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PRINCIPIA MATHEMATICA DESCRIPTION THEORY: THE CLASSICAL AND AN ALTERNATIVE NOTATION

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The class of description formulas is not rigorously determined in *Principia Mathematica.* And the syntax of ℩-formulas therein countenanced ill-suits their semantics. Correctives for both defects (if the second condition be a defect) are presented below.[1] First, rules for the language of a base theory $\mathcal{J}$ along with relevant ancillary definitions are set out. Next, these rules are augmented in order to secure the generation of ℩-formulas, and ancillary definitions are amended. Then, a simpler notation is introduced. Next, a theory $\mathcal{R}$ is explained the language of which embraces both notations, and it is shown how they are interchangeable and how the new could indeed be employed as a scaffolding for the introduction of the old. Last, comments are made regarding the semantic misleadingness of the classical notation, and regarding a well-known property of this notation, *viz.*, the term-like behaviour of 'proper' ℩-descriptions, that mitigates this misleadingness. These comments are facilitated by the presence in the base theory, and thus in its extension $\mathcal{R}$, of 'Fregean' descriptions that are genuine terms.

1. The base theory $\mathcal{J}$. The framework assumed is the theory of formal languages set out by Donald Kalish and Richard Montague, and I take as my base theory their description calculus.[2] Thus the vocabulary of $\mathcal{J}$ consists of the logical constants $\bigwedge$, $\bigvee$, ℩, $\sim$, $\rightarrow$, $\vee$, $\wedge$, $\longleftrightarrow$, and $=$, lower-case italicized Latin letters to serve as variables, parentheses, superscripted letters 'A' through 'E' to serve as n-place operation letters, and superscripted letters

1. Alfred North Whitehead and Bertrand Russell, *Principia Mathematica to* *56 (Cambridge, 1962). Several of the ideas contained in this note took shape in conversations with Richard Montague and David Kaplan, and I am also indebted to Bas van Fraassen for comments on an intermediate draft.

2. Donald Kalish and Richard Montague, *Logic: Techniques of Formal Reasoning* (New York, 1964). The "description calculus" is presented in Chapter VII; Chapters VIII and IX contain the general framework, formal languages and theories.

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'H' through 'Z' to serve as n-place predicate letters. A letter's superscript indicates the number of the letter's places. The letters 'F' and 'G' are set aside for use in schemata.

The *terms* and *formulas* of $\mathcal{J}$ are generated by applications of the following rules:

(1) A variable is a term.
(2) If ζ is an n-place operation letter and $\alpha_1 \ldots \alpha_n$ are n not necessarily distinct terms, then $\zeta\ \alpha_1 \ldots \alpha_n$ is a term.
(3) If π is an n-place predicate letter and $\alpha_1 \ldots \alpha_n$ are n not necessarily distinct terms, then $\pi\ \alpha_1 \ldots \alpha_n$ is a formula.
(4) If α and β are terms, then $\alpha = \beta$ is a formula.
(5) If ϕ, ψ are formulas, then $\sim\phi$, $(\phi \rightarrow \psi)$, $(\phi \vee \psi)$, $(\phi \wedge \psi)$, and $(\phi \leftrightarrow \psi)$ are formulas.
(6) If ϕ is a formula and α a variable, then $\bigwedge\alpha\phi$ and $\bigvee\alpha\phi$ are formulas, and $\urcorner\alpha\phi$ is a term.

Ancillary definitions: An occurrence of a *variable* α is *bound* in a term or formula ϕ just in case it stands within an occurrence in ϕ of $\bigwedge\alpha\psi$, $\bigvee\alpha\psi$, or $\urcorner\alpha\psi$ where ψ is a formula. And an occurrence of a *term* β is *bound* in a term or formula ϕ just in case β contains an occurrence of a variable α that is a bound occurrence of α in ϕ and a free occurrence of α in β. Formula ψ comes from formula ϕ by *proper substitution of a term β for a variable α* just in case ψ is like ϕ except for having free occurrences of β wherever ϕ has free occurrences of α. Formula ψ comes from formula ϕ by *proper substitution of a formula (term) χ for an n-place predicate (operation) letter π* just in case ψ is obtained from ϕ by,

