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POSSIBILITY PRE-SUPPOSITION FREE LOGICS

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Presupposition-free logics are usually taken to be free of *existential* presuppositions. Allowing individual constants (what we will call 'free variables' in this paper) not to designate is usually taken as allowing them not to designate some individual in a domain of existing individuals. One standard move that has been used in such a situation is then to have the non-existence designating free variables designate something else such as an individual in a domain disjoint from the domain of existing individuals. This move could be seen as a move to allowing non-existence designating free variables to designate imaginary, or fictional, or possible but non-existent individuals.

The question can then arise as to whether or not an existential presupposition free logic has a possibility presupposition. It would certainly be so if the free variables must designate an individual either in the domain of existing individuals or in the domain of possible but non-existent individuals. Since presupposition free logics were first designed to eliminate existential presupposition it would be interesting to see what are the results of designing possibility presupposition-free logics.

One crucial feature of standard presupposition free logics is that the quantifiers range only over the domain of existing individuals, whereas free variables may designate any individual in either the existence domain or the domain of possible but non-existent individuals. In order to proceed towards a set of possibility presupposition-free logics, we could introduce quantifiers to range over a domain of possible individuals, which would include the set of existent individuals, and we could have the free variables designating any individual in either the domain of possible individuals or in a domain of impossible individuals (which would be disjoint from the possibility domain).

As far as the truth of statements is concerned we could assume that all statements without free variables or with free variables designating possible individuals will come under the standard truth conditions. Statements containing free variables that designate impossible individuals could then be treated in a range of ways parallel to those set out by Leblanc and

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Thomason.[1] We could then consider the three options such that (a) all atomic statements are arbitrarily assigned T or F and the truth value of non-atomic ones is determined by the standard semantical rules of truth, (b) all statements containing free variables designating impossible individuals are T, or (c) all statements containing free variables designating impossible individuals are F.

Although we could proceed in such a fashion, we will not. Instead, we will relate some syntactic systems to the model system type of semantics advocated by Hintikka. So we will be concerned about whether statements are satisfiable or not satisfiable. A set of formulae is satisfiable if and only if it can be imbedded in a set which satisfies certain conditions. Any particular formula is satisfiable if and only if its unit set is.

The conditions referred to can be seen as criteria of consistency, or as criteria for a coherent state description. A satisfiable statement is then one which is consistent, or one which can be part of a coherent state description. In particular we will be concerned with imbedding sets of formulae in maximal sets, or maximally extended state descriptions. These latter have been called maximal consistent novels.

As far as the satisfiability of statements is concerned we will assume that all statements without free variables fall under standard conditions (C . ~) and (C .⊃) set out below.[2] Our main interest will be with statements containing free variables, in particular, with the statements of a range of formal systems $\mathbf{QH}\overset{i}{=}$ $(0 \leqslant i \leqslant 15)$. In all of $\mathbf{QH}\overset{i}{=}$ statements of the form $(\Sigma x)(x = a)$ translate as: "a is possible" or, "there is at least one possible individual identical with a"; those of the form $\sim(\Sigma x)(x = a)$ translate as: "a is impossible."

If a statement contains the free individual variable a there will be several options open as regards the satisfiability of the statement. These options can be outlined in terms of trying to imbed the unit set of the statement in a maximal set containing either $(\Sigma x)(x=a)$ and not $\sim(\Sigma x)(x=a)$, or $\sim(\Sigma x)(x = a)$, or neither $(\Sigma x)(x = a)$ nor $\sim(\Sigma x)(x = a)$. In the case of the first alternative, where a is possible, we will consider satisfiability under purely standard conditions. A statement and its negation cannot both be satisfiable. In the case of the second alternative, where a is impossible, we will consider two kinds of satisfiability: (a) if the unit set of any formula A can be imbedded in λ, then the unit set of $\sim A$ cannot; (b) the unit set of any formula containing a can be imbedded in a set λ. In the case of the third alternative; (c) the unit set of any formula containing a cannot be imbedded in a set λ.

One way of seeing these options would be to say that (a) is where statements about impossible objects are consistent or inconsistent in exactly the same way as statements about possible objects; (b) is where

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1. Leblanc, H., and R. H. Thomason, "Completeness theorems for some presupposition-free logics," *Fundamenta Mathematicae*, vol. 62 (1968), p. 126.

