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NORMAL AND SKEW SYSTEMS

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Preface Suppose the formal system T has a binary connective $\rightarrow$. If T has:

$$\textsf{MP}: \frac{A, A \rightarrow B}{B}$$

among its rules of inference, it is natural to ask: "Which theorems of T are still provable (without repetitions) if one refuses to use **MP** except when A precedes $A \rightarrow B$?" Roughly speaking, we will say that T is normal if the answer to this question is "All of them."

In forming a precise definition a certain difficulty becomes apparent. One might be able to augment the sequence $\ldots, A \rightarrow B, \ldots, A, \ldots, B$ by new formulas to obtain $\ldots, A \rightarrow B, \ldots, A, \ldots, C, \ldots, C \rightarrow B, \ldots, B$, and thus avoid the issue. For example, if T had the axiom schemes:

A1) $A \rightarrow (B \rightarrow A)$
A2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,

then $\ldots, A \rightarrow B, \ldots, A, \ldots, B$ could be replaced by:

$\vdots$	
$A \rightarrow B$	
$\vdots$	
A	
$\vdots$	
F	
$A \rightarrow (F \rightarrow A)$	A1
$F \rightarrow A$	**MP**
$(A \rightarrow B) \rightarrow (F \rightarrow (A \rightarrow B))$	A1
$F \rightarrow (A \rightarrow B)$	**MP**
$(F \rightarrow (A \rightarrow B)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow B))$	A2
$(F \rightarrow A) \rightarrow (F \rightarrow B)$	**MP**
$F \rightarrow B$	**MP**

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B **MP**

where F is any axiom.

In view of this, we adopt a definition which insists that A precede $A \to B$ regardless of the other formulas which appear. This definition has the additional advantage that it does not refer to the rules of inference.

1 *Preliminaries* Let T be any formal system with a binary connective $\to$. (To increase readability we sometimes write $\Rightarrow$.)

Definition 1: The sequence $s = A_1 \ldots A_n$ of formulas of $\mathsf{L}(T)$ is a *normal sequence* if, whenever $k < j$ for a pair of formulas $A = A_j$, $A \to B = A_k$ in s, there is an $i < k$ such that $A = A_i$. (Equivalently, the first appearance of A precedes the first appearance of $A \to B$.) We say that s is a *skew sequence* if, whenever $j < k$ for such a pair, there is an $i < j$ such that $A \to B = A_i$.

Definition 2: A *normal (skew) theorem* is one that has a proof which is a normal (skew) sequence. A *normal (skew) system* is one whose theorems all are normal (skew). And a system is *strongly normal (skew)* if every equivalent system which has modus ponens as its only rule of inference is normal (skew).

2 *A Skew Axiomatization for the Propositional Calculus* Let **P** denote the propositional calculus with axiom schemes:

A1) $A \to (B \to A)$
A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
A3) $(\neg A \to \neg B) \to ((\neg A \to B) \to A)$,

and the single rule of inference:

$$\textbf{MP}: \frac{A, A \to B}{B}$$

An interesting conjecture is that **P** is strongly normal, but not strongly skew. The question is open[1] and, in fact, we do not even know if **P** is normal. In the present paper, we show that **P** is skew, and list a few by-products of the method of proof.

Theorem 1: *Let B be a theorem of* **P**. *Then for each proposition letter p, B has a proof $s = A_1 \ldots A_n$ with the properties:*

(a) *p appears in each A_i, $1 \leq i < n$*
(b) *there exist A_j, A_k in s such that A_k is $A_j \to A_n$*
(c) *s is a skew sequence.*

Proof: By induction on theorems. First, suppose that B is $A \to (C \to A)$. Consider the following sequence of formulas:

1. $((C \to (F \to A)) \to ((C \to F) \to (C \to A)))$
   $\Rightarrow (A \to ((C \to (F \to A)) \to ((C \to F) \to (C \to A))))$ **A1**
2. $(C \to (F \to A)) \Rightarrow ((C \to F) \to (C \to A))$ **A2**

1. We now know that **P** is neither strongly normal nor strongly skew.

3.	$(A \to ((C \to (F \to A)) \to ((C \to F) \to (C \to A))))$ $\to ((A \to (C \to (F \to A))) \to (A \to ((C \to F) \to (C \to A))))$	A2
4.	$A \to ((C \to (F \to A)) \to ((C \to F) \to (C \to A)))$	MP 1, 2
5.	$(A \to (C \to (F \to A))) \to (A \to ((C \to F) \to (C \to A)))$	MP 3, 4
6.	$((F \to A) \to (C \to (F \to A))) \to (A \to ((F \to A) \to (C \to (F \to A))))$	A1
7.	$(F \to A) \to (C \to (F \to A))$	A1
8.	$(A \to ((F \to A) \to (C \to (F \to A))))$ $\to ((A \to (F \to A)) \to (A \to (C \to (F \to A))))$	A2
9.	$A \to ((F \to A) \to (C \to (F \to A)))$	MP 6, 7
10.	$(A \to (F \to A)) \to (A \to (C \to (F \to A)))$	MP 8, 9
11.	$A \to (F \to A)$	A1
12.	$A \to (C \to (F \to A))$	MP 10, 11
13.	$(A \to ((C \to F) \to (C \to A))) \to ((A \to (C \to F)) \to (A \to (C \to A)))$	A2
14.	$A \to ((C \to F) \to (C \to A))$	MP 5, 12
15.	$(A \to (C \to F)) \to (A \to (C \to A))$	MP 13, 14
16.	$(F \to (C \to F)) \to (A \to (F \to (C \to F)))$	A1
17.	$F \to (C \to F)$	A1
18.	$(A \to (F \to (C \to F))) \to ((A \to F) \to (A \to (C \to F)))$	A2
19.	$A \to (F \to (C \to F))$	MP 16, 17
20.	$(A \to F) \to (A \to (C \to F))$	MP 18, 19
21.	$F \to (A \to F)$	A1
22.	F	
23.	$A \to F$	MP 21, 22
24.	$A \to (C \to F)$	MP 20, 23
25.	$A \to (C \to A)$	MP 15, 24

where F is any axiom in which p appears, except that $F \neq A$ and $A \neq F \to (C \to F)$. Clearly, this sequence has properties (a) and (b). To see that it also has property (c), it will suffice to show that there are no pairs of formulas of the form $D, D \to E$ with D preceding $D \to E$ in the sequence. Hence the choice of F; for otherwise we would have the following such pairs:

if $F = A$,	22, 23;	22, 24;	22, 25;
if $F = A = C$,	11, 15;	11, 16;	12, 18;

and

if $A = F \to (C \to F)$,	17, 19;	17, 23;	17, 24;	17, 25.

We conclude that there are no such pairs for our choice of F. Now suppose that B is $(A \to (C \to D)) \to ((A \to C) \to (A \to D))$. We present the following sequence:

1.	$((F \to (A \to (C \to D))) \to ((F \to A) \to (F \to (C \to D))))$ $\to ((A \to (C \to D))$ $\to ((F \to (A \to (C \to D))) \to ((F \to A) \to (F \to (C \to D)))))$	A1
2.	$(F \to (A \to (C \to D))) \to ((F \to A) \to (F \to (C \to D)))$	A2
3.	$((A \to (C \to D))$ $\to ((F \to (A \to (C \to D))) \to ((F \to A) \to (F \to (C \to D)))))$	

$\Rightarrow (((A \to (C \to D)) \to (F \to (A \to (C \to D))))$
$\to ((A \to (C \to D)) \to ((F \to A) \to (F \to (C \to D)))))$ A2

4. $(A \to (C \to D))$
$\Rightarrow ((F \to (A \to (C \to D))) \to ((F \to A) \to (F \to (C \to D))))$ MP 1, 2

5. $((A \to (C \to D)) \to (F \to (A \to (C \to D))))$
$\Rightarrow ((A \to (C \to D)) \to ((F \to A) \to (F \to (C \to D))))$ MP 3, 4

6. $(A \to (C \to D)) \Rightarrow (F \to (A \to (C \to D)))$ A1

7. $((F \to (C \to D)) \to ((F \to C) \to (F \to D)))$
$\Rightarrow ((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D))))$ A1

8. $(((F \to A) \to ((F \to C) \to (F \to D)))$
$\to ((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to ((F \to A) \to ((F \to C) \to (F \to D))$
$\to ((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))))$ A1

9. $((F \to A) \to ((F \to C) \to (F \to D)))$
$\Rightarrow ((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))$ A1

10. $((F \to (A \to C)) \to ((F \to A) \to (F \to C)))$
$\Rightarrow ((A \to C) \to ((F \to (A \to C)) \to ((F \to A) \to (F \to C))))$ A1

11. $(F \to (C \to D)) \Rightarrow ((F \to C) \to (F \to D))$ A2

12. $((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D))))$
$\Rightarrow ((A \to (C \to D))$
$\to ((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D)))))$ A1

13. $(((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D))))$
$\to (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to (((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D))))$
$\to (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D))))))$ A1

14. $((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D))))$
$\Rightarrow (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D))))$ A2

15. $(F \to A) \Rightarrow ((F \to (C \to D)) \to ((F \to C) \to (F \to D)))$ MP 7, 11

16. $((A \to (C \to D))$
$\to (((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D))))$
$\to (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D))))))$
$\Rightarrow (((A \to (C \to D))$
$\to ((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D)))))$
$\to ((A \to (C \to D))$
$\to (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D))))))$ A2