(i) replacing each occurrence of π in ϕ by $\{\chi\}$

and then,

(ii) replacing each expression of the form

$$\{\chi\}\ \alpha_1 \ldots \alpha_n,$$

where $\alpha_1 \ldots \alpha_n$ are terms, by the expression obtained from χ by replacing all free occurrences of 'a' by α_1, 'b' by α_2, and so on up to the n'th variable whose free occurrences are to be replaced by α_n;

it is required that χ and ϕ have no variables in common, and that no variable occurs both free and bound in χ. Formula ψ is an *instance* of ϕ just in case ψ can be obtained from ϕ by some sequence of proper substitutions on variables, predicate letters, and operation letters. The following are pairs of *immediate alphabetic variants*, $\langle\bigwedge\alpha\phi, \bigwedge\alpha'\phi'\rangle$, $\langle\bigvee\alpha\phi, \bigvee\alpha'\phi'\rangle$, and $\langle\urcorner\alpha\phi, \urcorner\alpha'\phi'\rangle$, where α and α' are variables, ϕ and ϕ' formulas, ϕ' comes from ϕ by proper substitution of α' for α, and ϕ comes from ϕ' by proper substitution of α for α'. Formula ψ is an *alphabetic variant* of formula ϕ if ψ can be obtained from ϕ by a sequence of interchanges of immediate alphabetic variants.

$\mathcal{J}$ is axiom-free: its inference rules and derivation procedures are summarized on pages 264-270 in "Kalish and Montague." Explicit notice is taken here only of the rules that govern the description operator ℩: if α and γ are variables, ϕ a formula, β a variable not free in ϕ, and ψ comes from ϕ by proper substitution of $\text{℩}\alpha\phi$ for α, then from

$$\vee\beta\wedge\alpha\ (\phi \leftrightarrow \alpha = \beta) \text{ infer } \psi,$$

and from

$$\sim\vee\beta\wedge\alpha\ (\phi \leftrightarrow \alpha = \beta) \text{ infer } \text{℩}\alpha\phi = \text{℩}\gamma\sim\gamma = \gamma.$$

A ℩-definite description, i.e., a 'closed' term $\text{℩}\alpha\phi$ in which no variable occurs free, *denotes*: if proper, the unique thing that satisfies ϕ; if improper, that thing that every improper description denotes.

2. The classical notation. Add to the vocabulary of $\mathcal{J}$ brackets and the variable binding formula-maker ι. Add to the formation rules of $\mathcal{J}$ the following clause, thus producing simultaneous new definitions of 'term,' 'formula,' *and* 'bound occurrence of a variable' (as well as several other phrases useful for exposition). Clause (7):

(a) An expression $\iota\alpha\phi$, α a variable and ϕ a formula or pseudo-formula, is *an* ι*-description the variable of which is* α.
(b) An expression ϕ is a *pseudo-term (pseudo-formula)* just in case a term (formula) ϕ' is like ϕ except for having, in place of one or more occurrences in ϕ of ι-descriptions, occurrences of their variables. A term (formula) related to a pseudo-term (pseudo-formula) ϕ in this manner is *an associated term (formula) of* ϕ.
(c) *An occurrence of a variable* α *is bound* in a term or formula π just in case *either* it stands within an occurrence in π of an expression χ such that (i) χ is $\wedge\alpha\phi$, $\vee\alpha\phi$, $\text{℩}\alpha\phi$, $\bar{\mathrm{T}}\alpha\phi\psi$, or $[\iota\alpha\phi]\psi$, (ii) ϕ is a formula or pseudo-formula, (iii) ψ is a formula or pseudo-formula, and (iv) either χ is a term or formula or χ' is an associated term or formula of χ and the occurrence of α in question does not stand in χ in an ι-description that is supplanted by its variable in χ'. An occurrence of a variable α is *free* in a term or formula π just in case it stands in π and is not bound in π.
(d) If ϕ and ψ are formulas, α is a variable, and ψ' an expression that comes from ψ by putting $\iota\alpha\phi$ in place of each free occurrence of α in ψ, then $[\iota\alpha\phi]\psi'$ is a *formula.*

The following are formulas by rule (7):

$$[\iota x\mathrm{H}^1x]\mathrm{I}^1\iota x\mathrm{H}^1x,\ [\iota x\mathrm{H}^1x]\mathrm{P}^0,\ [\iota y\mathrm{H}^1y]\mathrm{I}^2x\iota y\mathrm{H}^1y,\ [\iota y\mathrm{H}^1y]\wedge x\mathrm{I}^2x\iota y\mathrm{H}^1y,$$

and

$$[\iota y\mathrm{J}^1y]\,[\iota x\mathrm{H}^1x]\mathrm{I}^2\iota y\mathrm{J}^1y\iota x\mathrm{H}^1x.$$