2. See page 56 below.

any statement about impossible objects is consistent; (c) is where no statement about impossible objects is consistent.

Amongst the sixteen $\mathbf{QH}^{i}_{=}$ there are seven where a is never impossible. (i = 0 or 3 or 4 or 7 or 8 or 11 or 14), and so they all fall under the first alternative. There are three (i = 1 or 2 or 12) where statements are handled in the spirit of (a) above. There are three (i = 5 or 6 or 13) where statements are handled in the spirit of (b) above. There are three (i = 9 or 10 or 15) where statements are handled in the spirit of (c) above.

NOTE: Case (a) presents us with certain difficulties of interpretation if we see this case as outlined above. It would be contended that for some predicate P: $(Pa \mathbin{\&} \sim Pa) \equiv \sim(\Sigma x)(x = a)$. In such a situation we can hardly accept our interpretation of case (a).[3] Routley suggests that such problematic situations could be dealt with by means of predicate negation: $(Pa \mathbin{\&} \overline{P}a) \equiv \sim(\Sigma x)(x = a)$, where $^{-}$ represents predicate negation.

1 Primitive Symbols:
improper symbols $\supset \sim \Pi$.( )
bound variables $x_0, y_0, z_0, x_1, y_1, z_1, x_2, \ldots$
free variables $a_0, b_0, c_0, a_1, b_1, c_1, a_2, \ldots$
propositional variables $p_0, q_0, r_0, p_1, q_1, r_1, p_2, \ldots$
n-ary predicate variables ($n \geq 1$) $F^n_0, G^n_0, H^n_0, F^n_1, G^n_1, H^n_1, F^n_2, \ldots$
predicate constants =, E.

2 Formation Rules:
(i) A propositional variable standing alone is a wff.
(ii) If F^n is an n-ary predicate variable, and if $a_1, \ldots, a_n$ are n free variables (not necessarily distinct) then $F^n_{a_1a_2\ldots a_n}$ is a wff.
(iii) If a is any free variable then Ea is a wff.
(iv) If a and b are any free variables (not necessarily distinct) then $a = b$ is a wff.
Wffs according to (i)-(iv) are atomic wffs.
(v) If A is a wff, so is $\sim A$.
(vi) If A and B are wffs, so is $(A \supset B)$.
(vii) If A is a wff and x is any bound variable, then $(\Pi x)(A(x/a))$ is a wff, where: If A is a wff and X is a variable free or bound and Y is a variable free or bound, then $A(X/Y)$ is the result of substituting X for zero or more occurrences of Y in A (we will also use $A(X/Y)$ where $A(X/Y)$ is the result of substituting X for every occurrence of Y in A).

3 Axiom Schemata:
1 $A \supset (B \supset A)$
2 $A \supset (B \supset C) \supset. A \supset B \supset. A \supset C$
3 $\sim A \supset \sim B \supset. B \supset A$
4 $A \supset (\Pi x)A$ *provided x does not occur in A*
5 $(\Pi x)(A \supset B) \supset. (\Pi x) A \supset (\Pi x)B$

3. R. Routley, "Some things do not exist," *Notre Dame Journal of Formal Logic*, vol. 7 (1966), p. 260.

5a $(\Pi x)(A \supset B) \supset. (\Pi x) A \supset (\Pi x) B$, *provided x does occur in B*
6 $(\Sigma y)(y = a) \supset. (\Pi x)A \supset A(a/x)$
6a $(\Sigma y)(y = a) \supset. (\Pi x)A \supset A(a/x)$, *provided x does occur in A*
7 $(\Pi x)A \supset A(a/x)$
7a $(\Pi x)A \supset A(a/x)$, *provided x does occur in A*
8 $(\Pi x)(\Sigma y)(y = x)$
9 $a = a$
10 $a = b \supset. A \supset A(b//a)$
11 $Ea \supset. (\Sigma y)(y = a)$[4]

Rules:

R1. $A, A \supset B \rightarrow B$
R1.a $A, A \supset B \rightarrow B$ *provided every free variable in A is in B*
R2. $A \rightarrow (\Pi x)(A(x/a))$ *provided x does not occur in A*
R3. $A, \sim A \rightarrow B$