17. $(A \to (C \to D))$
$\Rightarrow (((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D))))$
$\to (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D)))))$ MP 13, 14

18. $((A \to (C \to D))$
$\to ((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D)))))$ **MP** 16, 17

19. $(A \to (C \to D))$
$\Rightarrow ((F \to A) \to ((F \to (C \to D)) \to ((F \to C) \to (F \to D))))$ **MP** 12, 15

20. $((A \to (C \to D))$
$\to (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D)))))$
$\Rightarrow (((A \to (C \to D)) \to ((F \to A) \to (F \to (C \to D))))$
$\to ((A \to (C \to D)) \to ((F \to A) \to ((F \to C) \to (F \to D)))))$ A2

21. $(A \to (C \to D))$
$\Rightarrow (((F \to A) \to (F \to (C \to D)))$
$\to ((F \to A) \to ((F \to C) \to (F \to D))))$ **MP** 18, 19

22. $((A \to (C \to D)) \to ((F \to A) \to (F \to (C \to D))))$
$\Rightarrow ((A \to (C \to D)) \to ((F \to A) \to ((F \to C) \to (F \to D))))$ **MP** 20, 21

23. $(A \to (C \to D)) \Rightarrow ((F \to A) \to (F \to (C \to D)))$ **MP** 5, 6

24. $((A \to (C \to D))$
$\to (((F \to A) \to ((F \to C) \to (F \to D)))$
$\to ((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))))$
$\Rightarrow (((A \to (C \to D)) \to ((F \to A) \to ((F \to C) \to (F \to D))))$
$\to ((A \to (C \to D))$
$\to ((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))))$ A2

25. $(A \to (C \to D))$
$\Rightarrow (((F \to A) \to ((F \to C) \to (F \to D)))$
$\to ((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D)))))$ **MP** 8, 9

26. $((A \to (C \to D)) \to ((F \to A) \to ((F \to C) \to (F \to D))))$
$\Rightarrow ((A \to (C \to D))$
$\to ((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D)))))$ **MP** 24, 25

27. $(A \to (C \to D)) \Rightarrow ((F \to A) \to ((F \to C) \to (F \to D)))$ **MP** 22, 23

28. $(F \to (A \to C)) \Rightarrow ((F \to A) \to (F \to C))$ A2

29. $((A \to C) \to ((F \to (A \to C)) \to ((F \to A) \to (F \to C))))$
$\Rightarrow (((A \to C) \to (F \to (A \to C)))$
$\to ((A \to C) \to ((F \to A) \to (F \to C))))$ A2

30. $(A \to C) \Rightarrow ((F \to (A \to C)) \to ((F \to A) \to (F \to C)))$ **MP** 10, 28

31. $((A \to C) \to (F \to (A \to C)))$
$\Rightarrow ((A \to C) \to ((F \to A) \to (F \to C)))$ **MP** 29, 30

32. $(A \to C) \Rightarrow (F \to (A \to C))$ A1

33. $((A \to C) \to ((F \to A) \to (F \to C)))$
$\Rightarrow ((A \to (C \to D)) \to ((A \to C) \to ((F \to A) \to (F \to C))))$ A1

34. $(((F \to A) \to (F \to D)) \to (A \to ((F \to A) \to (F \to D))))$
$\Rightarrow ((A \to C) \to (((F \to A) \to (F \to D)) \to (A \to ((F \to A) \to (F \to D)))))$
A1

35. $((F \to A) \to (F \to D)) \Rightarrow (A \to ((F \to A) \to (F \to D)))$ A1

36. $((A \to C) \to (((F \to A) \to (F \to D)) \to (A \to ((F \to A) \to (F \to D)))))$
$\Rightarrow (((A \to C) \to ((F \to A) \to (F \to D)))$
$\to ((A \to C) \to (A \to ((F \to A) \to (F \to D)))))$ A2

37. $(A \to C) \Rightarrow (((F \to A) \to (F \to D)) \to (A \to ((F \to A) \to (F \to D))))$ **MP** 34, 35

38. $(((A \to C) \to ((F \to A) \to (F \to D)))$
$\to ((A \to C) \to (A \to ((F \to A) \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to (((A \to C) \to ((F \to A) \to (F \to D)))$
$\to ((A \to C) \to (A \to ((F \to A) \to (F \to D))))))$ A1

39. $((A \to C) \to ((F \to A) \to (F \to D)))$
$\Rightarrow ((A \to C) \to (A \to ((F \to A) \to (F \to D))))$ **MP** 36, 37

40. $(A \to C) \Rightarrow ((F \to A) \to (F \to C))$ **MP** 31, 32

41. $(((F \to A) \to ((F \to C) \to (F \to D)))$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D))))$
$\Rightarrow ((A \to C)$
$\to (((F \to A) \to ((F \to C) \to (F \to D)))$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D)))))$ A1

42. $((F \to A) \to ((F \to C) \to (F \to D)))$
$\Rightarrow (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D)))$ A2

43. $((A \to C)$
$\to (((F \to A) \to ((F \to C) \to (F \to D)))$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D)))))$
$\Rightarrow (((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))$
$\to ((A \to C)$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D)))))$ A2

44. $(A \to C)$
$\Rightarrow (((F \to A) \to ((F \to C) \to (F \to D)))$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D))))$ **MP** 41, 42

45. $(((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))$
$\to ((A \to C)$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to (((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))$
$\to ((A \to C)$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D))))))$ A1

46. $((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))$
$\Rightarrow ((A \to C)$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D))))$ **MP** 43, 44

47. $((A \to (C \to D))$
$\to (((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))$
$\to ((A \to C)$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D))))))$
$\Rightarrow (((A \to (C \to D))$
$\to ((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D)))))$
$\to ((A \to (C \to D))$
$\to ((A \to C)$
$\to (((F \to A) \to (F \to C)) \to ((F \to A) \to (F \to D))))))$ A2

48. $(A \to (C \to D))$
$\Rightarrow (((A \to C) \to ((F \to A) \to ((F \to C) \to (F \to D))))$

$\rightarrow ((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D)))))$ **MP** 45, 46

49. $((A \rightarrow (C \rightarrow D))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow ((F \rightarrow C) \rightarrow (F \rightarrow D)))))$
$\rightarrow ((A \rightarrow (C \rightarrow D))$
$\rightarrow ((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D)))))$ **MP** 47, 48

50. $(A \rightarrow (C \rightarrow D))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow ((F \rightarrow C) \rightarrow (F \rightarrow D))))$ **MP** 26, 27

51. $(((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))$
$\rightarrow (((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow C)))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D)))))$
$\rightarrow ((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))$
$\rightarrow (((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow C)))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))))$ A1

52. $((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))$
$\rightarrow (((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow C)))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))$ A2

53. $((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))$
$\rightarrow (((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow C)))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))))$
$\rightarrow (((A \rightarrow (C \rightarrow D))$
$\rightarrow ((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D)))))$
$\rightarrow ((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow C)))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))))$ A2

54. $(A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))$
$\rightarrow (((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow C)))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D)))))$ **MP** 51, 52

55. $((A \rightarrow (C \rightarrow D))$
$\rightarrow ((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D)))))$
$\rightarrow ((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow C)))$
$\rightarrow ((A \rightarrow C) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D)))))$ **MP** 53, 54

56. $(A \rightarrow (C \rightarrow D))$
$\rightarrow ((A \rightarrow C)$
$\rightarrow (((F \rightarrow A) \rightarrow (F \rightarrow C)) \rightarrow ((F \rightarrow A) \rightarrow (F \rightarrow D))))$ **MP** 49, 50

57. $((A \to (C \to D))$
$\to (((A \to C) \to ((F \to A) \to (F \to C)))$
$\to ((A \to C) \to ((F \to A) \to (F \to D)))))$
$\Rightarrow (((A \to (C \to D)) \to ((A \to C) \to ((F \to A) \to (F \to C))))$
$\to ((A \to (C \to D)) \to ((A \to C) \to ((F \to A) \to (F \to D)))))$ A2

58. $(A \to (C \to D))$
$\Rightarrow (((A \to C) \to ((F \to A) \to (F \to C)))$
$\to ((A \to C) \to ((F \to A) \to (F \to D))))$ MP 55, 56

59. $((A \to (C \to D)) \to ((A \to C) \to ((F \to A) \to (F \to C))))$
$\Rightarrow ((A \to (C \to D)) \to ((A \to C) \to ((F \to A) \to (F \to D))))$ MP 57, 58

60. $((A \to (C \to D))$
$\to (((A \to C) \to ((F \to A) \to (F \to D)))$
$\to ((A \to C) \to (A \to ((F \to A) \to (F \to D))))))$
$\Rightarrow (((A \to (C \to D)) \to ((A \to C) \to ((F \to A) \to (F \to D))))$
$\to ((A \to (C \to D)) \to ((A \to C) \to (A \to ((F \to A) \to (F \to D))))))$ A2

61. $(A \to (C \to D))$
$\Rightarrow (((A \to C) \to ((F \to A) \to (F \to D)))$
$\to ((A \to C) \to (A \to ((F \to A) \to (F \to D)))))$ MP 38, 39

62. $((A \to (C \to D)) \to ((A \to C) \to ((F \to A) \to (F \to D))))$
$\Rightarrow ((A \to (C \to D)) \to ((A \to C) \to (A \to ((F \to A) \to (F \to D)))))$
MP 60, 61

63. $(A \to (C \to D)) \Rightarrow ((A \to C) \to ((F \to A) \to (F \to C)))$ MP 33, 40

