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A SHORT POSTULATE-SYSTEM FOR ORTHOLATTICES

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By definition, *cf.*, [1], p. 52, an ortholattice is a lattice with universal bounds and a unary operation $^\perp$ satisfying:

L1 $[a]: a \epsilon A . \supset . a \cap a^\perp = 0$
L2 $[a]: a \epsilon A . \supset . a \cup a^\perp = 1$
L3 $[ab]: a, b \epsilon A . \supset . (a \cup b)^\perp = a^\perp \cap b^\perp$
L4 $[ab]: a, b \epsilon A . \supset . (a \cap b)^\perp = a^\perp \cup b^\perp$
L5 $[a]: a \epsilon A . \supset . a = (a^\perp)^\perp$

In this note it will be proved that:

(**A**) *Any algebraic system*

$$\mathfrak{A} = \langle A, \cup, \cap, ^\perp \rangle$$

where $\cup$ *and* $\cap$ *are two binary operations and* $^\perp$ *is a unary operation defined on the carrier set A, is an ortholattice, if it satisfies the following four mutually independent postulates:*

B1 $[ab]: a, b \epsilon A . \supset . a \cup b = b \cup a$
B2 $[ab]: a, b \epsilon A . \supset . a = a \cap (a \cup b)$
B3 $[ab]: a, b \epsilon A . \supset . a = a \cup (b \cap b^\perp)$
B4 $[abc]: a, b, c \epsilon A . \supset . (a \cup b) \cup c = ((c^\perp \cap b^\perp) \cap a^\perp)^\perp$[1]

Proof:

1 Since it is self-evident that formulas *B1-B4* hold in the field of any ortholattice, only a converse should be proved. Hence, let us assume *B1-B4*. Then:

B5 $[ab]: a, b \epsilon A . \supset . a \cap a^\perp = b \cap b^\perp$
$[B3, a/a \cap a^\perp; B1, a/a \cap a^\perp, b/b \cap b^\perp; B3, a/b \cap b^\perp, b/a]$
D1 $[a]: a \epsilon A . \supset . a \cap a^\perp = 0$ $[B5]$
B6 $[a]: a \epsilon A . \supset . a = a \cup 0$ $[B3; D1, a/b]$
B7 $[a]: a \epsilon A . \supset . a = a \cap a$ $[B2, b/0; B6]$
B8 $[a]: a \epsilon A . \supset . a = 0 \cup a$ $[B6; B1, b/0]$

1. Of course, in this postulate-system, the operations $\cup$, $\cap$, and $^\perp$ are not mutually independent.

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B9 $[a]: a \epsilon A . \supset . 0 = 0 \cap a$ $[B2, a/0, b/a; B8]$

B10 $0 = (0^{\perp})^{\perp}$

PR $0 = 0 \cup 0 = (0 \cup 0) \cup 0$ $[B6, a/0; B6, a/0 \cup 0]$

$= ((0^{\perp} \cap 0^{\perp}) \cap 0^{\perp})^{\perp}$ $[B4, a/0, b/0, c/0]$

$= (0^{\perp} \cap 0^{\perp})^{\perp} = (0^{\perp})^{\perp}$ $[B7, a/0^{\perp}; B7, a/0^{\perp}]$

D2 $0^{\perp} = 1$ $[D1; B5]$

B11 $1^{\perp} = 0$ $[D2; B10]$

B12 $[a]: a \epsilon A . \supset . a \cup 1 = 1$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$a \cup 1 = (a \cup 0) \cup 1 = ((1^{\perp} \cap 0^{\perp}) \cap a^{\perp})^{\perp}$ $[1; B6; B4, b/0, c/1]$

$= ((0 \cap 0^{\perp}) \cap a^{\perp})^{\perp} = (0 \cap a^{\perp})^{\perp}$ $[B11; B9, a/0^{\perp}]$

$= (0)^{\perp} = 1$ $[B9, a/a^{\perp}; D2]$

B13 $[a]: a \epsilon A . \supset . a = a \cap 1$ $[B2, b/1; B12]$

B14 $[ab]: a, b \epsilon A . \supset . a \cup b = (b^{\perp} \cap a^{\perp})^{\perp}$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a \cup b = (0 \cup a) \cup b = ((b^{\perp} \cap a^{\perp}) \cap 0^{\perp})^{\perp}$ $[1; B8; B4, a/0, b/a, c/b]$

