathsf{U}t_1}{=} t_2$ (there is no genuine asymmetry about this definition, for if $t_1 \underset{\mathsf{U}t_1}{=} t_2$, then $t_2 \underset{\mathsf{U}t_1}{=} t_1$ by 3.4.1, and $t_2\mathsf{U}_{t_2}$ by (9), so $t_2 \underset{\mathsf{U}t_2}{=} t_1$ by 3.4.5). All the theorems of **QT** = with identity ([9], Ch. 2, §8) are then provable in **SQT**.

Metatheorem 6.2 *If* $\vdash_{\mathbf{QT=}} P$ *then* $\vdash_{\mathbf{SQT}} P$.

Proof: We need only show that the schemas $(\forall x)(x = x)$ and $x = y \supset \left(\phi \supset \phi\frac{x}{y}\right)$ are derivable in **SQT**. The first is by definition $(\forall x\mathsf{U}_x)(x \underset{\mathsf{U}x}{=} x)$, which is provable by Gen′ from the form $x \underset{\mathsf{U}x}{=} x$ of (9). The second is by definition $x \underset{\mathsf{U}x}{=} y \supset \left(\phi \supset \phi\frac{x}{y}\right)$, which is a case of (7), with the same restriction on substitutions.

We can also prove in **SQT** the standard equivalences relating unrestricted quantification and identity, as defined above, to sortally-restricted quantification and sortal-relative identity.

Theorem 6.3 $(\forall xS)\phi \equiv (\forall x)(xS \supset \phi)$.

Proof: If $(\forall xS)\phi$, then $xS \supset \phi$ by (5), hence $(\forall x\mathsf{U}_x)(xS \supset \phi)$ by Gen′, and DT applies. If $(\forall x\mathsf{U}_x)(xS \supset \phi)$ then $xS \supset \phi$ as in 6.1, hence $(\forall xS)\phi$ by Gen, and DT applies.

Theorem 6.4 $t_1 \underset{S}{=} t_2 \equiv t_1 = t_2 \ \&\ t_1S$.

Proof: If $t_1 \underset{S}{=} t_2$, then t_1S, and $t_1\mathsf{U}_{t_1}$ by (9), hence $t_1 \underset{\mathsf{U}t_1}{=} t_2$ by 3.4.5. If $t_1 \underset{\mathsf{U}t_1}{=} t_2$ and t_1S then $t_1 \underset{S}{=} t_2$ by 3.4.5.

7 *Second-Order Sortal Quantification Theory* We introduced ultimate sortals into our formal theory **SQT** by use of a primitive symbol U, which acts as a function from individual terms and sortal constants to ultimate sortals. The same effect can be achieved by the use of variables ranging over sortals, and appropriate axioms involving them. We define the theory **SQT 2** by the following amendments to **SQT**:

Symbols Add denumerably many sortal variables $S, S_1, S_2, \ldots$ The symbol U will be defined rather than primitive.

Sortal Terms A sortal term will now be simply either a sortal constant or a sortal variable.

Wffs

Sortal terms are as above.
If P is a wff, and S is any sortal variable, then $(\exists S)P$ is a wff.
$(\forall S)P$ is defined as $\sim(\exists S) \sim P$.
$\mathsf{u}(S)$ is defined as $(\forall S_1)(S \subseteq S_1 \supset S = S_1)$.

Logical Axioms Delete (9), (10), and (11), and add:

(9)′ $(\exists S)tS$ *for any individual term* t.
(10)′ $(\forall S)(\exists S_1)(S \subseteq S_1 \ \&\ \mathsf{u}(S_1))$.
(11)′ $S_1 \mathsf{I} S_2 \supset (\exists S_3)(S_1 \subseteq S_3 \ \&\ S_2 \subseteq S_3)$.
(12)′ $(\forall S)(P \supset Q) \supset (P \supset (\forall S)Q)$, *if* S *is not free in* P.
(13)′ $(\forall S)\phi \supset \phi^S_T$, *where* T *is any sortal term, and* ϕ^S_T *is the result of*

replacing S by T in ϕ, provided no such replacement yields a bound occurrence of T.

Rules of Inference Add *Gen-2*: $(\forall S)P$ follows from P, for any sortal variable S. Axiom schema (12) and (13), and the Rule Gen-2, simply introduce standard quantificational reasoning for the sortal variables. Axiom schema (9)′, (10′), and (11)′ embody the principles concerning sortals and ultimate sortals which we have been representing in **SQT** and in S-sets: that every individual falls under some sortal, that every sortal is subordinate to an ultimate sortal, and that intersecting sortals are subordinates of a common sortal. It is easy to prove from these axioms, just as we did for S-sets in section **4.1**, that any two ultimate sortals are disjoint, and hence that each individual falls under one and only one ultimate sortal and that each sortal is subordinate to one and only one ultimate sortal. Hence we can define the function U, which gives the ultimate sortal of each individual term and each sortal term, and thus derive **SQT** within **SQT 2**.

In this second-order theory **SQT 2** we can use an idea of Geach's ([4], §93, p. 154) to define unrestricted quantification in terms of a double sortal quantification of first and second order. We define $(\exists x)\phi$ as $(\exists S)(\exists xS)\phi$, where S is the first sortal variable which does not occur free in ϕ. We can also use an idea of Wiggins' ([20], p. 27) to define unrestricted identity, defining $t_1 = t_2$ as $(\exists S)(t_1 \underset{S}{=} t_2)$. We can then show easily that **SQT 2** contains the whole of standard quantification theory with identity, by deriving the standard axioms and rules in **SQT 2** as we did in **SQT** in section **6**. The equivalences $(\forall xS)\phi \equiv (\forall x)(xS \supset \phi)$ and $t_1 \underset{S}{=} t_2 \equiv t_1 = t_2 \ \& \ t_1 S$ will be similarly derivable. **SQT 2** has the same semantics as **SQT**; it is just a less economical way of expressing the same principles concerning sortals.

8 *Concluding Remarks* What then does all this formal development amount to? Does the derivability of unrestricted quantification and identity, and the provability of the equivalences $(\forall xS)\phi \equiv (\forall x)(xS \supset \phi)$ and $(t_1 \underset{S}{=} t_2) \equiv (t_1 S \ \& \ t_1 = t_2)$, show that sortal quantification theory **SQT** is a mere notational alternative to orthodox quantification theory **QT**, and so is of no philosophical importance? (This is the view that Quine takes of the standard many-sorted theories, [12], pp. 92, 96.)

On our view, every individual has a criterion of identity, so every individual falls under some sortal term; and the notion of criterion of identity as given by ultimate sortals involves the structure represented in **SQT** and in S-sets. **QT** is a calculus which deals with the unrestricted notions of quantification, identity, and individual. If it is accepted on philosophical grounds that the fundamental notions of quantification, identity, and individual are sortal-relative or sortally-restricted, and that the unrestricted notions are definable in terms of the restricted ones (just as the property 'is loved' is definable as 'is loved by someone'—in terms of the relation 'is loved by'), then the above equivalences will be taken as showing that **QT** is definable in terms of **SQT**, rather than the reverse. We can thus offer a foundation for **QT** rather in the way that Frege and Russell offered a foundation for arithmetic.

The motivation for investigating foundations is often ontological, and this case is no exception. Much philosophical controversy has centered around the supposed ontological commitments of **QT** or theories expressed in **QT**; light may be cast on this if we define the ontological commitment of a sortal quantification theory as the ultimate sorts of the individual and sortal terms in its theorems. But the philosophical defence of such a definition lies beyond the scope of this paper.

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