64. $(A \to (F \to A)) \Rightarrow ((A \to C) \to (A \to (F \to A)))$ A1

65. $((A \to (F \to D)) \to ((A \to F) \to (A \to D)))$
$\Rightarrow ((A \to C) \to ((A \to (F \to D)) \to ((A \to F) \to (A \to D))))$ A1

66. $(A \to (F \to D)) \Rightarrow ((A \to F) \to (A \to D))$ A2

67. $A \Rightarrow (F \to A)$ A1

68. $((A \to C) \to (A \to (F \to A)))$
$\Rightarrow ((A \to (C \to D)) \to ((A \to C) \to (A \to (F \to A))))$ A1

69. $((A \to C) \to ((A \to (F \to D)) \to ((A \to F) \to (A \to D))))$
$\Rightarrow (((A \to C) \to (A \to (F \to D)))$
$\to ((A \to C) \to ((A \to F) \to (A \to D))))$ A2

70. $(A \to C) \Rightarrow ((A \to (F \to D)) \to ((A \to F) \to (A \to D)))$ MP 65, 66

71. $(((A \to C) \to (A \to (F \to D)))$
$\to ((A \to C) \to ((A \to F) \to (A \to D))))$
$\Rightarrow ((A \to (C \to D))$
$\to (((A \to C) \to (A \to (F \to D)))$
$\to ((A \to C) \to ((A \to F) \to (A \to D)))))$ A1

72. $((A \to C) \to (A \to (F \to D)))$
$\Rightarrow ((A \to C) \to ((A \to F) \to (A \to D)))$ MP 69, 70

73. $(A \to C) \Rightarrow (A \to (F \to A))$ MP 64, 67

74. $F \Rightarrow ((A \to C) \to F)$ A1

75. $(F \to (A \to F)) \Rightarrow ((A \to C) \to (F \to (A \to F)))$ A1

76. $F \Rightarrow (A \to F)$ A1

77. F

78. $((A \to C) \to (F \to (A \to F))) \Rightarrow (((A \to C) \to F) \to ((A \to C) \to (A \to F)))$
A2

79. $(A \to C) \Rightarrow (F \to (A \to F))$ MP 75, 76
80. $((A \to C) \to F) \Rightarrow ((A \to C) \to (A \to F))$ MP 78, 79
81. $(A \to C) \Rightarrow F$ MP 74, 77
82. $((A \to C) \to (A \to F)) \Rightarrow ((A \to (C \to D)) \to ((A \to C) \to (A \to F)))$ A1
83. $(A \to C) \Rightarrow (A \to F)$ MP 80, 81
84. $(A \to (C \to D)) \Rightarrow ((A \to C) \to ((F \to A) \to (F \to D)))$ MP 59, 63
85. $((A \to ((F \to A) \to (F \to D)))$
$\to ((A \to (F \to A)) \to (A \to (F \to D))))$
$\Rightarrow ((A \to C)$
$\to ((A \to ((F \to A) \to (F \to D)))$
$\to ((A \to (F \to A)) \to (A \to (F \to D)))))$ A1
86. $(A \to ((F \to A) \to (F \to D)))$
$\Rightarrow ((A \to (F \to A)) \to (A \to (F \to D)))$ A2
87. $((A \to C)$
$\to ((A \to ((F \to A) \to (F \to D)))$
$\to ((A \to (F \to A)) \to (A \to (F \to D)))))$
$\Rightarrow (((A \to C) \to (A \to ((F \to A) \to (F \to D))))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D)))))$ A2
88. $(A \to C)$
$\Rightarrow ((A \to ((F \to A) \to (F \to D)))$
$\to ((A \to (F \to A)) \to (A \to (F \to D))))$ MP 85, 86
89. $(((A \to C) \to (A \to ((F \to A) \to (F \to D))))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to (((A \to C) \to (A \to ((F \to A) \to (F \to D))))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))))$ A1
90. $((A \to C) \to (A \to ((F \to A) \to (F \to D))))$
$\Rightarrow ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))$ MP 87, 88
91. $((A \to (C \to D))$
$\to (((A \to C) \to (A \to ((F \to A) \to (F \to D))))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))))$
$\Rightarrow (((A \to (C \to D)) \to ((A \to C) \to (A \to ((F \to A) \to (F \to D)))))$
$\to ((A \to (C \to D))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))))$ A2
92. $(A \to (C \to D))$
$\Rightarrow (((A \to C) \to (A \to ((F \to A) \to (F \to D))))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D)))))$ MP 89, 90
93. $((A \to (C \to D)) \to ((A \to C) \to (A \to ((F \to A) \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D)))))$ MP 91, 92
94. $(A \to (C \to D)) \Rightarrow ((A \to C) \to (A \to ((F \to A) \to (F \to D))))$ MP 62, 84
95. $(((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))$
$\to (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))$
$\to (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D)))))$ A1

96. $((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))$
$\Rightarrow (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D))))$ A2

97. $((A \to (C \to D))$
$\to (((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))$
$\to (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D))))))$
$\Rightarrow (((A \to (C \to D))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D)))))$
$\to ((A \to (C \to D))$
$\to (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D))))))$ A2

98. $(A \to (C \to D))$
$\Rightarrow (((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))$
$\to (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D)))))$ MP 95, 96

99. $((A \to (C \to D))$
$\to ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D)))))$
$\Rightarrow ((A \to (C \to D))$
$\to (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D)))))$ MP 97, 98

100. $(A \to (C \to D))$
$\Rightarrow ((A \to C) \to ((A \to (F \to A)) \to (A \to (F \to D))))$ MP 93, 94

101. $((A \to (C \to D))$
$\to (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D)))))$
$\Rightarrow (((A \to (C \to D)) \to ((A \to C) \to (A \to (F \to A))))$
$\to ((A \to (C \to D)) \to ((A \to C) \to (A \to (F \to D)))))$ A2

102. $(A \to (C \to D))$
$\Rightarrow (((A \to C) \to (A \to (F \to A)))$
$\to ((A \to C) \to (A \to (F \to D))))$ MP 99, 100

103. $((A \to (C \to D)) \to ((A \to C) \to (A \to (F \to A))))$
$\Rightarrow ((A \to (C \to D)) \to ((A \to C) \to (A \to (F \to D))))$ MP 101, 102

104. $((A \to (C \to D))$
$\to (((A \to C) \to (A \to (F \to D)))$
$\to ((A \to C) \to ((A \to F) \to (A \to D)))))$
$\Rightarrow (((A \to (C \to D)) \to ((A \to C) \to (A \to (F \to D))))$
$\to ((A \to (C \to D)) \to ((A \to C) \to ((A \to F) \to (A \to D)))))$ A2

105. $(A \to (C \to D))$
$\Rightarrow (((A \to C) \to (A \to (F \to D)))$
$\to ((A \to C) \to ((A \to F) \to (A \to D))))$ MP 71, 72

106. $((A \to (C \to D)) \to ((A \to C) \to (A \to (F \to D))))$
$\Rightarrow ((A \to (C \to D)) \to ((A \to C) \to ((A \to F) \to (A \to D))))$ MP 104, 105

107. $(A \to (C \to D)) \Rightarrow ((A \to C) \to (A \to (F \to A)))$ MP 68, 73

108. $(A \to (C \to D)) \Rightarrow ((A \to C) \to (A \to (F \to D)))$ MP 103, 107

109. $(((A \to C) \to ((A \to F) \to (A \to D)))$
$\to (((A \to C) \to (A \to F)) \to ((A \to C) \to (A \to D))))$

$\rightarrow ((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow ((A \rightarrow F) \rightarrow (A \rightarrow D)))$
$\rightarrow (((A \rightarrow C) \rightarrow (A \rightarrow F)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D)))))$ A1

110. $((A \rightarrow C) \rightarrow ((A \rightarrow F) \rightarrow (A \rightarrow D)))$
$\rightarrow (((A \rightarrow C) \rightarrow (A \rightarrow F)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D)))$ A2

111. $((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow ((A \rightarrow F) \rightarrow (A \rightarrow D)))$
$\rightarrow (((A \rightarrow C) \rightarrow (A \rightarrow F)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D)))))$
$\rightarrow (((A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow ((A \rightarrow F) \rightarrow (A \rightarrow D))))$
$\rightarrow ((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow (A \rightarrow F)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D)))))$ A2

112. $(A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow ((A \rightarrow F) \rightarrow (A \rightarrow D)))$
$\rightarrow (((A \rightarrow C) \rightarrow (A \rightarrow F)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D))))$ MP 109, 110

113. $((A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow ((A \rightarrow F) \rightarrow (A \rightarrow D))))$
$\rightarrow ((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow (A \rightarrow F)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D))))$ MP 111, 112

114. $(A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow ((A \rightarrow F) \rightarrow (A \rightarrow D)))$ MP 106, 108

115. $((A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow (A \rightarrow F)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D))))$
$\rightarrow (((A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow F)))$
$\rightarrow ((A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D))))$ A2

116. $(A \rightarrow (C \rightarrow D))$
$\rightarrow (((A \rightarrow C) \rightarrow (A \rightarrow F)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D)))$ MP 113, 114

117. $((A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow F)))$
$\rightarrow ((A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D)))$ MP 115, 116