The following are *not* formulas:

$$\mathrm{I}^1\iota x\mathrm{H}^1x,\ [\iota y\,\mathrm{H}^1y]\mathrm{I}^1\iota x\mathrm{H}^1x,\ [\iota x\mathrm{J}^1x]\mathrm{I}^1\iota x\mathrm{H}^1x,$$
$$[\iota x\mathrm{H}^1x]\mathrm{I}^2x\iota x\mathrm{H}^1x,\ [\iota x\mathrm{H}^1x]\wedge x\mathrm{I}^2y\iota x\mathrm{H}^1x,$$

and

$$[\iota x J^1 x]\,[\iota x H^1 x] I^2 \iota x J^1 x \iota x H^1 x.$$

We have here a superficial departure from the language of *Principia Mathematica*. Its authors would wish to countenance as a formula $[\iota x J^1 x]\,[\iota x H^1 x] I^2 \iota x J^1 x \iota x H^1 x$. I exclude this string in order to secure equivalence of a sort between ι and a new, grammatically simpler operator $\bar{\mathsf{T}}$: see below. If desired, it seems one could modify the grammar of ι in order to allow this string while securing a kind of equivalence with a somewhat less simple operator $\bar{\bar{\mathsf{T}}}$ that would enter formulas $\bar{\bar{\mathsf{T}}}\alpha\beta\phi\psi$, α and β distinct variables, ϕ and ψ formulas. To illustrate clause 7 (c): each occurrence of x and z is bound in, and each occurrence of w is free in, $[\iota x H^2 xw] \wedge z I^2 z \iota x H^2 xw$. Similarly, the first and fourth occurrences of z are free in, and all other occurrences of variables are bound in, $[\iota x H^2 xz] \wedge z I^2 z \iota x H^2 xz$. In contrast, all occurrences of variables are bound in $\wedge z[\iota x H^2 xz] I^2 z \iota x H^2 xz$. The *scope* of an occurrence of an ι-description $\iota\alpha\phi$ is the shortest formula, or pseudo-formula, $[\iota\alpha\phi]\psi'$ in which it stands, and the initial occurrence of $[\iota\alpha\phi]$ therein is its scope *indicator*. One consequence of the revised definition of bondage is that an occurrence of a variable β that stands in an occurrence in a term or formula π of an ι-description is bound in π just in case the 'corresponding' occurrence of β in this ι-description's scope indicator is bound in π.

As an aid to the next definition I note that, as a consequence of clauses (1)-(7), for each pseudo-term (pseudo-formula) there is exactly one associated term (formula). The pair $\langle[\iota\alpha\phi]\psi, [\iota\alpha'\phi']\psi'\rangle$ is a *pair of immediate alphabetic variants* if α and α' are variables, ϕ, ψ, ϕ', and ψ' are formulas or pseudo-formulas, ϕ_1 is a formula and ϕ_1 either is ϕ or is the associated formula of ϕ, ϕ_1' is similarly related to ϕ', ϕ_1' comes from ϕ_1 by proper substitution α' for α and ϕ_1 comes from ϕ_1' by proper substitution of α for α', ψ' is like ψ except that in place of each occurrence of $\iota\alpha\phi$ in ψ that does not stand in or to the right of an occurrence in ψ of $[\iota\alpha\phi]$ there stands in ψ' an occurrence of $\iota\alpha'\phi'$. And the pairs $\langle\wedge\alpha\phi, \wedge\alpha'\phi'\rangle$, $\langle\vee\alpha\phi, \vee\alpha'\phi'\rangle$, and $\langle\urcorner\alpha\phi, \urcorner\alpha'\phi'\rangle$ are *pairs of immediate* alphabetic variants if α and α' are variables, ϕ and ϕ' are formulas or pseudo-formulas, ϕ_1 is a formula and ϕ_1 either is ϕ or is the associated formula of ϕ, ϕ_1' is similarly related to ϕ', ϕ_1' comes from ϕ_1 by proper substitution of α' for α, and ϕ_1 comes from ϕ_1' by proper substitution of α for α'. Finally, the pair $\langle \bar{\mathsf{T}}\alpha\phi\psi, \bar{\mathsf{T}}\alpha'\phi'\psi'\rangle$ is a *pair of immediate alphabetic variants* if α and α' are variables; $\langle\phi, \phi'\rangle$ and $\langle\psi, \psi'\rangle$ are pairs of formulas or pseudo-formulas; ϕ_1 is a formula and ϕ_1 either is ϕ or is the associated formula of ϕ; ϕ_1', ψ_1, and ψ_1' are similarly related to ϕ', ψ, and ψ' respectively; ϕ_1' comes from ϕ_1 by proper substitution of α' for α, and ϕ_1 comes from ϕ_1' by proper substitution of α for α'; and ψ_1' and ψ_1 are similarly related.

In *proper substitution* on predicate letters and operation letters, pseudo-terms associated with replaced letters are to be 'brought into' substituends in the second step as if they were genuine terms.