We adopt the convention that if $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ are the distinct free variables in A, then $\alpha(A)$ is: $(\Sigma y)(y = a_1) \supset. (\Sigma y)(y = a_2) \supset. \ldots \supset. (\Sigma y)(y = a_n) \supset A$

R4: $\alpha(\alpha(A)) \rightarrow \alpha(A)$

4 The $\mathbf{QH}^{\underline{i}}_{=}$ Systems: Where $(0 \leq i \leq 15)$ the systems $\mathbf{QH}^{\underline{i}}_{=}$ can be axiomatized using sets of axiom schema and rules as follows (we shall use $\alpha(1)$, $\alpha(2), \ldots, \alpha(11)$ to stand for $\alpha(A), \alpha(B), \ldots, \alpha(K)$ where $A, B, \ldots, K$ are axiom schema 1, 2, . . ., 11 respectively): Let

$S = \{1, 2, 3, 4, 5, 9, 10, 11\}$
$S' = \{1, 2, 3, 5a, 9, 10, 11\}$
$S'' = \{\alpha(1), \alpha(2), \alpha(3), \alpha(4), \alpha(5), \alpha(9), \alpha(10), \alpha(11)\}$
$S''' = \{\alpha(1), \alpha(2), \alpha(3), \alpha(5a), \alpha(9), \alpha(10), \alpha(11)\}$

$\mathbf{QH}^{\underline{0}}_{=} = S \cup \{7; R1, R2; 8\}$
$\mathbf{QH}^{\underline{1}}_{=} = S \cup \{6; R1, R2\}$
$\mathbf{QH}^{\underline{2}}_{=} = S \cup \{6a; R1, R2\}$
$\mathbf{QH}^{\underline{3}}_{=} = S \cup \{7; R1a, R2; 8\}$
$\mathbf{QH}^{\underline{4}}_{=} = S \cup \{7a; R1a, R2; 8\}$
$\mathbf{QH}^{\underline{5}}_{=} = S \cup \{6; R1a, R2\}$
$\mathbf{QH}^{\underline{6}}_{=} = S \cup \{6a; R1a, R2\}$
$\mathbf{QH}^{\underline{7}}_{=} = S'' \cup \{\alpha(7); R1, R2, R3, R4; 8\}$
$\mathbf{QH}^{\underline{8}}_{=} = S'' \cup \{\alpha(7a); R1, R2, R3, R4; 8\}$
$\mathbf{QH}^{\underline{9}}_{=} = S'' \cup \{\alpha(6); R1, R2, R3, R4\}$
$\mathbf{QH}^{\underline{10}}_{=} = S'' \cup \{\alpha(6a); R1, R2, R3, R4\}$
$\mathbf{QH}^{\underline{11}}_{=} = S' \cup \{7; R1, R2; 8\}$
$\mathbf{QH}^{\underline{12}}_{=} = S' \cup \{6a; R1, R2\}$

4. The convention for bracketing is as in: Church, A., *Introduction to Mathematical Logic*, vol. 1, Princeton University Press, New Jersey (1956), p. 42.

$QH^{13}_{=} = S' \cup \{6a;\ R1a,\ R2\}$
$QH^{14}_{=} = S''' \cup \{\alpha(7a);\ R1,\ R2,\ R3,\ R4;\ 8\}$
$QH^{15}_{=} = S''' \cup \{\alpha(6a);\ R1,\ R2,\ R3,\ R4\}$
(We use $\cup$ for set theoretic union and $\cap$ for set theoretic intersection.)

5A The consistency of a formula with respect to a given system S, i.e., S-consistency

A formula, A, of any system S (e.g., $QH^{i}_{=}$) is consistent with respect to S, S-consistent, iff $\sim A$ is not a thesis of S. ('$\dashv_S A$' is 'A is not a thesis of S'; '$\vdash_S A$' is 'A is a thesis of S': so S-consistent (A) .≡. $\dashv_S \sim A$.)

The consistency of a set of formulae of S, not S-consistency

A finite set of formulae of S, $\{A_1, \ldots, A_n\}$ is consistent iff $\dashv_S \sim (A_1 \ \& \ \ldots \ \& \ A_n)$.