118. $(A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow F))$ MP 82, 83

119. $(A \rightarrow (C \rightarrow D)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow D))$ MP 117, 118

where F is any axiom in which p appears, except that $F \neq A$, $F \neq C$, $F \neq A \rightarrow C$, $F \neq A \rightarrow (C \rightarrow D)$, $C \neq F \rightarrow A$, $C \neq F \rightarrow D$, $C \neq (F \rightarrow A) \rightarrow (F \rightarrow D)$, and $A \neq (F \rightarrow (C \rightarrow D)) \rightarrow ((F \rightarrow C) \rightarrow (F \rightarrow D))$. Again, this sequence clearly has properties (a) and (b), and to see that it also has property (c) we search throughout for pairs of the form D, $D \rightarrow E$ with D preceding $D \rightarrow E$. A careful (and tedious!) examination of the sequence reveals the choice of F to have been a fortuitous one. Indeed, F was so chosen precisely to avoid the following such pairs:

if $F = A$, 67, 75; 73, 78; 84, 113;

if $F = A$ and $C = D$, 28, 65; 30, 69; 32, 72; 40, 110; 63, 113;

if $F = A$ and $C = A \rightarrow F$, 76, 79; 76, 81; 76, 83; 76, 88;

if $F = A$ and $D = F \rightarrow A$, 11, 103;

if $F = A = C$, 11, 65; 23, 106; 27, 113; 42, 109;
21, 104; 27, 62; 32, 68;

if $F = A = C$ and $D = A \rightarrow C$, 32, 35;

if $F = A = D$, 11, 117;

if $F = A = D$ and $C = F \to A$, 75, 96;

if $F = A = C = D$,
23, 103; 40, 42; 76, 84; 76, 98; 76, 105; 76, 112; 76, 118;
27, 59; 75, 103; 76, 92; 76, 100; 76, 107; 76, 114; 76, 119;
28, 117; 75, 106; 76, 94; 76, 102; 76, 108; 76, 116;

if $F = C$, 32, 78;

if $F = C$ and $A = D$, 64, 103; 67, 92; 67, 100; 67, 107; 67, 114; 67, 119;
64, 106; 67, 94; 67, 102; 67, 108; 67, 116;
67, 84; 67, 98; 67, 105; 67, 112; 67, 118;

if $F = C$ and $D = F \to A$, 66, 103;

if $F = C = D$, 66, 117;

if $F = A \to C$, 77, 79; 77, 81; 77, 83; 77, 88;

if $F = A \to C$ and $A = C = D$, 6, 117;

if $F = A \to (C \to D)$, 77, 84; 77, 98; 77, 105; 77, 112; 77, 118;
77, 92; 77, 100; 77, 107; 77, 114; 77, 119;
77, 94; 77, 102; 77, 108; 77, 116;

if $C = F \to A$, 67, 70; 67, 73; 67, 79; 67, 81; 67, 83; 67, 88;

if $C = F \to D$, 66, 110;

if $C = (F \to A) \to (F \to D)$, 86, 96;

and

if $A = (F \to (C \to D)) \to ((F \to C) \to (F \to D))$, 11, 67.

Next, suppose that B is $(\neg A \to \neg C) \to ((\neg A \to C) \to A)$. Consider the following sequence:

1.	$F \to (\neg A \to F)$	A1
2.	$(\neg C \to (\neg\neg F \to \neg C)) \to (\neg A \to (\neg C \to (\neg\neg F \to \neg C)))$	A1
3.	$((\neg\neg F \to \neg C) \to ((\neg\neg F \to C) \to \neg F))$	
	$\to (\neg A \to ((\neg\neg F \to \neg C) \to ((\neg\neg F \to C) \to \neg F)))$	A1
4.	$(\neg\neg F \to \neg C) \to ((\neg\neg F \to C) \to \neg F)$	A3
5.	$(\neg A \to ((\neg\neg F \to \neg C) \to ((\neg\neg F \to C) \to \neg F)))$	
	$\to ((\neg A \to (\neg\neg F \to \neg C)) \to (\neg A \to ((\neg\neg F \to C) \to \neg F)))$	A2
6.	$\neg A \to ((\neg\neg F \to \neg C) \to ((\neg\neg F \to C) \to \neg F))$	MP 3, 4
7.	$((\neg A \to (\neg\neg F \to \neg C)) \to (\neg A \to ((\neg\neg F \to C) \to \neg F)))$	
	$\to ((\neg A \to \neg C)$	
	$\to ((\neg A \to (\neg\neg F \to \neg C)) \to (\neg A \to ((\neg\neg F \to C) \to \neg F))))$	A1
8.	$(\neg A \to (\neg\neg F \to \neg C)) \to (\neg A \to ((\neg\neg F \to C) \to \neg F))$	MP 5, 6
9.	$((\neg A \to \neg C)$	
	$\to ((\neg A \to (\neg\neg F \to \neg C)) \to (\neg A \to ((\neg\neg F \to C) \to \neg F))))$	
	$\to (((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C)))$	
	$\to ((\neg A \to \neg C) \to (\neg A \to ((\neg\neg F \to C) \to \neg F))))$	A2

10. $(\neg A \to \neg C)$
$\Rightarrow ((\neg A \to (\neg\neg F \to \neg C)) \to (\neg A \to ((\neg\neg F \to C) \to \neg F)))$ **MP** 7, 8
11. $((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C)))$
$\Rightarrow ((\neg A \to \neg C) \to (\neg A \to ((\neg\neg F \to C) \to \neg F)))$ **MP** 9, 10
12. $\neg C \Rightarrow (\neg\neg F \to \neg C)$ A1
13. $(\neg A \to (\neg C \to (\neg\neg F \to \neg C))) \Rightarrow ((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C)))$ A2
14. $\neg A \Rightarrow (\neg C \to (\neg\neg F \to \neg C))$ **MP** 2, 12
15. $((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C)))$
$\Rightarrow (F \to ((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C))))$ A1
16. $(\neg A \to \neg C) \Rightarrow (\neg A \to (\neg\neg F \to \neg C))$ **MP** 13, 14
17. $(F \to ((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C))))$
$\Rightarrow ((F \to (\neg A \to \neg C)) \to (F \to (\neg A \to (\neg\neg F \to \neg C))))$ A2
18. $F \Rightarrow ((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C)))$ **MP** 15, 16
19. $F \Rightarrow ((\neg A \to \neg C) \to F)$ A1
20. $(C \to (\neg\neg F \to C)) \Rightarrow (\neg A \to (C \to (\neg\neg F \to C)))$ A1
21. $C \Rightarrow (\neg\neg F \to C)$ A1
22. $(\neg A \to (C \to (\neg\neg F \to C))) \Rightarrow ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C)))$ A2
23. $\neg A \Rightarrow (C \to (\neg\neg F \to C))$ **MP** 20, 21
24. $((\neg A \to C) \to (\neg A \to (\neg\neg F \to C)))$
$\Rightarrow (F \to ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C))))$ A1
25. $((\neg A \to C) \to (\neg A \to (\neg\neg F \to C)))$
$\Rightarrow ((\neg A \to \neg C) \to ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C))))$ A1
26. $(\neg A \to C) \Rightarrow (\neg A \to (\neg\neg F \to C))$ **MP** 22, 23
27. $(F \to ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C))))$
$\Rightarrow ((F \to (\neg A \to C)) \to (F \to (\neg A \to (\neg\neg F \to C))))$ A2
28. $F \Rightarrow ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C)))$ **MP** 24, 26
29. $F \Rightarrow ((\neg A \to C) \to F)$ A1
30. F
31. $(\neg A \to F) \Rightarrow ((\neg A \to C) \to (\neg A \to F))$ A1
32. $\neg A \Rightarrow F$ **MP** 1, 30
33. $((\neg A \to C) \to (\neg A \to F)) \Rightarrow ((\neg A \to \neg C) \to ((\neg A \to C) \to (\neg A \to F)))$ A1
34. $(\neg A \to C) \Rightarrow (\neg A \to F)$ **MP** 31, 32
35. $((F \to (\neg A \to \neg C)) \to (F \to (\neg A \to (\neg\neg F \to \neg C))))$
$\Rightarrow ((\neg A \to \neg C)$
$\to ((F \to (\neg A \to \neg C)) \to (F \to (\neg A \to (\neg\neg F \to \neg C)))))$ A1
36. $(F \to (\neg A \to \neg C)) \Rightarrow (F \to (\neg A \to (\neg\neg F \to \neg C)))$ **MP** 17, 18
37. $((\neg A \to \neg C)$
$\to ((F \to (\neg A \to \neg C)) \to (F \to (\neg A \to (\neg\neg F \to \neg C)))))$
$\Rightarrow (((\neg A \to \neg C) \to (F \to (\neg A \to \neg C)))$
$\to ((\neg A \to \neg C) \to (F \to (\neg A \to (\neg\neg F \to \neg C)))))$ A2
38. $(\neg A \to \neg C)$
$\Rightarrow ((F \to (\neg A \to \neg C)) \to (F \to (\neg A \to (\neg\neg F \to \neg C))))$ **MP** 35, 36
39. $((\neg A \to \neg C) \to (F \to (\neg A \to \neg C)))$
$\Rightarrow ((\neg A \to \neg C) \to (F \to (\neg A \to (\neg\neg F \to \neg C))))$ **MP** 37, 38
40. $(\neg A \to \neg C) \Rightarrow (F \to (\neg A \to \neg C))$ A1
41. $((\neg A \to \neg C) \to (F \to (\neg A \to (\neg\neg F \to \neg C))))$
$\Rightarrow (((\neg A \to \neg C) \to F) \to ((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C))))$ A2