3. A new notation. For a new and simpler notation, interchangeable with the classical, add to the vocabulary of $\mathcal{J}$ the variable-binding operator $\bar{\mathsf{T}}$, a constant which, like $\mathsf{1}$, is a formula-maker not a term-maker. $\bar{\mathsf{T}}$ enters formulas in accordance with the following clause:

(8) If ϕ and ψ are formulas and α a variable, then $\bar{\mathsf{T}}\alpha\phi\psi$ is a formula.

Ancillary definitions, as amended in section **2**, anticipated the introduction of $\bar{\mathsf{T}}$. Much simpler definitions, in particular, of 'bondage' and 'alphabetic variance' would be possible but for the presence of $\mathsf{1}$. We state here for future reference simple extensions, suitable for $\mathsf{1}$-free expressions, of definitions set out in section **1**: An occurrence of a variable α is *bound* in a term or formula χ if it stands in an occurrence in χ of $\bar{\mathsf{T}}\alpha\phi\psi$, α a variable and ϕ and ψ formulas. And the pair $\langle\bar{\mathsf{T}}\alpha\phi\psi, \bar{\mathsf{T}}\alpha'\phi'\psi'\rangle$ is a pair of *immediate alphabetic variants*, α and α' variables and ϕ, ψ, ϕ', and ψ' formulas, if ϕ' and ψ' come from ϕ and ψ respectively by proper substitution of α' for α, and ϕ and ψ come from ϕ' and ψ' respectively by proper substitution of α for α'.

4. The theory $\mathcal{R}$ and the interchangeability of the classical and new notations. Let the terms and formulas of $\mathcal{R}$ be all and only results of successive applications of clauses (1)-(8), and ancillary definitions as amended in section **2**. *Axioms* of $\mathcal{R}$ are all instances in $\mathcal{R}$ of the axiom schemata,

AS1: $(\bar{\mathsf{T}}x\mathrm{F}^1x\mathrm{G}^1x \longleftrightarrow \bigvee y(\bigwedge x(\mathrm{F}^1x \longleftrightarrow x = y) \wedge \mathrm{G}^1y))$

and

AS2: $([\mathsf{1}x\mathrm{F}^1x]\,\mathrm{G}^1\mathsf{1}x\mathrm{F}^1x \longleftrightarrow \bigvee y(\bigwedge x(\mathrm{F}^1x \longleftrightarrow x = y) \wedge \mathrm{G}^1y))$.

The inference rules and derivation procedures are those of $\mathcal{J}$ augmented by a rule permitting interchange of alphabetic variants. Observe that the theory $\mathcal{R}$ contains three distinct description operators. One, the *term-maker* $\mathsf{7}$, has no counterpart in *Principia Mathematica.* The other two, $\bar{\mathsf{T}}$ and $\mathsf{1}$, are formula-makers and provide alternative and interchangeable notations for *Principia Mathematica* description theory.

Interchangeability metatheorem: Let ϕ and ϕ' be formulas such that ϕ' comes from ϕ by either translation rule $\mathsf{1}/\bar{\mathsf{T}}$ or translation rule $\bar{\mathsf{T}}/\mathsf{1}$, stated below. Then $(\phi \longleftrightarrow \phi')$ is a theorem of $\mathcal{R}$. The translation rules:

ϕ' comes from ϕ by *translation rule* $\mathsf{1}/\bar{\mathsf{T}}$ if, in place of an occurrence of $\bar{\mathsf{T}}\alpha\psi\chi$ in ϕ there stands in ϕ' an occurrence of $[\mathsf{1}\alpha\psi]\chi'$, α a variable, ψ, χ, and χ' formulas or pseudo-formulas, and χ' related to χ so that, where (i) $\bar{\mathsf{T}}\alpha\psi_1\chi_1$ is a formula and either is $\bar{\mathsf{T}}\alpha\psi\chi$ or is the associated formula of $\bar{\mathsf{T}}\alpha\psi\chi$, and (ii) $[\mathsf{1}\alpha\psi_1]\chi_1'$ is a formula that is similarly related to $[\mathsf{1}\alpha\psi]\chi'$, χ_1' comes from χ_1 by replacing each free occurrence of α in χ_1 by an occurrence of $\mathsf{1}\alpha\psi_1$.

ϕ' comes from ϕ by *translation rule* $\bar{\mathsf{T}}/\mathsf{1}$ if, in place of an occurrence in ϕ of $[\mathsf{1}\alpha\psi]\chi$ there stands in ϕ' an occurrence of $\bar{\mathsf{T}}\alpha\psi\chi'$, α a variable, ψ and χ

formulas or pseudo-formulas, and χ' a formula or pseudo-formula that comes from χ by replacing each occurrence in χ of $\iota\alpha\psi$ that does not stand in or to the right of an occurrence in χ of $[\iota\alpha\psi]$ by an occurrence of α.