$= ((b^{\perp} \cap a^{\perp}) \cap 1)^{\perp} = (b^{\perp} \cap a^{\perp})^{\perp}$ $[D2; B13, a/b^{\perp} \cap a^{\perp}]$

B15 $[a]: a \epsilon A . \supset . a = (a^{\perp})^{\perp}$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$a = 0 \cup a = (a^{\perp} \cap 0^{\perp})^{\perp} = (a^{\perp} \cap 1)^{\perp} = (a^{\perp})^{\perp}$

$[1; B8; B14, a/0, b/a; D2; B13, a/a^{\perp}]$

B16 $[a]: a \epsilon A . \supset . a \cup a^{\perp} = 1$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$a \cup a^{\perp} = ((a^{\perp})^{\perp} \cap a^{\perp})^{\perp} = (a \cap a^{\perp})^{\perp}$ $[1; B14; b/a^{\perp}; B15]$

$= 0^{\perp} = 1$ $[D1; D2]$

B17 $[a]: a \epsilon A . \supset . a = a \cup a$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$a = (a^{\perp})^{\perp} = (a^{\perp} \cap a^{\perp})^{\perp} = a \cup a$ $[1; B15; B7, a/a^{\perp}; B14, b/a]$

B18 $[ab]: a, b \epsilon A . \supset . (a \cup b)^{\perp} = a^{\perp} \cap b^{\perp}$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$(a \cup b)^{\perp} = (b \cup a)^{\perp} = ((a^{\perp} \cap b^{\perp})^{\perp})^{\perp}$ $[1; B1; B14, a/b, b/a]$

$= a^{\perp} \cap b^{\perp}$ $[B15, a/a^{\perp} \cap b^{\perp}]$

B19 $[ab]: a, b \epsilon A . \supset . a \cap b = b \cap a$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a \cap b = (a^{\perp})^{\perp} \cap (b^{\perp})^{\perp} = (a^{\perp} \cup b^{\perp})^{\perp}$ $[1; B15; B15, a/b; B18, a/a^{\perp}, b/b^{\perp}]$

$= (b^{\perp} \cup a^{\perp})^{\perp} = (b^{\perp})^{\perp} \cap (a^{\perp})^{\perp}$ $[B1, a/a^{\perp}, b/b^{\perp}; B18, a/b^{\perp}, b/a^{\perp}]$

$= b \cap a$ $[B15, a/b; B15]$

B20 $[ab]: a, b \epsilon A . \supset . (a \cap b)^{\perp} = a^{\perp} \cup b^{\perp}$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$(a \cap b)^{\perp} = (b \cap a)^{\perp} = ((b^{\perp})^{\perp} \cap (a^{\perp})^{\perp})^{\perp}$ $[1; B19; B15, a/b; B15]$

$= a^{\perp} \cup b^{\perp}$ $[B14, a/a^{\perp}, b/b^{\perp}]$

B21 $[ab]: a, b \epsilon A . \supset . a = a \cup (a \cap b)$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$a = (a^{\perp})^{\perp} = (a^{\perp} \cap (a^{\perp} \cup b^{\perp}))^{\perp}$ $[1; B15; B2, a/a^{\perp}, b/b^{\perp}]$

$= (a^{\perp})^{\perp} \cup (a^{\perp} \cup b^{\perp})^{\perp}$ $[B20, a/a^{\perp}, b/a^{\perp} \cup b^{\perp}]$

$= a \cup ((a^{\perp})^{\perp} \cap (b^{\perp})^{\perp}) = a \cup (a \cap b)$ $[B15; B18, a/a^{\perp}, b/b^{\perp}; B15; B15, a/b]$

B22 $[abc]: a, b, c \epsilon A . \supset . (a \cup b) \cup c = a \cup (b \cup c)$

PR $[abc]:\mathrm{Hp}(1).\supset.$

$(a \cup b) \cup c = ((c^{\perp} \cap b^{\perp}) \cap a^{\perp})^{\perp}$	$[1; B4]$
$= (a^{\perp} \cap (c^{\perp} \cap b^{\perp}))^{\perp}$	$[B19, a/c^{\perp} \cap b^{\perp}, b/a^{\perp}]$
$= (a^{\perp})^{\perp} \cup (c^{\perp} \cap b^{\perp})^{\perp}$	$[B20, a/a^{\perp}, b/c^{\perp} \cap b^{\perp}]$
$= a \cup (b \cup c)$	$[B15; B14, a/b, b/c]$