42. $(\neg A \to \neg C) \to (F \to (\neg A \to (\neg\neg F \to \neg C)))$ **MP** 39, 40
43. $((\neg A \to \neg C) \to F) \to ((\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C)))$ **MP** 41, 42
44. $(\neg A \to \neg C) \to F$ **MP** 19, 30
45. $((F \to (\neg A \to C)) \to (F \to (\neg A \to (\neg\neg F \to C))))$
$\to ((\neg A \to C) \to ((F \to (\neg A \to C)) \to (F \to (\neg A \to (\neg\neg F \to C)))))$ **A1**
46. $(F \to (\neg A \to C)) \to (F \to (\neg A \to (\neg\neg F \to C)))$ **MP** 27, 28
47. $((\neg A \to C) \to ((F \to (\neg A \to C)) \to (F \to (\neg A \to (\neg\neg F \to C)))))$
$\to (((\neg A \to C) \to (F \to (\neg A \to C)))$
$\to ((\neg A \to C) \to (F \to (\neg A \to (\neg\neg F \to C)))))$ **A2**
48. $(\neg A \to C) \to ((F \to (\neg A \to C)) \to (F \to (\neg A \to (\neg\neg F \to C))))$ **MP** 45, 46
49. $((\neg A \to C) \to (F \to (\neg A \to C)))$
$\to ((\neg A \to C) \to (F \to (\neg A \to (\neg\neg F \to C))))$ **MP** 47, 48
50. $(\neg A \to C) \to (F \to (\neg A \to C))$ **A1**
51. $((\neg A \to C) \to (F \to (\neg A \to (\neg\neg F \to C))))$
$\to (((\neg A \to C) \to F) \to ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C))))$ **A2**
52. $(\neg A \to C) \to (F \to (\neg A \to (\neg\neg F \to C)))$ **MP** 49, 50
53. $((\neg A \to C) \to F) \to ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C)))$ **MP** 51, 52
54. $(\neg A \to C) \to F$ **MP** 29, 30
55. $(\neg A \to C) \to (\neg A \to (\neg\neg F \to C))$ **MP** 53, 54
56. $(\neg A \to \neg C) \to (\neg A \to (\neg\neg F \to \neg C))$ **MP** 13, 14
57. $((\neg A \to ((\neg\neg F \to C) \to \neg F))$
$\to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F)))$
$\to ((\neg A \to \neg C)$
$\to ((\neg A \to ((\neg\neg F \to C) \to \neg F))$
$\to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))))$ **A1**
58. $(\neg A \to ((\neg\neg F \to C) \to \neg F))$
$\to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))$ **A2**
59. $((\neg A \to \neg C)$
$\to ((\neg A \to ((\neg\neg F \to C) \to \neg F))$
$\to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))))$
$\to (((\neg A \to \neg C) \to (\neg A \to ((\neg\neg F \to C) \to \neg F)))$
$\to ((\neg A \to \neg C) \to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))))$ **A2**
60. $(\neg A \to \neg C)$
$\to ((\neg A \to ((\neg\neg F \to C) \to \neg F))$
$\to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F)))$ **MP** 57, 58
61. $((\neg A \to \neg C) \to (\neg A \to ((\neg\neg F \to C) \to \neg F)))$
$\to ((\neg A \to \neg C) \to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F)))$ **MP** 59, 60
62. $(\neg A \to \neg C) \to (\neg A \to ((\neg\neg F \to C) \to \neg F))$ **MP** 11, 56
63. $(((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))$
$\to ((\neg A \to C) \to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))))$
$\to ((\neg A \to \neg C)$
$\to (((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))$
$\to ((\neg A \to C) \to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F)))))$ **A1**
64. $((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))$
$\to ((\neg A \to C) \to ((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F)))$ **A1**
65. $((\neg A \to \neg C)$
$\to (((\neg A \to (\neg\neg F \to C)) \to (\neg A \to \neg F))$

$\rightarrow ((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))))$
$\Rightarrow (((\neg A \rightarrow \neg C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))$
$\rightarrow ((\neg A \rightarrow \neg C)$
$\rightarrow ((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))))$ A2

66. $(\neg A \rightarrow \neg C)$
$\Rightarrow (((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F))))$ **MP** 63, 64

67. $((\neg A \rightarrow \neg C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))$
$\Rightarrow ((\neg A \rightarrow \neg C)$
$\rightarrow ((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F))))$ **MP** 65, 66

68. $(\neg A \rightarrow \neg C) \Rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F))$ **MP** 61, 62

69. $(((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))$
$\rightarrow (((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C)))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F))))$
$\Rightarrow ((\neg A \rightarrow \neg C)$
$\rightarrow (((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))$
$\rightarrow (((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C)))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F)))))$ A1

70. $((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))$
$\Rightarrow (((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C)))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F)))$ A2

71. $((\neg A \rightarrow \neg C)$
$\rightarrow (((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))$
$\rightarrow (((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C)))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F)))))$
$\Rightarrow (((\neg A \rightarrow \neg C)$
$\rightarrow ((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F))))$
$\rightarrow ((\neg A \rightarrow \neg C)$
$\rightarrow (((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C)))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F)))))$ A2

72. $(\neg A \rightarrow \neg C)$
$\Rightarrow (((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))$
$\rightarrow (((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C)))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F))))$ **MP** 69, 70

73. $((\neg A \rightarrow \neg C)$
$\rightarrow ((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F))))$
$\Rightarrow ((\neg A \rightarrow \neg C)$
$\rightarrow (((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C)))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F))))$ **MP** 71, 72

74. $(\neg A \rightarrow \neg C)$
$\Rightarrow ((\neg A \rightarrow C) \rightarrow ((\neg A \rightarrow (\neg\neg F \rightarrow C)) \rightarrow (\neg A \rightarrow \neg F)))$ **MP** 67, 68

75. $((\neg A \rightarrow \neg C)$
$\rightarrow (((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C)))$
$\rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F))))$
$\Rightarrow (((\neg A \rightarrow \neg C) \rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow (\neg\neg F \rightarrow C))))$
$\rightarrow ((\neg A \rightarrow \neg C) \rightarrow ((\neg A \rightarrow C) \rightarrow (\neg A \rightarrow \neg F))))$ A2

76.	$(\neg A \to \neg C)$	
	$\vdash ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C)))$	
	$\to ((\neg A \to C) \to (\neg A \to \neg F)))$	**MP** 73, 74
77.	$((\neg A \to \neg C) \to ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C))))$	
	$\vdash ((\neg A \to \neg C) \to ((\neg A \to C) \to (\neg A \to \neg F)))$	**MP** 75, 76
78.	$(\neg A \to \neg C) \vdash ((\neg A \to C) \to (\neg A \to (\neg\neg F \to C)))$	**MP** 25, 55
79.	$((\neg A \to \neg F) \to ((\neg A \to F) \to A))$	
	$\vdash ((\neg A \to C) \to ((\neg A \to \neg F) \to ((\neg A \to F) \to A)))$	**A1**
80.	$(\neg A \to \neg F) \vdash ((\neg A \to F) \to A)$	**A3**
81.	$((\neg A \to C) \to ((\neg A \to \neg F) \to ((\neg A \to F) \to A)))$	
	$\vdash ((\neg A \to C) \to (\neg A \to \neg F))$	
	$\to ((\neg A \to C) \to ((\neg A \to F) \to A)))$	**A2**
82.	$(\neg A \to C) \vdash ((\neg A \to \neg F) \to ((\neg A \to F) \to A))$	**MP** 79, 80
83.	$(((\neg A \to C) \to (\neg A \to \neg F))$	
	$\to ((\neg A \to C) \to ((\neg A \to F) \to A)))$	
	$\vdash ((\neg A \to \neg C)$	
	$\to (((\neg A \to C) \to (\neg A \to \neg F))$	
	$\to ((\neg A \to C) \to ((\neg A \to F) \to A))))$	**A1**
84.	$((\neg A \to C) \to (\neg A \to \neg F))$	
	$\vdash ((\neg A \to C) \to ((\neg A \to F) \to A))$	**MP** 81, 82
85.	$((\neg A \to \neg C)$	
	$\to (((\neg A \to C) \to (\neg A \to \neg F))$	
	$\to ((\neg A \to C) \to ((\neg A \to F) \to A))))$	
	$\vdash (((\neg A \to \neg C) \to ((\neg A \to C) \to (\neg A \to \neg F)))$	
	$\to ((\neg A \to \neg C) \to ((\neg A \to C) \to ((\neg A \to F) \to A))))$	**A2**
86.	$(\neg A \to \neg C)$	
	$\vdash (((\neg A \to C) \to (\neg A \to \neg F))$	
	$\to ((\neg A \to C) \to ((\neg A \to F) \to A)))$	**MP** 83, 84
87.	$((\neg A \to \neg C) \to ((\neg A \to C) \to (\neg A \to \neg F)))$	
	$\vdash ((\neg A \to \neg C) \to ((\neg A \to C) \to ((\neg A \to F) \to A)))$	**MP** 85, 86
88.	$(\neg A \to \neg C) \vdash ((\neg A \to C) \to (\neg A \to \neg F))$	**MP** 77, 78
89.	$(((\neg A \to C) \to ((\neg A \to F) \to A))$	
	$\to (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A)))$	
	$\vdash ((\neg A \to \neg C)$	
	$\to (((\neg A \to C) \to ((\neg A \to F) \to A))$	
	$\to (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A))))$	**A1**
90.	$((\neg A \to C) \to ((\neg A \to F) \to A))$	
	$\vdash (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A))$	**A2**
91.	$((\neg A \to \neg C)$	
	$\to (((\neg A \to C) \to ((\neg A \to F) \to A))$	
	$\to (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A))))$	
	$\vdash (((\neg A \to \neg C) \to ((\neg A \to C) \to ((\neg A \to F) \to A)))$	
	$\to ((\neg A \to \neg C)$	
	$\to (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A))))$	**A2**
92.	$(\neg A \to \neg C)$	
	$\vdash (((\neg A \to C) \to ((\neg A \to F) \to A))$	
	$\to (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A)))$	**MP** 89, 90