Also, if Λ is an infinite set of formulae of S, Λ is consistent iff it contains no inconsistent finite subset of formulae, i.e., if for all Λ^1, $\Lambda^1 \subseteq \Lambda$ and $\Lambda^1 = \{A_1, \ldots, A_n\}$ then, if $\vdash_S \sim (A_1 \ \& \ \ldots \ \& \ A_n)$, then Λ is inconsistent (in this paper we shall simplify '$\vdash_{QH^{i}_{=}}$' to '$\vdash_i$', and '$\dashv_{QH^{i}_{=}}$' to '$\dashv_i$').

5B The α consistency of a formula with respect to the systems $QH^{i}_{=}$ $(7 \leqslant i \leqslant 10$ or $i = 14$ or $15)$:

A formula, A, of any $QH^{i}_{=}$ $(7 \leqslant i \leqslant 10$ or $i = 14$ or $15)$ system is α consistent with respect to the relevant $QH^{i}_{=}$, $QH^{i}_{=}$ α consistent, iff $\alpha(\sim A)$ is not a thesis of $QH^{i}_{=}$: $QH^{i}_{=}$ α consistent (A) .≡. $\dashv_i \alpha(\sim A)$. It follows that if A contains no free variables that $QH^{i}_{=}$ α consistent (A) .≡. $QH^{i}_{=}$ consistent (A).

The α consistency of a set of formulae of $QH^{i}_{=}$, not $QH^{i}_{=}$ α consistency:

A finite set of formulae of $QH^{i}_{=}$, $\{A_1, \ldots, A_n\}$ is α consistent iff $\dashv_i \alpha(\sim (A_1 \ \& \ \ldots \ \& \ A_n))$.

Also, if Λ is an infinite set of formulae of $QH^{i}_{=}$, Λ is α consistent iff it contains no not α consistent finite subset of formulae, i.e., if for all Λ', $\Lambda' \subseteq \Lambda$ and $\Lambda' = \{A_1, \ldots, A_n\}$ then, if $\vdash_i \alpha(\sim (A_1 \ \& \ \ldots \ \& \ A_n))$, then Λ is not α consistent.

5C A T-consistent set of formulae of $QH^{i}_{=}$ is constructed by forming the union of a consistent set of formulae of $QH^{i}_{=}$ with a set of all the formulae of $QH^{i}_{=}$ containing any free variable a for which there is a formula of the form $\sim(\Sigma y)(y = a)$ in the consistent set. Similarly, but conversely, an F-consistent set of formulae of $QH^{i}_{=}$ is that subset of a consistent set which remains when all the formulae containing any free variable a, for which there is a formula of the form $\sim(\Sigma y)(y = a)$ in the set, are removed.

6 In order to show a system S complete, we show that if a formula A is S-valid, then $\vdash_S A$; or, for every S-consistent formula there is a verifying (or satisfying) S Model; or, in the case of systems $QH^{i}_{=}$ $(7 \leqslant i \leqslant 10$ or $i = 14$ or $15)$ for every $QH^{i}_{=}$ α consistent formula there is a satisfying S Model.

Procedure: for the systems $QH^{i}_{=}$ $(0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13)$ we construct a maximal consistent set of formulae of $QH^{\underline{i}}$ which contains A; and in the case of $0 \leqslant i \leqslant 2$ or $i = 11$ or 12 we form a $QH^{\underline{i}}$ Model which satisfies every formula of the set, and hence A itself; and in the case $3 \leqslant i \leqslant 6$ or $i = 13$ we construct a maximal T-consistent set of formulae of $QH^{\underline{i}}$ which contains A, and form a $QH^{\underline{i}}$ Model which satisfies every formula of the set, and hence A itself; for the systems $QH^{\underline{i}}$ $(7 \leqslant i \leqslant 10$ or $i = 14$ or 15) we construct a maximal α consistent set of formulae of $QH^{\underline{i}}$ and thence a maximal Fα consistent set of formulae which contains $QH^{\underline{i}}$ α consistent A, and form a $QH^{\underline{i}}$ Model which satisfies every formula of the set, and hence A itself.

