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CONCERNING THE PROPER AXIOMS OF S4.02

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In [4] it has been established that the addition of the following formula

Ł1 *ℭℭℭpLppCLMLpp*

as a new axiom, to S4 generates a system, called S4.02, which is a proper extension of S4. And obviously, *cf.* [6], in the field of S4, **Ł1** is inferentially equivalent to

Ł2 *ℭℭℭpLppLCLMLpp*

In this note it will be shown that in the field of S4 each of the following two formulas

Ł3 *ℭℭℭpLpLpCLMLpLp*

and

Ł4 *ℭℭℭpLpLpCLMLpp*

is inferentially equivalent to **Ł1**.

Proof:

1 Assume S4 and **Ł3**. Then, obviously, we have **Ł4**. Now, S4 yields the following formulas:

Z1 *ℭLpLLp*
Z2 *ℭℭpqℭLpLq*

Whence,

Z3 *ℭℭLℭpLpLpCLMLpp* [**Ł3**; *Z1*]
Ł1 *ℭℭℭpLppCLMLpp* [*Z2, p/ℭpLp, q/p*; *Z3*; S1°]

Thus, in the field of S4: {**Ł3**} → {**Ł4**} → {**Ł1**}.

2 Now, let us assume S4 and **Ł1**. Then:

Z1 *ℭℭvℭqrℭℭℭprsℭvℭℭpqs* [S4]
Z2 *ℭℭpqℭℭvℭℭprsℭvℭℭpqs* [S4]

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Z3 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}\mathfrak{C}rs\mathfrak{C}\mathfrak{C}qr\mathfrak{C}ps$ [S3°]

Z4 $\mathfrak{C}\mathfrak{C}ts\mathfrak{C}\mathfrak{C}pq\mathfrak{C}\mathfrak{C}v\mathfrak{C}\mathfrak{C}prt\mathfrak{C}v\mathfrak{C}\mathfrak{C}qrs$ [S4]

Z5 $\mathfrak{C}\mathfrak{C}pCqr\mathfrak{C}\mathfrak{C}rs\mathfrak{C}pCqs$ [S3°]

Z6 $\mathfrak{C}\mathfrak{C}pCqr\mathfrak{C}\mathfrak{C}pCqs\mathfrak{C}\mathfrak{C}rCst\mathfrak{C}pCqt$ [S3°]

Z7 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqCrp\mathfrak{C}\mathfrak{C}pqCrq$ [S2]

Z8 $\mathfrak{C}CqrCCpqCpr$ [S1°]

Z9 $\mathfrak{C}\mathfrak{C}p\mathfrak{C}pq\mathfrak{C}pq$ [S2]

Z10 $\mathfrak{C}pCNpq$ [S1°]

Z11 $\mathfrak{C}NpCpq$ [S1°]

Z12 $\mathfrak{C}LpCNpq$ [S2]

Z13 $\mathfrak{C}\mathfrak{C}NppCpLp$ [S2°]

Z14 $\mathfrak{C}\mathfrak{C}pLq\mathfrak{C}pq$ [S2]

Z15 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqr\mathfrak{C}\mathfrak{C}NrLNpLr$ [S3°]

Z16 $\mathfrak{C}CMpLq\mathfrak{C}pLq$ [S4°; *cf.* [3]]

Z17 $\mathfrak{C}CpLpCCMpp\mathfrak{C}pLp$

[*Z5*, $p/CpLp$, $q/CMpp$, $r/CMpLp$, $s/\mathfrak{C}pLp$; *Z8*, q/p, p/Mp, r/Lp; *Z16*, q/p]

Z18 $\mathfrak{C}CNpLNqCMqp$ [S1°]

Z19 $\mathfrak{C}CLMLCqLprCLMLpr$ [S4°]

Z20 $\mathfrak{C}CLMLCNpqrCLMLpr$ [S2°]

Z21 $\mathfrak{C}CLMLCNpqCNrLNsCLMLpCMsr$

[*Z5*, $p/CLMLCNpqCNrLNs$, $q/LMLp$, $r/CNrLNs$, $s/CMsr$; *Z20*, $r/CNrLNs$; *Z18*, p/r, q/s]

Z22 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLp\mathfrak{C}\mathfrak{C}Np\mathfrak{C}pLpCpLp$

[*Z1*, $v/\mathfrak{C}\mathfrak{C}pLpLp$, $q/\mathfrak{C}pLp$, r/p, p/Np, $s/CpLp$; *Z14*, $p/\mathfrak{C}pLp$, q/p; *Z13*]

Z23 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLp\mathfrak{C}\mathfrak{C}CpLp\mathfrak{C}pLpCpLp$

[*Z2*, p/Np, $q/CpLp$, $v/\mathfrak{C}\mathfrak{C}pLpLp$, $r/\mathfrak{C}pLp$, $s/CpLp$; *Z11*, q/Lp; *Z22*]