93. $((\neg A \to \neg C) \to ((\neg A \to C) \to ((\neg A \to F) \to A)))$
$\Rightarrow ((\neg A \to \neg C)$
$\to (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A)))$ **MP** 91, 92

94. $(\neg A \to \neg C) \Rightarrow ((\neg A \to C) \to ((\neg A \to F) \to A))$ **MP** 87, 88

95. $((\neg A \to \neg C)$
$\to (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A)))$
$\Rightarrow (((\neg A \to \neg C) \to ((\neg A \to C) \to (\neg A \to F)))$
$\to ((\neg A \to \neg C) \to ((\neg A \to C) \to A)))$ A2

96. $(\neg A \to \neg C)$
$\Rightarrow (((\neg A \to C) \to (\neg A \to F)) \to ((\neg A \to C) \to A))$ **MP** 93, 94

97. $((\neg A \to \neg C) \to ((\neg A \to C) \to (\neg A \to F)))$
$\Rightarrow ((\neg A \to \neg C) \to ((\neg A \to C) \to A))$ **MP** 95, 96

98. $(\neg A \to \neg C) \Rightarrow ((\neg A \to C) \to (\neg A \to F))$ **MP** 33, 34

99. $(\neg A \to \neg C) \Rightarrow ((\neg A \to C) \to A)$ **MP** 97, 98

where F is any axiom in which p appears, except that $F \neq \neg A$, $F \neq C$, $F \neq \neg C$, $F \neq \neg A \to C$, $\neg F \neq C$, and $C \neq F \to (\neg A \to F)$. This sequence clearly has properties (a) and (b), and as before, we search for pairs of the form D, $D \to E$ with D preceding $D \to E$. But E was chosen just to avoid such pairs, as we see from the following list:

if $F = \neg A$, 30, 32;

if $F = \neg A$ and $C = \neg A \to F$, 1, 26; 1, 48; 1, 52; 1, 55;
1, 34; 1, 50; 1, 54; 1, 82;

if $F = \neg A$ and $\neg C = \neg A \to F$, 1, 10; 1, 44; 1, 68; 1, 86; 1, 98;
1, 16; 1, 56; 1, 72; 1, 88; 1, 99;
1, 38; 1, 60; 1, 74; 1, 92;
1, 40; 1, 62; 1, 76; 1, 94;
1, 42; 1, 66; 1, 78; 1, 96;

if $F = C$, 1, 46; 32, 48; 32, 52; 32, 55;
32, 34; 32, 50; 32, 54; 32, 82;

if $F = C$ and $A = \neg F$, 1, 20; 21, 46;

if $F = \neg C$, 1, 36; 32, 42; 32, 62; 32, 74; 32, 88; 32, 98;
31, 97; 32, 44; 32, 66; 32, 76; 32, 92; 32, 99;
32, 38; 32, 56; 32, 68; 32, 78; 32, 94;
32, 40; 32, 60; 32, 72; 32, 86; 32, 96;

if $F = \neg C$ and $A = \neg F$, 1, 2; 12, 36;

if $F = \neg A \to C$, 1, 33; 30, 34; 30, 50; 30, 54; 30, 82;
28, 51; 30, 48; 30, 52; 30, 55;

if $\neg F = C$, 80, 90;

and

if $C = F \to (\neg A \to F)$, 1, 21.

Finally, suppose the theorem is true about A and about $A \to B$. Let q be a proposition letter which does not appear in either A or B. Then A has a proof $s_1 = A_1 \ldots A_n$ with the properties:

(a_1) q appears in each A_i, $1 \leq i < n$
(b_1) there exist A_{j_1}, A_{k_1} in s_1 such that A_{k_1} is $A_{j_1} \to A_n$
(c_1) s_1 is a skew sequence,

and $A \to B$ has a proof $s_2 = B_1 \ldots B_m$ with the properties:

(a_2) q appears in each B_i, $1 \leq i < m$
(b_2) there exist B_{j_2}, B_{k_2} in s_2 such that B_{k_2} is $B_{j_2} \to B_m$
(c_2) s_2 is a skew sequence.

We will use s_1 and s_2 to build the desired proof of B, but first we need to introduce some notation and define a trio of formulas for future use. Throughout the rest of the proof we write C^* for D if $C = D \to E$. Also, we write $C(F)$ for that formula obtained from C by replacing each occurrence of q by F. Now let F_1 be any formula in which p appears, except that:

$$\begin{aligned}
&A_{j_1}(F_1) \neq A,\\
&A_{j_1}(F_1) \neq A \to B,\\
&A_i(F_1) \neq B^*,\ 1 \leq i \leq j_1.
\end{aligned}$$

Let F_2 be any formula in which p appears, except that:

$$\begin{aligned}
&A_{j_1}(F_1) \neq B_{j_2}(F_2) \to (A \to B),\\
&B_{j_2}(F_2) \neq A,\\
&B_{j_2}(F_2) \neq A_{j_1}(F_1),\\
&B_{j_2}(F_2) \neq A_{j_1}(F_1) \to A,\\
&B_i(F_2) \neq A_{j_1}(F_1),\ 1 \leq i \leq j_2,\\
&B_i(F_2) \neq A_{j_1}(F_1) \to A,\ 1 \leq i < k_2 < i \leq j_2,\\
&B_i(F_2) \neq (A_j(F_1))^*,\ 1 \leq i \leq j_2,\ 1 \leq j \leq j_1,\\
&B_i(F_2) \neq B^*,\ 1 \leq i < k_2 < i \leq j_2.
\end{aligned}$$

And let F be any axiom in which p appears, except that:

$$\begin{aligned}
&F \neq A_i(F_1),\ 1 \leq i \leq j_1,\\
&F \neq B^*,\\
&F \neq B_{j_2}(F_2),\\
&F \neq B_{j_2}(F_2) \to (A \to B),\\
&F \neq B_i(F_2),\ 1 \leq i \leq j_2,\\
&F^* \neq A_i(F_1),\ 1 \leq i \leq j_1,\\
&F^* \neq B_i(F_2),\ 1 \leq i \leq j_2,\\
&A_{j_1}(F_1) \neq F \to A,\\
&A_{j_1}(F_1) \neq F \to (A \to B),\\
&A_{j_1}(F_1) \neq F \to B_{j_2}(F_2),\\
&A_{j_1}(F_1) \neq (F \to A) \to (F \to B),\\
&A_{j_1}(F_1) \neq F \to (B_{j_2}(F_2) \to (A \to B)),\\
&A_{j_1}(F_1) \neq B_{j_2}(F_2) \to (F \to B_{j_2}(F_2)),\\
&A_{j_1}(F_1) \neq (F \to B_{j_2}(F_2)) \to (F \to (A \to B)),
\end{aligned}$$