B23 $[abc]:a,b,c \in A.\supset.\ (a \cap b) \cap c = a \cap (b \cap c)$

PR $[abc]:\mathrm{Hp}(1).\supset.$

$(a \cap b) \cap c = (((a \cap b) \cap c)^{\perp})^{\perp}$	$[1; B15, a/(a \cap b) \cap c]$
$= ((a \cap b)^{\perp} \cup c^{\perp})^{\perp}$	$[B20, a/a \cap b, b/c]$
$= ((a^{\perp} \cup b^{\perp}) \cup c^{\perp})^{\perp}$	$[B20]$
$= (a^{\perp} \cup (b^{\perp} \cup c^{\perp}))^{\perp}$	$[B22, a/a^{\perp}, b/b^{\perp}, c/c^{\perp}]$
$= (a^{\perp})^{\perp} \cap (b^{\perp} \cup c^{\perp})^{\perp}$	$[B18, a/a^{\perp}, b/b^{\perp} \cup c^{\perp}]$
$= a \cap ((b^{\perp})^{\perp} \cap (c^{\perp})^{\perp})$	$[B15; B18, a/b^{\perp}, b/c^{\perp}]$
$= a \cap (b \cap c)$	$[B15, a/b; B15, a/c]$

Since it has been proved above that formulas *B17, B7, B1, B22, B23, B21, B2, D1, B16, B18, B20,* and *B15* are the consequences of the postulates *B1-B4*, we know that system 𝔄 is an ortholattice.

2 The mutual independence of axioms *B1-B4* is established by using the following algebraic tables:

𝔐1

$\cup$	α	β
α	α	α
β	β	β

$\cap$	α	β
α	α	β
β	α	β

x	$x^{\perp}$
α	β
β	α

𝔐2

$\cup$	α	β
α	α	β
β	β	α

$\cap$	α	β
α	β	α
β	α	β

x	$x^{\perp}$
α	β
β	α

𝔐3

$\cup$	α	β	γ
α	γ	γ	γ
β	γ	γ	γ
γ	γ	γ	γ

$\cap$	α	β	γ
α	α	α	α
β	β	β	β
γ	γ	γ	γ

x	$x^{\perp}$
α	β
β	γ
γ	γ

𝔐4

$\cup$	α	β	γ
α	α	β	α
β	β	β	β
γ	α	β	γ

$\cap$	α	β	γ
α	α	α	γ
β	α	β	γ
γ	γ	γ	γ

x	$x^{\perp}$
α	γ
β	γ
γ	γ

Namely:

(a) 𝔐1 verifies *B2, B3,* and *B4,* but falsifies *B1* for a/α and b/β: (i) $\alpha \cup \beta = \alpha$, (ii) $\beta \cup \alpha = \beta$.
(b) 𝔐2 verifies *B1, B3,* and *B4,* but falsifies *B2* for a/α and b/α: (i) $\alpha = \alpha$, (ii) $\alpha \cap (\alpha \cup \alpha) = \alpha \cap \alpha = \beta$.
(c) 𝔐3 verifies *B1, B2,* and *B4,* but falsifies *B3* for a/α and b/α: (i) $\alpha = \alpha$, (ii) $\alpha \cup (\alpha \cap \alpha^{\perp}) = \alpha \cup (\alpha \cap \beta) = \alpha \cup \alpha = \gamma$.

(d) $\mathfrak{M}4$ verifies *B1*, *B2*, and *B3*, but falsifies *B4* for a/α, b/α, and c/α: (i) $(\alpha \cup \alpha) \cup \alpha = \alpha \cup \alpha = \alpha$, (ii) $((\alpha^{\perp} \cap \alpha^{\perp}) \cap \alpha^{\perp})^{\perp} = ((\gamma \cap \gamma) \cap \gamma)^{\perp} = (\gamma \cap \gamma)^{\perp} = \gamma^{\perp} = \gamma$.

Thus, the proof of (**A**) is complete.

REFERENCE

[1] Birkhoff, G., *Lattice Theory*, Third (new) Edition. American Mathematical Society Colloquium Publications, volume XXV. Providence, Rhode Island, 1967.

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