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A SUBSTITUTION PROPERTY

ALAN C. WILDE

Let $(X, +, \cdot)$ be a commutative ring with identity 1, and let $X' = \{x \in X | x^2 = x\}$. Also, let $x \cup y = x + y - xy$, $x \cap y = xy$, and $\overline{x} = 1 - x$. Then it is known that $\langle X', \cup, \cap, -\rangle$ is a Boolean algebra. We want to explore a property of a function u: $(X')^m \rightarrow X'$ and a function h: $(X')^n \rightarrow X'$ that are of the form $u(x_1, \ldots, x_m) = \sum a_{i_1 \ldots i_m} x_1^{(i_1)} \ldots x_m^{(i_m)}$ and $h(y_1, \ldots, y_n) = \sum b_{i_1 \ldots i_n} y_1^{(i_1)} \ldots y_n^{(i_n)}$ where i_k, $a_{i_1 \ldots i_m}$, $b_{i_1 \ldots i_n} \in \{0, 1\}$, $x_i^{(1)} = x_i$, and $x_i^{(0)} = \overline{x_i}$ (i.e., disjunctive normal form). If we let $v(x_i) = u(x_1, \ldots, x_i, \ldots, x_m)$ with all variables constant except x_i, we can prove the following:

Theorem $v(h(y_1, \ldots, y_n)) = v(1)\,v(0) \prod_{k=1}^{n} v(y_k) + v(1)\,\overline{v(0)} \sum b_{i_1 \ldots i_n} \prod_{k=1}^{n} v^{(i_k)}(y_k)$
$+ \overline{v(1)}\,v(0) \sum \overline{b_{i_1 \ldots i_n}} \prod_{k=1}^{n} v^{(1-i_k)}(y_k)$. (1)

Proof: We verify equation (1) by truth-value analysis, and then the Stone-Representation Theorem shows that (1) holds in a Boolean algebra. Substituting truth values $j_1, \ldots, j_n$ for $y_1, \ldots, y_n$, we see that all terms vanish except those in which $i_k = j_k (1 \leq k \leq n)$. Thus it reduces to

$$v(h(j_1, \ldots, j_n)) = v(1)\,v(0) + b_{j_1 \ldots j_n} v(1)\,\overline{v(0)} + \overline{b_{j_1 \ldots j_n}}\,\overline{v(1)}\,v(0). \qquad (*)$$

If $b_{j_1 \ldots j_n} = 1$, the RHS[1] of (*) becomes $v(1)\,v(0) + v(1)\,\overline{v(0)} = v(1)$; if $b_{j_1 \ldots j_n} = 0$, the RHS of (*) equals $v(1)\,v(0) + \overline{v(1)}\,v(0) = v(0)$. But the LHS of (*) equals $v(b_{j_1 \ldots j_n})$ in any case. Thus we did the first part, and we are done.

Since $v(1)\,v(0)\,\overline{v(y)} = 0$ and $\overline{v(1)\,v(0)}\,v(y) = 0$ (by truth-value analysis and the Stone-Representation Theorem), terms of the form $v^{(i)}(1)\,v^{(i)}(0) \prod_{k=1}^{n} v^{(i_k)}(y_k)$ where one i_k does not equal i also equal 0. The question occurs as to what are all the forms of $v(h(y_1, \ldots, y_n))$ that are linear combinations of conjunctions containing $v(1)$, $v(0)$, $v(y_1), \ldots, v(y_n)$. All conjunctions are of the kind we discussed above and also of the form

1. The abbreviations RHS and LHS stand for "right-hand side" and "left-hand side" respectively.

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$v(1)\,v(0)\prod_{k=1}^{n} v(y_k)$, $v^{(i)}(1)\,v^{(1-i)}(0)\prod_{k=1}^{n} v^{(i_k)}(y_k)$, and $\overline{v(1)\,v(0)}\prod_{k=1}^{n} v(y_k)$. The middle kind are not 0 for all u and occur in (1), but the last kind is not 0 for all u and does not occur. We have then this:

Corollary *An expression for* $v(h(y_1, \ldots, y_n))$ *as a linear combination of conjunctions containing* $v(1), v(0), v(y_1), \ldots, v(y_n)$ *is the* RHS *of* (1) *plus a linear combination of terms of the form* $v^{(i)}(1)\,v^{(i)}(0)\prod_{k=1}^{n} v^{(i_k)}(y_k)$ *where at least one* i_k *does not equal* i.

A simpler way of writing (1) is the following:

$$\begin{aligned} v(h(y_1, \ldots, y_n)) = {} & v(1)\,v(0)\prod_{k=1}^{n} v(y_k) + v(1)\,\overline{v(0)}\,h(v(y_1), \ldots, v(y_n)) \\ & + \overline{v(1)}\,v(0)\,\bar{h}(\overline{v(y_1)}, \ldots, \overline{v(y_n)}), \end{aligned} \tag{1'}$$

where $\bar{h}(y_1, \ldots, y_n) = 1 - h(y_1, \ldots, y_n)$. This substitution property holds for all u and h.

The theorem has as a corollary the "generalized distributive law," see [1]:

$$u(x, y \equiv z) = u(x, y) \equiv u(x, z) \equiv u(x, 0). \tag{2}$$

Noting again that $v(1)\,v(0)\,\overline{v(y)} = 0$ and $\overline{v(1)\,v(0)}\,v(y) = 0$, and using the notation of this paper, we see that (2) takes the form

$$\begin{aligned} v(y\bar{z} + \bar{y}z) &= [v(y)\,\overline{v(z)} + \overline{v(y)}\,v(z)]\,\overline{v(0)} + [v(y)\,v(z) + \overline{v(y)}\,\overline{v(z)}]\,v(0) \\ &= v(y)\,\overline{v(z)}\,v(1)\,\overline{v(0)} + \overline{v(y)}\,v(z)\,v(1)\,\overline{v(0)} + v(y)\,v(z)\,v(1)\,v(0) \\ &\quad + v(y)\,v(z)\,\overline{v(1)}\,v(0) + v(y)\,v(z)\,v(1)\,v(0), \end{aligned}$$

which we get also by (1).

REFERENCE

[1] Wilde, Alan C., "Generalizations of the distributive and associative laws," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 491-493.

The University of Michigan
Ann Arbor, Michigan