$A_{j_1}(F_1) \neq (F \to (A \to B)) \to ((F \to A) \to (F \to B))$,
$A_{j_1}(F_1) \neq (B_{j_2}(F_2) \to (A \to B)) \to (F \to (B_{j_2}(F_2) \to (A \to B)))$,
$A_{j_1}(F_1) \neq (F \to (B_{j_2}(F_2) \to (A \to B)))$
$\qquad \to ((F \to B_{j_2}(F_2)) \to (F \to (A \to B)))$,
$A_i(F_1) \neq F \to A_{j_1}(F_1)$, $1 \leq i < j_1$,
$(A_i(F_1))^* \neq F \to (A \to B)$, $1 \leq i \leq j_1$,
$(A_i(F_1))^* \neq F \to B_{j_2}(F_2)$, $1 \leq i \leq j_1$,
$(A_i(F_1))^* \neq (F \to A) \to (F \to B)$, $1 \leq i \leq j_1$,
$(A_i(F_1))^* \neq A_{j_1}(F_1) \to (F \to A_{j_1}(F_1))$, $1 \leq i < k_1 < i < j_1$,
$(A_i(F_1))^* \neq B_{j_2}(F_2) \to (F \to B_{j_2}(F_2))$, $1 \leq i \leq j_1$,
$(A_i(F_1))^* \neq F \to (B_{j_2}(F_2) \to (A \to B))$, $1 \leq i \leq j_1$,
$(A_i(F_1))^* \neq (F \to B_{j_2}(F_2)) \to (F \to (A \to B))$, $1 \leq i \leq j_1$,
$(A_i(F_1))^* \neq (A_{j_1}(F_1) \to A) \to (F \to (A_{j_1}(F_1) \to A))$, $1 \leq i < k_1 < i < j_1$,
$(A_i(F_1))^* \neq (F \to (A \to B)) \to ((F \to A) \to (F \to B))$, $1 \leq i \leq j_1$,
$(A_i(F_1))^* \neq (F \to (A_{j_1}(F_1) \to A)) \to ((F \to A_{j_1}(F_1)) \to (F \to A))$,
$\qquad 1 \leq i < k_1 < i < j_1$,
$(A_i(F_1))^* \neq (B_{j_2}(F_2) \to (A \to B)) \to (F \to (B_{j_2}(F_2) \to (A \to B)))$, $1 \leq i \leq j_1$,
$(A_i(F_1))^* \neq (F \to (B_{j_2}(F_2) \to (A \to B)))$
$\qquad \to ((F \to B_{j_2}(F_2)) \to (F \to (A \to B)))$, $1 \leq i \leq j_1$,
$B^* \neq F \to A$,
$B^* \neq F \to A_{j_1}(F_1)$,
$B^* \neq F \to B_{j_2}(F_2)$,
$B^* \neq F \to (A_{j_1}(F_1) \to A)$,
$B^* \neq A_{j_1}(F_1) \to (F \to A_{j_1}(F_1))$,
$B^* \neq B_{j_2}(F_2) \to (F \to B_{j_2}(F_2))$,
$B^* \neq (F \to A_{j_1}(F_1)) \to (F \to A)$,
$B^* \neq (A_{j_1}(F_1) \to A) \to (F \to (A_{j_1}(F_1) \to A))$,
$B^* \neq (F \to (A_{j_1}(F_1) \to A)) \to ((F \to A_{j_1}(F_1)) \to (F \to A))$,
$B_i(F_2) \neq F \to A$, $1 \leq i < k_2 < i \leq j_2$,
$B_i(F_2) \neq F \to A_{j_1}(F_1)$, $1 \leq i \leq j_2$,
$B_i(F_2) \neq F \to B_{j_2}(F_2)$, $1 \leq i < j_2$,
$B_i(F_2) \neq F \to (A_{j_1}(F_1) \to A)$, $1 \leq i \leq j_2$,
$(B_i(F_2))^* \neq B_{j_2}(F_2) \to (F \to B_{j_2}(F_2))$, $1 \leq i < k_2 < i < j_2$,
$(B_i(F_2))^* \neq (F \to (A \to B)) \to ((F \to A) \to (F \to B))$, $1 \leq i \leq j_2$,
$(B_i(F_2))^* \neq (B_{j_2}(F_2) \to (A \to B)) \to (F \to (B_{j_2}(F_2) \to (A \to B)))$,
$\qquad 1 \leq i < k_2 < i < j_2$,
$(B_i(F_2))^* \neq (F \to (B_{j_2}(F_2) \to (A \to B)))$
$\qquad \to ((F \to B_{j_2}(F_2)) \to (F \to (A \to B)))$, $1 \leq i < k_2 < i < j_2$.

Having chosen specific formulas F_1, F_2, and F, consider the following sequence:

1.	$(B_{j_2}(F_2) \to (A \to B)) \Rightarrow (F \to (B_{j_2}(F_2) \to (A \to B)))$	A1
2.	$(F \to (B_{j_2}(F_2) \to (A \to B))) \Rightarrow ((F \to B_{j_2}(F_2)) \to (F \to (A \to B)))$	A2
3.	$B_{j_2}(F_2) \Rightarrow (F \to B_{j_2}(F_2))$	A1
4.	$(F \to (A \to B)) \Rightarrow ((F \to A) \to (F \to B))$	A2
5.	$B_1(F_2)$	
⋮	⋮	

$k_2 + 4.$	$B_{j_2}(F_2) \Rightarrow (A \to B)$	
$\vdots$	$\vdots$	
$j_2 + 4.$	$B_{j_2}(F_2)$	
$j_2 + 5.$	$F \Rightarrow (B_{j_2}(F_2) \to (A \to B))$	**MP** $1, k_2 + 4$
$j_2 + 6.$	$(F \to B_{j_2}(F_2)) \Rightarrow (F \to (A \to B))$	**MP** $2, j_2 + 5$
$j_2 + 7.$	$F \Rightarrow B_{j_2}(F_2)$	**MP** $3, j_2 + 4$
$j_2 + 8.$	$F \Rightarrow (A \to B)$	**MP** $j_2 + 6, j_2 + 7$
$j_2 + 9.$	$(F \to A) \Rightarrow (F \to B)$	**MP** $4, j_2 + 8$
$j_2 + 10.$	$(A_{j_1}(F_1) \to A) \Rightarrow (F \to (A_{j_1}(F_1) \to A))$	A1
$j_2 + 11.$	$(F \to (A_{j_1}(F_1) \to A)) \Rightarrow ((F \to A_{j_1}(F_1)) \to (F \to A))$	A2
$j_2 + 12.$	$A_{j_1}(F_1) \Rightarrow (F \to A_{j_1}(F_1))$	A1
$j_2 + 13.$	$A_1(F_1)$	
$\vdots$	$\vdots$	
$k_1 + j_2 + 12.$	$A_{j_1}(F_1) \Rightarrow A$	
$\vdots$	$\vdots$	
$j_1 + j_2 + 12.$	$A_{j_1}(F_1)$	
$j_1 + j_2 + 13.$	$F \Rightarrow (A_{j_1}(F_1) \to A)$	**MP** $j_2 + 10, k_1 + j_2 + 12$
$j_1 + j_2 + 14.$	$(F \to A_{j_1}(F_1)) \Rightarrow (F \to A)$	**MP** $j_2 + 11, j_1 + j_2 + 13$
$j_1 + j_2 + 15.$	$F \Rightarrow A_{j_1}(F_1)$	**MP** $j_2 + 12, j_1 + j_2 + 12$
$j_1 + j_2 + 16.$	$F \Rightarrow A$	**MP** $j_1 + j_2 + 14, j_1 + j_2 + 15$
$j_1 + j_2 + 17.$	$F \Rightarrow B$	**MP** $j_2 + 9, j_1 + j_2 + 16$
$j_1 + j_2 + 18.$	F	
$j_1 + j_2 + 19.$	B	**MP** $j_1 + j_2 + 17, j_1 + j_2 + 18,$

where $B_1(F_2) \ldots B_{j_2}(F_2) \to (A \to B) \ldots B_{j_2}(F_2)$ and $A_1(F_1) \ldots A_{j_1}(F_1) \to A \ldots A_{j_1}(F_1)$ are just $B_1(F_2) \ldots B_{j_2}(F_2)$ and $A_1(F_1) \ldots A_{j_1}(F_1)$ written more explicitly. (We use here properties (b_2), (c_2), (b_1), and (c_1), and the fact that q does not appear in A or B.) We claim that the above sequence will do for s, i.e., that it has properties (a), (b), and (c).

Indeed, s clearly has property (a), because by (a_1) and (a_2) at least one of F_1, F_2, and F is a subformula of each formula in s. And certainly s has property (b). So to complete the proof it remains only to show that s has property (c). As before, we search in s for pairs of the form D, $D \to E$ with D preceding $D \to E$. But F_1, F_2, and F were chosen precisely to avoid such pairs, as we see from the following list:

if $(B_{j_2}(F_2) \to (A \to B)) \to (F \to (B_{j_2}(F_2) \to (A \to B))) = (B_i(F_2))^*$,
$1 \leq i < k_2 < i < j_2$, $1, i + 4$;

if $(B_{j_2}(F_2) \to (A \to B)) \to (F \to (B_{j_2}(F_2) \to (A \to B))) = A_{j_1}(F_1)$, $1, j_2 + 12$;

if $(B_{j_2}(F_2) \to (A \to B)) \to (F \to (B_{j_2}(F_2) \to (A \to B))) = (A_i(F_1))^*$, $1 \leq i \leq j_1$,
$1, i + j_2 + 12$;

if $F = B_{j_2}(F_2) \to (A \to B)$ and $A_{j_1}(F_1) = F \to (B_{j_2}(F_2) \to (A \to B))$,
$1, j_1 + j_2 + 14$;

if $(F \to (B_{j_2}(F_2) \to (A \to B))) \to ((F \to B_{j_2}(F_2)) \to (F \to (A \to B))) = (B_i(F_2))^*$,
$1 \leq i < k_2 < i < j_2$, $2, i + 4$;

if $(F \to (B_{j_2}(F_2) \to (A \to B))) \to ((F \to B_{j_2}(F_2)) \to (F \to (A \to B))) = A_{j_1}(F_1)$, $2, j_2 + 12$;

if $(F \to (B_{j_2}(F_2) \to (A \to B))) \to ((F \to B_{j_2}(F_2)) \to (F \to (A \to B))) = (A_i(F_1))^*$, $1 \leq i \leq j_1$, $2, i + j_2 + 12$;

if $F = B_{j_2}(F_2) = A = B$, $3, 4$;

if $B_{j_2}(F_2) \to (F \to B_{j_2}(F_2)) = (B_i(F_2))^*$, $1 \leq i < k_2 < i < j_2$, $3, i + 4$;

if $F = B_{j_2}(F_2)$ and $A = F \to B_{j_2}(F_2)$, $3, j_2 + 9$;

if $A_{j_1}(F_1) = B_{j_2}(F_2)$ and $A = F \to B_{j_2}(F_2)$, $3, j_2 + 10$;

if $F = B_{j_2}(F_2) = A_{j_1}(F_1) = A$, $3, j_2 + 11$; $j_2 + 7, j_2 + 10$;

if $B_{j_2}(F_2) \to (F \to B_{j_2}(F_2)) = A_{j_1}(F_1)$, $3, j_2 + 12$;

if $B_{j_2}(F_2) \to (F \to B_{j_2}(F_2)) = (A_i(F_1))^*$, $1 \leq i \leq j_1$, $3, i + j_2 + 12$;

if $F = B_{j_2}(F_2)$ and $A_{j_1}(F_1) = F \to B_{j_2}(F_2)$, $3, j_1 + j_2 + 14$;

