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A SOLE SUFFICIENT OPERATOR

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Generations of students have been asked to prove (as an exercise) that the Sheffer stroke operator is a sole sufficient operator to define all of the monadic and dyadic operators in a two-valued space. A two-place functionally complete operator has come to be called a Sheffer operator [1]. We define a three-place operator S suggested by the work of A. A. Markov [2] in the theory of algorithms and prove that this operator is functionally complete over any finite-valued space. The proof is constructive.

Let $X(n)$ be the space of values $\mathsf{T} = 1, 2, \ldots, n = \mathsf{F}$. Over $X(n)$ define:

$$(1) \qquad Sxyz = \begin{cases} z, & \text{if } x = y; \\ x, & \text{if } x \neq y. \end{cases}$$

Consider, as an example, the two-valued case, $\mathsf{T} = 1, 2 = \mathsf{F}$. Negation, implication, conjunction, alternation, and the Sheffer stroke are defined by:

(2) $Nx = S\mathsf{T}x\mathsf{F}$; $Cxy = S\mathsf{T}xy$; $Kxy = Sx\mathsf{T}y$; $Axy = Sx\mathsf{F}y$; $Dxy = x/y = S\mathsf{T}Sx\mathsf{T}y\mathsf{F}$.

From this it is clear that S is a sole sufficient operator in the two-valued case.

In the general case we define n operators V_j, $1 \leqslant j \leqslant n$, such that V_jx has the value 1 if $x = j$, and V_jx has the value n if $x \neq j$.

$$(3) \qquad V_jx = \begin{cases} S1S1xnn, & \text{if } j = 1; \\ SSjx1jn, & \text{if } 2 \leqslant j \leqslant n. \end{cases}$$

If $x = j = 1$, $V_11 = S1S11nn = S1nn = 1$.
If $x \neq j = 1$, $V_1x = S1S1xnn = S11n = n$.
If $x = j \neq 1$, $V_jx = SSjj1jn = S1jn = 1$.
If $x \neq j \neq 1$, $V_jx = SSjx1jn = Sjjn = n$.

Hence definition (3) has the desired property. Define:

(4) $Kxy = Sx1y$.

Note that $K11 = 1$ and that $K1n = Kn1 = Knn = n$.

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Now suppose that $x_1, x_2, \ldots, x_k$ are k variables with values in the space $X(n)$, and suppose that, among all of the n^k possible states of these variables, Q is the state defined by $x_1 = t_1, x_2 = t_2, \ldots, x_k = t_k$, where for each i such that $1 \leqslant i \leqslant k$, $t_i \in X(n)$. Define:

(5) $\chi_Q(\lambda) = KV_{t_1}x_1KV_{t_2}x_2K \ldots KV_{t_{k-1}}x_{k-1}V_{t_k}x_k;$

where λ varies over the space of all possible states of the k variables $x_1, \ldots, x_k$. Substitution of (3) and (4) into (5) produces an expression in which S is the sole operator. Each of the arguments $V_{t_j}x_j$ takes on only the values 1 or n. By the remark following definition (4), $\chi_Q(\lambda)$ takes on the values 1 if and only if $V_{t_1}x_1 = \ldots = V_{t_k}x_k = 1$; that is, if and only if, $x_1 = t_1, x_2 = t_2, \ldots, x_k = t_k$; that is, if and only if, $\lambda = Q$. In every other one of the possible states $\chi_Q(\lambda) = n$.

Next suppose that f is a k-adic operator and suppose that f operating on the k variables $x_1, \ldots, x_k$ in the state Q produces some result different from $r \in X(n)$. Suppose that we wish to define a k-adic operator f' which has the same effect as f in each of the $n^k - 1$ states other than Q and which produces the result r in the state Q. Define:

(6) $$f'(\lambda) = \begin{cases} SS1\ \chi_Q(\lambda)n1f(\lambda), & \text{if } r = n; \\ SS\ \chi_Q(\lambda)1rnf(\lambda), & \text{if } r \neq n. \end{cases}$$

If $\lambda = Q$ and $r = n$, $f'(\lambda) = SS11n1f(\lambda) = Sn1f(\lambda) = n$.
If $\lambda = Q$ and $r \neq n$, $f'(\lambda) = SS11rnf(\lambda) = Srnf(\lambda) = r$.
If $\lambda \neq Q$ and $r = n$, $f'(\lambda) = SS1nn1f(\lambda) = S11f(\lambda) = f(\lambda)$.
If $\lambda \neq Q$ and $r \neq n$, $f'(\lambda) = SSn1rnf(\lambda) = Snnf(\lambda) = f(\lambda)$.

If f is defined in terms of S alone, then f' is defined in terms of S alone.

Theorem *If f is a k-adic operator over $X(n)$ then f can be defined by an expression involving S as the sole operator.*

Proof: Let f_0 be an arbitrary k-adic operator over $X(n)$ defined by an expression with S as the sole operator. If $f_0 = f$ for each of the n^k possible states, then there is nothing to prove. If f_0 and f differ for some finite number of states (say h), then let Q_1 be one of these states and suppose that $f(Q_1) = r \neq f_0(Q_1)$. By (6) define a new operator f_1 such that in the states Q_1, f_1 produces the result r and in every other state f_1 produces the same result as f_0. This new operator f_1 differs from f in $h - 1$ states. Application of the process h times produces an operator f_h which has the same effect as f in each of the n^k possible states.

For example, consider the following definition of equivalence proposed by Łukasiewicz [3] for a three-valued logic:

E	1	2	3
1	1	2	3
2	2	1	2
3	3	2	1

We wish to express this in terms of S alone. A reasonable "first guess" is obtained from the definitions of (2), namely:

$$E_0xy = KCxyCyx = SS\mathbf{T}xy\mathbf{T}S\mathbf{T}yx = SS1xy1S1yx.$$

This has the truth table:

E_0	1	2	3
1	1	2	3
2	2	1	1
3	3	1	1

E_0 differs from E in the two states:

$$Q_1\colon\ x = 2,\ y = 3;\ \text{and}\ Q_2\colon\ x = 3,\ y = 2.$$

From (3), $V_2x = SS2x123$; $V_3x = SS3x133 = S3x1$.
From (4) and (5), $\chi_{Q_1}(\lambda) = KV_2xV_3y = SSS2x1231S3y1$.

We wish E_1 to differ from E_0 in the state Q_1 by taking on the value 2 in that state. Then,

$$E_1(\lambda) = SS\ \chi_{Q_1}(\lambda)123E_0(\lambda) = SSSSS2x1231S3y1123SS1xy1S1yx.$$

This differs from E only in the state Q_2.

$$\chi_{Q_2}(\lambda) = KV_3xV_2y = SS3x11SS2y123.$$

We wish E_2 to differ from E_1 in the state Q_2 by taking on the value 2 in that state. Then,

$$\begin{aligned} E_2(\lambda) &= SS\chi_{Q_2}(\lambda)123E_1(\lambda) \\ &= SSSS3x11SS2y123123SSSSS2x1231S3y1123SS1xy1S1yx. \end{aligned}$$

Finally, $Exy = E_2xy$.

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REFERENCES

[1] Martin, Norman M., "The Sheffer function of 3-valued logic," *The Journal of Symbolic Logic*, vol. 19 (1954), pp. 45-50.

[2] Markov, A. A., *Theory of Mathematical Algorithms*, Israel Program for Scientific Translations, Jerusalem (1962), pp. 192-222.

[3] Łukasiewicz, Jan, "On Three-valued Logic," in *Selected Works*, edited by L. Borkowski, North Holland Publishing Company, Amsterdam (1970), pp. 87-88.

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