Note that, principally in the light of clause (7)(d), a formula ϕ' comes from a formula ϕ by translation rule $\iota/\bar{\mathrm{T}}$ if and only if ϕ comes from ϕ' by translation rule $\bar{\mathrm{T}}/\iota$.

(A proof of the metatheorem for translation rule $\iota/\bar{\mathrm{T}}$. Two cases: (i) the rule is applied to an occurrence in ϕ of $\bar{\mathrm{T}}$ that does *not* stand in the scope of ι-description. In this case the rule calls for the replacement of an occurrence of a formula ψ by an occurrence of a formula ψ' where ψ and ψ', since alphabetic variants of corresponding instances of left-constituents of AS1 and AS2 respectively, are logically equivalent. (ii) The rule is applied to an occurrence in ϕ of $\bar{\mathrm{T}}$ that *does* stand in the scope of an ι-description. Let ψ be the last member of a sequence of formulas generated by applying the 'reverse' rule $\bar{\mathrm{T}}/\iota$ to the leftmost occurrence of a scope-indicator in ϕ, then to the leftmost in the result, and so on until all occurrences of scope indicators to the left of the occurrence of $\bar{\mathrm{T}}$ in question have been eliminated. Each formula in this sequence is logically equivalent to its predecessor, if any, for the reason spelled out under case (i). So $(\phi \leftrightarrow \psi)$ is a theorem of $\mathcal{R}$. If the sequence contains n formulas, let ψ' be the n'th member of a sequence generated in the same manner that has as its first member ϕ'. Then $(\phi' \leftrightarrow \psi')$ is a theorem of $\mathcal{R}$. And ψ' comes from ψ by an application of the rule to an occurrence of $\bar{\mathrm{T}}$ that does not stand in the scope of an ι-description. So, by case (i), $(\psi \leftrightarrow \psi')$ is a theorem of $\mathcal{R}$. Thus $(\phi \leftrightarrow \phi')$ is a theorem of $\mathcal{R}$. Q.E.D. A proof of the metatheorem for translation rule $\bar{\mathrm{T}}/\iota$ proceeds similarly.)

The pair $\langle \phi, \phi' \rangle$ is a translation pair, ϕ and ϕ' formulas, just in case there is a sequence of formulas beginning with ϕ and ending with ϕ' such that each member of the sequence comes from its predecessor, if any, by an application of one of the translation rules. The following are translation pairs:

$$\langle \bar{\mathrm{T}}x\mathrm{H}^1x\mathrm{P}^0,\ [\iota x\mathrm{H}^1x]\mathrm{P}^0\rangle,$$
$$\langle \bar{\mathrm{T}}x\mathrm{H}^1x\mathrm{I}^1x,\ [\iota x\mathrm{H}^1x]\mathrm{I}^1\iota x\mathrm{H}^1x\rangle,$$
$$\langle \bar{\mathrm{T}}x\mathrm{H}^1x\mathrm{I}^2xx,\ [\iota x\mathrm{H}^1x]\mathrm{I}^2\iota x\mathrm{H}^1x\iota x\mathrm{H}^1x\rangle,$$
$$\langle \bar{\mathrm{T}}x\mathrm{H}^1x\bar{\mathrm{T}}y\mathrm{I}^1y\mathrm{J}^2xy,\ [\iota x\mathrm{H}^1x]\,[\iota y\mathrm{I}^1y]\mathrm{J}^2\iota x\mathrm{H}^1x\iota y\mathrm{I}^1y\rangle,$$

and

$$\langle \bar{\mathrm{T}}x\mathrm{H}^1x(\mathrm{I}^1x \rightarrow \wedge x(\mathrm{J}^1x \wedge \bar{\mathrm{T}}x\mathrm{H}^1x\mathrm{K}^1x)),$$
$$[\iota x\mathrm{H}^1x]\,(\mathrm{I}^1\iota x\mathrm{H}^1x \rightarrow \wedge x(\mathrm{J}^1x \wedge [\iota x\mathrm{H}^1x]\,\mathrm{K}^1\iota x\mathrm{H}^1x))\rangle.$$

5. Alternative formation rules and definitions for $\mathcal{R}$. Here we take the $\bar{\mathrm{T}}$-notation as 'basic' and bring the ι-notation in as a 'variant.' The definitions in this section have the same extensions as their counterparts, if any, above:

(i) ϕ is a $\bar{\mathrm{T}}$-*term* ($\bar{\mathrm{T}}$-*formula*) just in case ϕ is generated by clauses (1)-(6) and (8) above.