7A A maximal consistent set is one such that for every formula A of S, at least one of A or $\sim A$ is in the set; or, a set of formulae of S is maximally consistent iff it is consistent, and every formula of S not in it is inconsistent with it.

If we are given an S-consistent formula, A, we can always construct a maximal consistent set, ${}^s\Gamma$, which contains A, (in this paper we shall denote a maximal consistent set for any system $QH^{\underline{i}}$ $(0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13)$ by ${}^i\Gamma$). We arrange all the formulae of S in order: $A_1, A_2, A_3, \ldots, A_n, \ldots$ $(1 \leqslant n)$. We construct ${}^s\Gamma$ by forming a series of sets, ${}^s\Gamma_0$, ${}^s\Gamma_1$, ${}^s\Gamma_2, \ldots$, ${}^s\Gamma_n, \ldots$ $(1 < n)$ as follows:

${}^s\Gamma_0 = \{A\}$
${}^s\Gamma_1 = \{A\} \cup \{A_1\}$, if $\dashv_s \sim(A \,\&\, A_1)$, but if $\vdash_s \sim(A \,\&\, A_1)$ then ${}^s\Gamma_1 = \{A\} = {}^s\Gamma_0$.

So, for each subsequent set: given ${}^s\Gamma_n$

${}^s\Gamma_{n+1} = {}^s\Gamma_n \cup \{A_{n+1}\}$ if $\dashv_s \sim(A_{\Gamma n} \,\&\, A_{n+1})$, where $A_{\Gamma n}$ is the conjunction of wffs in ${}^s\Gamma_n$, otherwise
${}^s\Gamma_{n+1} = {}^s\Gamma_n$.

Finally, ${}^s\Gamma$ is the set of all the formulae in ${}^s\Gamma_0, {}^s\Gamma_1, {}^s\Gamma_2, \ldots, {}^s\Gamma_n, \ldots$ $(1 \leqslant n)$. Clearly, each set in the series is consistent because ${}^s\Gamma_0$ is consistent, and each ${}^s\Gamma_{n+1}$ is consistent if ${}^s\Gamma_n$ is.

Proof: Let ${}^s\Lambda$ be any finite subset of ${}^s\Gamma$, and let A_n be the last formula to have been included in ${}^s\Lambda$. Then ${}^s\Lambda$ is a subset of ${}^s\Gamma_n$. But ${}^s\Gamma_n$ is consistent, therefore ${}^s\Lambda$ is consistent. Hence ${}^s\Gamma$ contains no finite inconsistent subsets, and is therefore itself consistent. Moreover, ${}^s\Gamma$ is maximal: consider any formula A_m which is consistent with ${}^s\Gamma$, since A_m is consistent with ${}^s\Gamma$ it is consistent with any subset of ${}^s\Gamma$, and in particular with ${}^s\Gamma_{m-1}$; therefore, in the construction of ${}^s\Gamma$ it will have been added to ${}^s\Gamma_{m-1}$ to form ${}^s\Gamma_m$, and so is in ${}^s\Gamma$.

7B A maximal T-consistent set is the union of a maximal consistent set and a set of all the formulae containing any free variable a for which there is a formula of the form $\sim(\Sigma y)(y = a)$ in the maximal consistent set.

If we are given an S Consistent formula, A, we can set about constructing ${}^s\Gamma$. But, when we order the formulae of S: $A_1, A_2, A_3, \ldots, A_n, \ldots$ $(1 \leqslant n)$, we begin with the formulae of the forms $(\Sigma y)(y = a)$ and $\sim(\Sigma y)(y = a)$ as follows: $(\Sigma y)(y = a_0)$, $\sim(\Sigma y)(y = a_0)$, $(\Sigma y)(y = a_1)$, $\sim(\Sigma y)(y = a_1)$, $(\Sigma y)(y = a_2)$,

$\sim(\Sigma y)(y = a_2)$ . . ., $(\Sigma y)(y = a_n)$, $\sim(\Sigma y)(y = a_n)$, . . . $(n \geqslant 0)$. So we construct a series of sets ${}^s\Gamma_0$, ${}^s\Gamma_{00}$, ${}^s\Gamma_{01}$, ${}^s\Gamma_{02}$, . . ., ${}^s\Gamma_{0n}$, . . . $(0 \leqslant n)$ as follows