Z24 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCLMLpCpLp$

[*Z3*, $p/\mathfrak{C}\mathfrak{C}pLpLp$, $q/\mathfrak{C}\mathfrak{C}CpLp\mathfrak{C}pLpCpLp$, $r/CLMLCpLpCpLp$, $s/CLMLpCpLp$; *Z23*; *Z19*, q/p, $r/CpLp$; **Ł1**, $p/CpLp$]

Z25 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLp\mathfrak{C}\mathfrak{C}NpLNpLp$ [*Z14*, $p/\mathfrak{C}pLp$, q/p; *Z15*, q/Lp, r/p; S1°]

Z26 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLp\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLp\mathfrak{C}\mathfrak{C}p\mathfrak{C}NpLNpLp$

[*Z1*, $v/\mathfrak{C}\mathfrak{C}pLpLp$, $q/\mathfrak{C}NpLNp$, r/Lp, s/Lp; *Z25*]

Z27 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLp\mathfrak{C}\mathfrak{C}p\mathfrak{C}NpLNpLp$ [*Z9*, $p/\mathfrak{C}\mathfrak{C}pLpLp$, $q/\mathfrak{C}\mathfrak{C}p\mathfrak{C}NpLNpLp$; *Z26*]

Z28 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLp\mathfrak{C}\mathfrak{C}CNpLNp\mathfrak{C}NpLNpCNpLNp$

[*Z4*, t/Lp, $s/CNpLNp$, $q/CNpLNp$, $v/\mathfrak{C}\mathfrak{C}pLpLp$, $r/\mathfrak{C}NpLNp$; *Z12*, q/LNp; *Z11*, q/LNp; *Z27*]

Z29 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCLMLpCMpp$

[*Z3*, $p/\mathfrak{C}\mathfrak{C}pLpLp$, $q/\mathfrak{C}\mathfrak{C}CNpLNp\mathfrak{C}NpLNpCNpLNp$, $r/CLMLCNpLNpCNpLNp$, $s/CLMLpCMpp$; *Z28*; *Z21*, q/LNp, r/p, s/p; **Ł1**, $p/CNpLNp$]

Z30 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCLMLp\mathfrak{C}pLp$

[*Z6*, $p/\mathfrak{C}\mathfrak{C}pLpLp$, $q/LMLp$, $r/CpLp$, $s/CMpp$, $t/\mathfrak{C}pLp$; *Z24*; *Z29*; *Z17*]

Ł3 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCLMLpLp$ [*Z7*, $p/\mathfrak{C}pLp$, q/Lp, $r/LMLp$; *Z30*]

Thus, in the field of S4: {**Ł1**} → {**Ł3**}. Hence, we have proved

{S4.02} ⇄ {S4; **Ł1**} ⇄ {S4; **Ł2**} ⇄ {S4; **Ł3**} ⇄ {S4; **Ł4**}

Remarks:

1 It should be noted that the proof given above is strictly analogous to the deductions which I presented in [5], pp. 366-367, section **1.2.2.**[1] Namely, in that paper a logical proof was given of Schumm's result, *cf.* [1], which he had obtained metalogically that in the field of S4 the so-called Diodorian modal formulas

N1 *ℭℭℭpLppCMLpp*

and

M1 *ℭℭℭpLpLpCMLpLp*

are inferentially equivalent. Obviously, an analogy existing between the proofs given in [5] and in this note is due to the fact that **N1** and **M1** have syntactical structures very similar to those which **Ł1** and **Ł3** possess respectively.

2 Recently, *cf.* [2], Schumm has proved metalogically that, in the field of S3, the formulas **Ł1** and **Ł2** are inferentially equivalent. It is an interesting open problem whether, in the field of S3, each of the following formulas **Ł3**, **Ł4** and

Ł5 *ℭℭℭpLpLpLCLMLpLp*
Ł6 *ℭℭℭpLpLpLCLMLpp*

is inferentially equivalent to **Ł1**. A similar open problem is also worth investigating. Namely, whether in the field of S3 all the known proper axioms of S4.1 are mutually equivalent.

REFERENCES

[1] Schumm, G. F., "Solutions to four modal problems of Sobociński," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 335-340.

[2] Schumm, G. F., "S3.02 = S3.03," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 147-148.

[3] Sobociński, B., "A note on modal systems," *Notre Dame Journal of Formal Logic*, vol. IV (1963), pp. 355-357.

[4] Sobociński, B., "A proper subsystem of S4.04," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 381-384.

1. In the paper mentioned here two obvious misprints appear. Viz., on p. 366, line 14, formula *Z17* should have the form: *ℭℭvℭqrℭℭℭprsℭvℭℭpqs* and on the same page, line 28, in the line proof of *Z26* a condition "S2" is missing.

[5] Sobociński, B., "Concerning some extensions of S4," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 363-370.

[6] Sobociński, B., "Modal system S3 and the proper axioms of S4.02 and S4.04," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 415-418.

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