(ii) If ϕ is a $\bar{\mathrm{T}}$-term ($\bar{\mathrm{T}}$-formula), then ϕ is a *term (formula).*
(iii) Clause 7 (a) above.
(iv) Clause 7 (b) above.
(v) If ϕ is a term (formula), then an *immediate ancestor* of ϕ is a term (formula) ϕ' that comes from ϕ by an application of rule $\bar{\mathrm{T}}/\iota$. If ϕ' is an immediate ancestor of ϕ, then ϕ is an immediate descendant of ϕ'.

Note that if ϕ' is an immediate ancestor of a term or formula ϕ then each occurrence of a symbol other than a bracket in ϕ can be associated with a unique occurrence of a symbol in ϕ' in the manner here illustrated.

$$(\mathrm{P}^0 \to (\bar{\mathrm{T}}x\mathrm{F}^1x[\iota y\mathrm{G}^1y]\mathrm{H}^2x\iota y\mathrm{G}^1y \wedge [\iota y\mathrm{G}^1y]\mathrm{F}^1\iota y\mathrm{G}^1y))$$

$$(\mathrm{P}^0 \to (\bar{\mathrm{T}}x\mathrm{F}^1x\ \ \bar{\mathrm{T}}y\mathrm{G}^1y\ \ \mathrm{H}^2xy\ \ \wedge [\iota y\mathrm{G}^1y]\mathrm{F}^1\iota y\mathrm{G}^1y))$$

In particular, for each occurrence of a variable α in ϕ there *corresponds* exactly one occurrence of α in ϕ'. We make use of this fact in the following key clause:

(vi) An occurrence of a variable α is *bound* in a term (formula) ϕ just in case either (a) ϕ is a $\bar{\mathrm{T}}$-term ($\bar{\mathrm{T}}$-formula) and the occurrence of α in question stands in an occurrence in ϕ of $\wedge\alpha\psi$, $\vee\alpha\psi$, $\urcorner\alpha\psi$, or $\bar{\mathrm{T}}\alpha\psi\chi$, ψ and χ formulas, *or* (b) ϕ' is an immediate ancestor of ϕ and the occurrence of α in ϕ that is in question corresponds to a bound occurrence of α in ϕ'. An occurrence of a variable α is *free* in a term (formula) ϕ just in case it stands in ϕ and is not bound in π.
(vii) If ϕ is a term (formula) and ϕ' comes from ϕ by an application of rule $\iota/\bar{\mathrm{T}}$, then ϕ' is a term (formula).

ϕ and ϕ_1 are *immediate alphabetic variants* just in case *either* (a) ϕ and ϕ_1 are $\bar{\mathrm{T}}$-terms or $\bar{\mathrm{T}}$-formulas and $\langle\phi, \phi_1\rangle$ is $\langle\wedge\alpha\psi, \wedge\alpha'\psi'\rangle$, $\langle\vee\alpha\psi, \vee\alpha'\psi'\rangle$, $\langle\urcorner\alpha\psi, \urcorner\alpha'\psi'\rangle$, or $\langle\bar{\mathrm{T}}\alpha\psi\chi, \bar{\mathrm{T}}\alpha'\psi'\chi'\rangle$, wherein α and α' are variables, ψ, ψ', χ, and χ' are $\bar{\mathrm{T}}$-formulas, ψ' comes from ψ by proper substitution of α' for α and ψ comes from ψ' by proper substitution of α for α', and χ' and χ are similarly related, *or* (b) ϕ' is an immediate ancestor of ϕ, ϕ_1' is an immediate ancestor of ϕ_1, and ϕ' and ϕ_1' are immediate alphabetic variants.

6. Remarks mainly on the classical notation. $\iota\alpha\phi$, α a variable ϕ a formula, is not a term. Of course it is not a formula either. These points taken together constitute, I think, part of the substance of the *Principia*-characterization of ι-descriptions as 'incomplete symbols.' Curiously there is another more literal sense in which $\iota\alpha\phi$ is not a complete symbol: in this sense, the 'complete' symbol, as can be gathered from clause (7)(d) above, is not $\iota\alpha\phi$ but, roughly,

$$[\iota\alpha\phi] \ldots \iota\alpha\phi \ldots$$

Thus $\mathrm{H}^1\iota x\mathrm{I}^1x$ is ill-formed and truncated somewhat in the way in which $\mathrm{P} \to \mathrm{Q}$, in contrast with $(\mathrm{P} \to \mathrm{Q})$, is. The two related ways in which, for example, $\iota x\mathrm{H}^1x$ is 'incomplete' suggest a gloss on the expression $a = \iota x\mathrm{H}^1x$.