if $B_{j_2}(F_2) \to (F \to B_{j_2}(F_2)) = B^*$, $3, j_1 + j_2 + 19$;

if $(F \to (A \to B)) \to ((F \to A) \to (F \to B)) = (B_i(F_2))^*$, $1 \leq i \leq j_2$, $4, i + 4$;

if $(F \to (A \to B)) \to ((F \to A) \to (F \to B)) = A_{j_1}(F_1)$, $4, j_2 + 12$;

if $(F \to (A \to B)) \to ((F \to A) \to (F \to B)) = (A_i(F_1))^*$, $1 \leq i \leq j_1$, $4, j_1 + j_2 + 12$;

if $F = B_i(F_2)$, $1 \leq i \leq j_2$, $i + 4, j_2 + 5$; $i + 4, j_1 + j_2 + 13$;
$i + 4, j_2 + 7$; $i + 4, j_1 + j_2 + 15$;
$i + 4, j_2 + 8$; $i + 4, j_1 + j_2 + 16$;
$i + 4, j_1 + j_2 + 17$;

if $B_i(F_2) = F \to B_{j_2}(F_2)$, $1 \leq i < j_2$, $i + 4, j_2 + 6$;

if $B_i(F_2) = F \to A$, $1 \leq i < k_2 < i \leq j_2$, $i + 4, j_2 + 9$;

if $B_i(F_2) = A_{j_1}(F_1) \to A$, $1 \leq i < k_2 < i \leq j_2$, $i + 4, j_2 + 10$;

if $B_i(F_2) = F \to (A_{j_1}(F_1) \to A)$, $1 \leq i \leq j_2$, $i + 4, j_2 + 11$;

if $B_i(F_2) = A_{j_1}(F_1)$, $1 \leq i \leq j_2$, $i + 4, j_2 + 12$;

if $B_i(F_2) = (A_j(F_1))^*$, $1 \leq i \leq j_2$, $1 \leq j \leq j_1$, $i + 4, j + j_2 + 12$;

if $B_i(F_2) = F \to A_{j_1}(F_1)$, $1 \leq i \leq j_2$, $i + 4, j_1 + j_2 + 14$;

if $B_i(F_2) = F^*$, $1 \leq i \leq j_2$, $i + 4, j_1 + j_2 + 18$;

if $B_i(F_2) = B^*$, $1 \leq i < k_2 < i \leq j_2$, $i + 4, j_1 + j_2 + 19$;

if $F \to (B_{j_2}(F_2) \to (A \to B)) = A_{j_1}(F_1)$, $j_2 + 5, j_2 + 12$;

if $F \to (B_{j_2}(F_2) \to (A \to B)) = (A_i(F_1))^*$, $1 \leq i \leq j_1$, $j_2 + 5, i + j_2 + 12$;

if $A_{j_1}(F_1) = B_{j_2}(F_2) \to (A \to B)$, $j_2 + 5, j_1 + j_2 + 14$;

if $(F \to B_{j_2}(F_2)) \to (F \to (A \to B)) = A_{j_1}(F_1)$, $j_2 + 6$, $j_2 + 12$;

if $(F \to (B_{j_2}(F_2)) \to (F \to (A \to B)) = (A_i(F_1))^*$, $1 \leq i \leq j_1$, $j_2 + 6$, $i + j_2 + 12$;

if $B_{j_2}(F_2) = A$, $j_2 + 7$, $j_2 + 9$;

if $B_{j_2}(F_2) = A_{j_1}(F_1) \to A$, $j_2 + 7$, $j_2 + 11$;

if $F \to B_{j_2}(F_2) = A_{j_1}(F_1)$, $j_2 + 7$, $j_2 + 12$;

if $F \to B_{j_2}(F_2) = (A_i(F_1))^*$, $1 \leq i \leq j_1$, $j_2 + 7$, $i + j_2 + 12$;

if $A_{j_1}(F_1) = B_{j_2}(F_2)$, $j_2 + 7$, $j_1 + j_2 + 14$;

if $F \to B_{j_2}(F_2) = B^*$, $j_2 + 7$, $j_1 + j_2 + 19$;

if $A = A_{j_1}(F_1) = B$, $j_2 + 8$, $j_2 + 11$;

if $F \to (A \to B) = A_{j_1}(F_1)$, $j_2 + 8$, $j_2 + 12$;

if $F \to (A \to B) = (A_i(F_1))^*$, $1 \leq i \leq j_1$, $j_2 + 8$, $i + j_2 + 12$;

if $A_{j_1}(F_1) = A \to B$, $j_2 + 8$, $j_1 + j_2 + 14$;

if $A_{j_1}(F_1) = F \to A$ and $A = F \to B$, $j_2 + 9$, $j_2 + 10$;

if $(F \to A) \to (F \to B) = A_{j_1}(F_1)$, $j_2 + 9$, $j_2 + 12$;

if $(F \to A) \to (F \to B) = (A_i(F_1))^*$, $1 \leq i \leq j_1$, $j_2 + 9$, $i + j_2 + 12$;

if $(A_{j_1}(F_1) \to A) \to (F \to (A_{j_1}(F_1) \to A)) = (A_i(F_1))^*$, $1 \leq i < k_1 < i < j_1$,
$j_2 + 10$, $i + j_2 + 12$;

if $(A_{j_1}(F_1) \to A) \to (F \to (A_{j_1}(F_1) \to A)) = B^*$, $j_2 + 10$, $j_1 + j_2 + 19$;

if $(F \to (A_{j_1}(F_1) \to A)) \to ((F \to A_{j_1}(F_1)) \to (F \to A)) = (A_i(F_1))^*$,
$1 \leq i < k_1 < i < j_1$, $j_2 + 11$, $i + j_2 + 12$;

if $(F \to (A_{j_1}(F_1) \to A)) \to ((F \to A_{j_1}(F_1)) \to (F \to A)) = B^*$,
$j_2 + 11$, $j_1 + j_2 + 19$;

if $A_{j_1}(F_1) \to (F \to A_{j_1}(F_1)) = (A_i(F_1))^*$, $1 \leq i < k_1 < i < j_1$, $j_2 + 12$, $i + j_2 + 12$;

if $A_{j_1}(F_1) \to (F \to A_{j_1}(F_1)) = B^*$, $j_2 + 12$, $j_1 + j_2 + 19$;

if $F = A_i(F_1)$, $1 \leq i \leq j_1$, $i + j_2 + 12$, $j_1 + j_2 + 13$; $i + j_2 + 12$, $j_1 + j_2 + 16$;
$i + j_2 + 12$, $j_1 + j_2 + 15$; $i + j_2 + 12$, $j_1 + j_2 + 17$;

if $A_i(F_1) = F \to A_{j_1}(F_1)$, $1 \leq i < j_1$, $i + j_2 + 12$, $j_1 + j_2 + 14$;

if $A_i(F_1) = F^*$, $1 \leq i \leq j_1$, $i + j_2 + 12$, $j_1 + j_2 + 18$;

if $A_i(F_1) = B^*$, $1 \leq i \leq j_1$, $i + j_2 + 12$, $j_1 + j_2 + 19$;

if $F \to (A_{j_1}(F_1) \to A) = B^*$, $j_1 + j_2 + 13$, $j_1 + j_2 + 19$;

if $(F \to A_{j_1}(F_1)) \to (F \to A) = B^*$, $j_1 + j_2 + 14$, $j_1 + j_2 + 19$;

if $F \to A_{j_1}(F_1) = B^*$, $j_1 + j_2 + 15$, $j_1 + j_2 + 19$;

if $F \to A = B^*, j_1 + j_2 + 16, j_1 + j_2 + 19$;

and finally,

if $F = B^*, \ j_1 + j_2 + 18, j_1 + j_2 + 19$.

We conclude that there are no such pairs for our choice of F_1, F_2, and F, and this completes the proof. Q.E.D.

Corollary 1: *The system* **P** *is a skew system.*

Proof: By Theorem 1, part (c), every theorem of **P** has a skew proof. Q.E.D.

3 *Corollaries* The method used to prove that the system **P** is skew yields two interesting corollaries. They say, in effect, that no formula is essential to a proof in **P**.

Corollary 2: *Let B be a theorem of* **P**. *Then for any finite list of formulas $F_1 \ldots F_k$, B has a proof $s = A_1 \ldots A_n$ with the properties:*

1. *$A_i \neq F_j$ for $1 \leq i < n$, $1 \leq j \leq k$,*
2. *there exist A_i, A_j in s such that $A_i = A_j \to A_n$.*

Proof: Choose a proposition letter p which does not appear in any of the F_j, $1 \leq j \leq k$. By Theorem 1, parts (a) and (b), B has a proof with the desired properties. Q.E.D.

In some respects this result is a bit startling. For example, given a formula F and an axiom A of **P**, it seems natural to write the following simple proof of $F \to A$:

1.	A	
2.	$A \to (F \to A)$	A1
3.	$F \to A$	**MP** 1, 2

In fact, it is difficult to imagine a proof of $F \to A$ in which formulas 1. and 2. do not appear. But by Corollary 2 there is such a proof.

Corollary 3: *Let* **P*** *be obtained from* **P** *by omitting a finite number of axioms $A_1 \ldots A_k$. Then* **P*** *is equivalent to* **P**.

Proof: We must show that every theorem of **P** has a proof in **P***. Let B be a theorem of **P**, and let $F_1 \ldots F_k$ be $A_1 \ldots A_k$. Then Corollary 2 gives a proof of B in **P***. Q.E.D.

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