${}^s\Gamma_0 = \{A\}$
${}^s\Gamma_{00} = \{A\} \cup \{(\Sigma y)(y = a_0)\}$ if $\dashv_s \sim(A \,\&\, (\Sigma y)(y = a_0))$, but if $\vdash_s \sim(A \,\&\, (\Sigma y)(y = a_0))$ then
${}^s\Gamma_{00} = \{A\}$,
${}^s\Gamma_{01} = {}^s\Gamma_{00} \cup \{\sim(\Sigma y)(y = a_0)\}$ if $\dashv_s \sim(A \,\&\, \sim(\Sigma y)(y = a_0))$, but if $\vdash_s \sim(A \,\&\, \sim(\Sigma y)(y = a_0))$ then
${}^s\Gamma_{01} = {}^s\Gamma_{00}$.
So, given ${}^s\Gamma_{0\ 2n-1}$

${}^s\Gamma_{02n} = {}^s\Gamma_{0\ 2n-1} \cup \{(\Sigma y)(y = a_{2n})\}$ if $\dashv_s \sim(B' \,\&\, (\Sigma y)(y = a_{2n}))$, where B' is the conjunction of formulae in ${}^s\Gamma_{0\ 2n-1}$, but if $\vdash_s \sim(B' \,\&\, (\Sigma y)(y = a_{2n}))$ then
${}^s\Gamma_{02n} = {}^s\Gamma_{0\ 2n-1}$,
${}^s\Gamma_{0\ 2n+1} = {}^s\Gamma_{02n} \cup \{\sim(\Sigma y)(y = a_{2n})\}$ if B'' is the conjunction of formulae in ${}^s\Gamma_{02n}$, and if $\dashv_s \sim(B'' \,\&\, \sim(\Sigma y)(y = a_{2n}))$, but if $\vdash_s \sim(B'' \,\&\, \sim(\Sigma y)(y = a_{2n}))$ then
${}^s\Gamma_{0\ 2n+1} = {}^s\Gamma_{02n}$.

${}^s\Gamma_1$ is the set of all the formulae in ${}^s\Gamma_0$, ${}^s\Gamma_{00}$, ${}^s\Gamma_{01}$, ${}^s\Gamma_{02}$, . . ., ${}^s\Gamma_{0n}$, . . . $(0 \leqslant n)$. We then proceed to construct ${}^s\Gamma$ from ${}^s\Gamma_1$, ${}^s\Gamma_2$, ${}^s\Gamma_3$, . . ., ${}^s\Gamma_n$, . . . $(1 \leqslant n)$ as in **7A** above.

Furthermore, since it is the case that iff $\sim(\Sigma y)(y = a_n) \in {}^s\Gamma_1$, then $\sim(\Sigma y)(y = a_n) \in {}^s\Gamma_{02n}$, because of the ordering as set out above, we construct a set of formulae ${}^s\gamma$ in the following way:

Iff $\sim(\Sigma y)(y = a_n) \in {}^s\Gamma_{02n}$, there are in a set ${}^s\gamma_{0n}$ all the wffs of S in which occur the free variable a_n. ${}^s\gamma$ is the set of all the formulae in ${}^s\gamma_{01}$, ${}^s\gamma_{02}$, ${}^s\gamma_{03}$, . . ., ${}^s\gamma_{0n}$, . . . $(n \geqslant 1)$.

The set ${}^{Ts}\Gamma$, which is the union of ${}^s\Gamma$ and ${}^s\gamma$ is the maximal T-consistent set with respect to S, constructed from the given S consistent formula A. In this paper we will denote a maximal T-consistent set for any system $\mathrm{QH}\underline{^{i}}$ $(0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13)$ by ${}^{Ti}\Gamma$, where ${}^{Ti}\Gamma = {}^i\Gamma \cup {}^i\gamma$.

Similarly, a maximal **F** consistent set is that subset of a maximal consistent set which remains when all the formulae containing any free variable, a, for which there is a formula of the form $\sim(\Sigma y)(y = a)$, in a maximal consistent set are removed. We proceed as above to construct ${}^s\Gamma$ and ${}^s\gamma$.