First, this expression is not an identity formula, though it would be if $\text{ɿ}x\mathrm{H}^1x$ were a term and so not 'incomplete' in the first way. Second, if anything, $a = \text{ɿ}x\mathrm{H}^1x$ is, by the *Principia*-convention for deleting scope indicators, short for the formula $[\text{ɿ}x\mathrm{H}^1x]\ a = \text{ɿ}x\mathrm{H}^1x$. When scope indicators are made explicit and 'complete' ɿ-symbols are displayed, some misleading grammatical appearances are corrected: the formula $[\text{ɿ}x\mathrm{H}^1x]\ a = \text{ɿ}x\mathrm{H}^1x$ looks no more like an identity than does $\wedge y\ a = y$ or $\vee y\ a = y$. (*Cf.* comments in *Principia Mathematica*, p. 67, on the string $a = (\text{ɿ}x)(\phi x)$.)

Unbracketed ɿ-descriptions, though not terms, do of course occupy 'term positions.' Why, given that these expressions are not what they are apt to seem, are they countenanced? How is the attractiveness and naturalness of the classical notation to be explained? A part of the answer, I think, is that under a certain condition, the 'propriety condition,' an ɿ-description, if 'closed,' behaves like a term. Let closures of $\vee\beta\wedge\alpha\ (\phi \longleftrightarrow \alpha = \beta)$, α a variable, ϕ a formula, and β a variable not free in ϕ, be equivalent expressions of *the propriety condition* for the ɿ-description $\text{ɿ}\alpha\phi$. (Note that this definition does not cover the case of an ɿ-description $\text{ɿ}\alpha\phi$ in which ϕ is a pseudo-formula.) Let an ɿ-description $\text{ɿ}\alpha\phi$, α a variable and ϕ a formula in which no variable other than α occurs free, be a *closed* ɿ-description. And let us adopt the convention that, 'given' an expression of the propriety condition for a closed ɿ-description $\text{ɿ}\alpha\phi$, scope indicators for occurrences of $\text{ɿ}\alpha\phi$ can be omitted from formulas. The thus licensed $[\text{ɿ}\alpha\phi]$-free expressions are in some cases ambiguous, but when ambiguous their ambiguities given their license, the propriety condition, are 'harmless': no matter how they are resolved the results are equivalent. For example, the expression $\sim\mathrm{I}^1\text{ɿ}x\mathrm{H}^1x$ could be short for either

$$\sim[\text{ɿ}x\mathrm{H}^1x]\mathrm{I}^1\text{ɿ}x\mathrm{H}^1x$$

or

$$[\text{ɿ}x\mathrm{H}^1x]\sim\mathrm{I}^1\text{ɿ}x\mathrm{H}^1x,$$

but given $\vee y\wedge x\ (\mathrm{H}^1x \longleftrightarrow x = y)$ the material equivalence of these formulas can be deduced. More precisely and generally, a formula

$$\chi \rightarrow (\psi \longleftrightarrow \psi')$$

is a theorem of $\mathcal{R}$, if there are variables α and β and a formula ϕ such that (i) β is not free in ϕ, (ii) χ is a closure of $\vee\beta\wedge\alpha\ (\phi \longleftrightarrow \alpha = \beta)$, (iii) ψ' is a formula that comes from ψ by making larger or smaller the scope of one or more occurrences of $\text{ɿ}\alpha\phi$, and (iv) each occurrence of a variable that stands in an occurrence in ψ of $\text{ɿ}\alpha\phi$ is free in ψ if and only if the corresponding occurrence in ψ' of this variable is free in ψ'. (A demonstration of the need for proviso (iv) is bracketed below.) So the licensed $[\text{ɿ}\alpha\phi]$-free expressions even when ambiguous have only 'harmless' ambiguities. And one can feel that licensed $[\text{ɿ}\alpha\phi]$-free expressions though *especially* misleading in grammatical form (in them nothing relieves the term-like appearance of ɿ-descriptions) are not *seriously* misleading and are, indeed, *helpfully* misleading. It is true that in them some

℩-descriptions look like terms, but then, when $[℩\alpha\phi]$-free formulas are by the present license sanctioned, $℩\alpha\phi$ *behaves* like a term, *exactly* like the term $⅂\alpha\phi$. More precisely and more generally, a formula

$$\chi \rightarrow (\psi \longleftrightarrow \psi')$$

is a theorem of $\mathcal{R}$, if there are variables α and β and a formula ϕ such that (a) β is not free in ϕ, (b) χ is a closure of $\vee\beta\wedge\alpha(\phi \longleftrightarrow \alpha = \beta)$, (c) ψ' is a formula that comes from ψ by deleting all occurrences in ψ of the scope indicator $[℩\alpha\phi]$ and replacing each unbracketed occurrence in ψ of $℩\alpha\phi$ by an occurrence of $⅂\alpha\phi$, and (d) each occurrence of a variable that stands in an occurrence in ψ of an unbracketed occurrence of $℩\alpha\phi$ is free in ψ if and only if the corresponding occurrence in ψ' of this variable in the corresponding (i.e., replacing) occurrence in ψ' of $⅂\alpha\phi$ is free in ψ'. (Again, the need for the last proviso is demonstrated below.)