Then, the set ${}^{Fs}\Gamma = {}^s\Gamma - ({}^s\gamma \cap {}^s\Gamma) = {}^s\Gamma - {}^s\gamma$.

7C A set of formulae of S is maximally α consistent iff it is α consistent, and every formula of S not in it is not α consistent with it. Given an S α consistent formula, A, we can construct a maximal α consistent set, ${}^{\alpha s}\Gamma$, which contains A, (in this paper we shall denote a maximal α consistent set for systems $\mathrm{QH}\underline{^{i}}$ $(7 \leqslant i \leqslant 10$ or $i = 14$ or $15)$ by ${}^i\Gamma$) by steps parallel to those in **7A** for maximal consistent sets except that:

${}^{\alpha s}\Gamma_1 = \{A\} \cup \{A_1\}$ if $\dashv_s \alpha(\sim(A \,\&\, A_1))$, but if $\vdash_s \alpha(\sim(A \,\&\, A_1))$ then
${}^{\alpha s}\Gamma_1 = {}^{\alpha s}\Gamma_0$

and thence with the obvious modifications in both construction and proof. And in a manner parallel to **7B** we can construct a maximal Fα consistent set. We denote a maximal Fα consistent set for any system $\mathbf{QH}\overset{i}{=}$ ($7 \leqslant i \leqslant 10$ or $i = 14$ or 15) by ${}^{Fi}\Gamma$, where ${}^{Fi}\Gamma = {}^{i}\Gamma - ({}^{i}\gamma \cap {}^{i}\Gamma)$ and ${}^{i}\gamma$ has been constructed as ${}^{i}\Gamma$ was constructed. It also follows that if we construct ${}^{Fi}\Gamma$ by beginning with some $\mathbf{QH}\overset{i}{=}$ α Consistent formula A, then $A \epsilon\, {}^{Fi}\Gamma$ ($7 \leqslant i \leqslant 10$ or $i = 14$ or 15).

Proof: There are two cases to be considered:
(a) where A contains no free variables: since ${}^{i}\Gamma_0 = \{A\}$ and therefore $A \epsilon\, {}^{i}\Gamma$ and $A \notin {}^{i}\gamma$ then $A \epsilon\, {}^{Fi}\Gamma$;
(b) where A contains free variables: let the free variables in A be $a_1, a_2, \ldots, a_n$ ($n \geqslant 1$), if $A \epsilon\, {}^{i}\Gamma$ and $A \notin {}^{Fi}\Gamma$ then $A \epsilon\, {}^{i}\gamma$ so, by construction of ${}^{i}\gamma$, at least one of

$\sim(\Sigma y)(y = a_1), \sim(\Sigma y)(y = a_2), \ldots, \sim(\Sigma y)(y = a_n)$ ($n \geqslant 1$) is in ${}^{i}\Gamma$.

Therefore $\vdash_i \alpha(\sim(A \,\&\, (\Sigma y)(y = a_n)))$ because $\vdash_i \alpha(\sim((\Sigma y)(y = a_n) \,\&\, \sim(\Sigma y)(y = a_n)))$ and by the construction of ${}^{i}\Gamma$, $(\Sigma y)(y = a_n)$ is earlier in the ordering than $\sim(\Sigma y)(y = a_n)$. I.e., $\vdash_i \alpha((\Sigma y)(y = a_n) \supset \sim A)$, i.e., $\vdash_i (\Sigma y)(y = a_n) \supset \sim A$ by R4, i.e., not α Consistent (A) contrary to hypothesis.

8 We now show some properties of maximal consistent sets and maximal α consistent sets.

Lemma 1i (where $0 \leqslant i \leqslant 2$ or $i = 11$ or 12). *If ${}^{i}\Gamma$ is maximal consistent then for any wff A, A and $\sim A$ are not both in ${}^{i}\Gamma$.*

Proof: If A and $\sim A$ were both in ${}^{i}\Gamma$, and if B were the conjunction of formulae in some finite subset of ${}^{i}\Gamma$, then $\{B, A, \sim A\}$ is consistent, i.e., $\nvdash_i \sim(B \,\&\, (A \,\&\, \sim A))$, but this is not so, because from axioms 1, 2, 3 and R1 $\vdash_i \sim(B \,\&\, (A \,\&\, \sim A))$. So, not both A and $\sim A$ are in ${}^{i}\Gamma$.