(Proviso (d): While the formula

$$(\wedge z \vee y \wedge x(\mathrm{F}^2xz \longleftrightarrow x = y) \rightarrow (\wedge z[℩x\mathrm{F}^2xz]\mathrm{G}^2z℩x\mathrm{F}^2xz \longleftrightarrow \wedge z\, \mathrm{G}^2z⅂x\mathrm{F}^2xz))$$

is a theorem of $\mathcal{R}$, the formula

$$(\wedge z \vee y \wedge x(\mathrm{F}^2xz \longleftrightarrow x = y) \rightarrow ([℩x\mathrm{F}^2xz]\wedge z\mathrm{G}^2z℩x\mathrm{F}^2xz \longleftrightarrow \wedge z\mathrm{G}^2z⅂x\mathrm{F}^2xz))$$

does not satisfy (d) and is not a theorem of $\mathcal{R}$. Its closures are false under the following interpretation:

Domain: natural numbers. Improper designatum: 0.
F^2: $a = b + 1$. G^2: $a < b$.

Proviso (iv): The formula

$$(\wedge z \vee y \wedge x(\mathrm{F}^2xz \longleftrightarrow x = y) \rightarrow (\wedge z[℩x\mathrm{F}^2xz]\mathrm{G}^2z℩x\mathrm{F}^2xz \longleftrightarrow [℩x\mathrm{F}^2xz]\wedge z\mathrm{G}^2z℩x\mathrm{F}^2xz))$$

does not satisfy (iv) and is not a theorem of $\mathcal{R}$. Its closures are false under the interpretation just given.

Note that neither of the generalizations about theoremhood could be strengthened by dropping the requirement that χ be a closure of $\vee\beta\wedge\alpha(\phi \longleftrightarrow \alpha = \beta)$ and requiring instead that χ simply *be* $\vee\beta\wedge\alpha(\phi \longleftrightarrow \alpha = \beta)$. Each of the formulas

$$(\vee y \wedge x(\mathrm{F}^2xz \longleftrightarrow x = y) \rightarrow (\wedge z \sim[℩x\mathrm{F}^2xz]\mathrm{G}^1℩x\mathrm{F}^2xz \longleftrightarrow \wedge z[℩x\mathrm{F}^2xz] \sim \mathrm{G}^1℩x\mathrm{F}^2xz)),$$
$$(\vee y \wedge x(\mathrm{F}^2xz \longleftrightarrow x = y) \rightarrow (\wedge z \sim[℩x\mathrm{F}^2xz]\mathrm{G}^1℩x\mathrm{F}^2xz \longleftrightarrow \wedge z \sim \mathrm{G}^1⅂x\mathrm{F}^2xz))$$

has falsifiable closures, so neither is a theorem of $\mathcal{R}$.)

Since $℩\alpha\phi$ behaves like a term in licensed $[℩\alpha\phi]$-free expressions, that it looks like a term in these expressions cannot seriously mislead. Indeed in derivation contexts and contexts of logical calculation, the term-like appearance of $℩\alpha\phi$ in licensed $[℩\alpha\phi]$-free expressions can be positively helpful. This is part of the reason for the naturalness and attractiveness of the ℩-notation. The grammar and 'logic' of 'the' in English is another part. The classical notation has not inconsiderable virtues many of which derive precisely from its misleading syntax. Of course the new $\bar{\mathsf{T}}$-notation has, especially for theory-building and meta-theoretical purposes, certain clear advantages. It is more economical and calls for relatively simple amendments to the formation rules and ancillary definitions of $\mathcal{J}$. In particular, the standard definition of 'bondage' is easily adjusted to cover $\bar{\mathsf{T}}$-formulas, and the usual practise of first defining 'term' and 'formula' and then, with the full language determined, defining 'bondage' can be left undisturbed. And perhaps more importantly, though this seems a subjective point, the $\bar{\mathsf{T}}$-notation is less misleading as to 'logical form' and intended semantics. It has the admittedly mixed virtue of not involving expressions that even look like new sorts of terms.[3]

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3. It has come to my attention just recently that binary quantifiers analogous to $\bar{\mathsf{T}}$ have at least twice before now been proposed as improvements upon the classical scope-notation of *Principia Mathematica*: Schock, Rolf, "Some remarks on Russell's treatment of definite descriptions," *Logique et Analyse* (1962), p. 80; Routley, R., "Some things do not exist," *Notre Dame Journal of Formal Logic* (1966), p. 270.