Lemma 1i (where $3 \leqslant i \leqslant 6$ or $i = 13$). *If ${}^{i}\Gamma$ is maximal consistent then for any wff A which contains no free variables A and $\sim A$ are not both in ${}^{i}\Gamma$.*

Lemma 2i (where $0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13$). *If ${}^{i}\Gamma$ is maximal consistent then for any wff A either A or $\sim A$ is in ${}^{i}\Gamma$. The proof of this lemma may be retrieved from Hughes & Cresswell,[5] with suitable modifications for each* $\mathbf{QH}\overset{i}{=}$.

Similarly:

Lemma 2i (where $7 \leqslant i \leqslant 10$ or $i = 14$ or 15). *If ${}^{i}\Gamma$ is maximal α consistent then for any wff A either A or $\sim A$ is in ${}^{i}\Gamma$.*

Lemma 3i (where $0 \leqslant i \leqslant 2$ or $i = 11$ or 12). *If ${}^{i}\Gamma$ is maximal consistent relative to* $\mathbf{QH}\overset{i}{=}$ *then for any wffs A and B, if $A \epsilon\, {}^{i}\Gamma$ and $(A \supset B) \epsilon\, {}^{i}\Gamma$ then $B \epsilon\, {}^{i}\Gamma$.*

5. Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London (1968), p. 152.

The proof of this lemma may be retrieved from Hughes & Cresswell,[6] and then modified to prove:

Lemma 3*i* (where $3 \leqslant i \leqslant 6$ or $i = 13$). *If* ${}^i\Gamma$ *is maximal consistent relative to* $\mathsf{QH}^{\underline{i}}$ *then for any wffs A and B, provided that every free variable in A is in B, if* $A \epsilon\ {}^i\Gamma$ *and* $(A \supset B) \epsilon\ {}^i\Gamma$ *then* $B \epsilon\ {}^i\Gamma$.

Lemma 3*i* (where $7 \leqslant i \leqslant 10$ or $i = 14$ or 15). *If* ${}^i\Gamma$ *is maximal* α *consistent relative to* $\mathsf{QH}^{\underline{i}}$ *then for any wffs A and B, if* $A \epsilon\ {}^i\Gamma$ *and* $(A \supset B) \epsilon\ {}^i\Gamma$, *then* $B \epsilon\ {}^i\Gamma$.

Lemma 4*i* (where $i = 0$ or 3 or 4 or 11). *If* ${}^i\Gamma$ *is maximal consistent relative to* $\mathsf{QH}^{\underline{i}}$ *then* ${}^i\Gamma$ *contains no formulae of the form* $\sim(\Sigma y)(y = a)$.

Proof: Since $\vdash_i \sim(\sim(\Sigma y)(y = a) \ \&\ a = a)$ there will be a subset of every maximal consistent set with which $\sim(\Sigma y)(y = a)$ is inconsistent.

Similarly:

Lemma 4*i* (where $i = 7$ or 8 or 14). *If* ${}^i\Gamma$ *is maximal* α *consistent relative to* $\mathsf{QH}^{\underline{i}}$ *then* ${}^i\Gamma$ *contains no formulae of the form* $\sim(\Sigma y)(y = a)$.

From this lemma and the respective Lemma 2*i*, it follows that every axiom is in every ${}^i\Gamma$. Furthermore that for every free variable $a, (\Sigma y)(y = a) \epsilon\ {}^i\Gamma$, when $i = 0$ or 3 or 4 or 7 or 8 or 11 or 14.

Lemma 4.1*i* (when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15). *If* ${}^i\Gamma$ *is maximal* α *consistent, then not both* $(\Sigma y)(y = a)$ *and* $\sim(\Sigma y)(y = a)$ *are in* ${}^i\Gamma$, *provided* $(\Sigma y)(y = a)$ *occurs before* $\sim(\Sigma y)(y = a)$ *in the ordering of* $\mathsf{QH}^{\underline{i}}$ *formulae.*

Proof: If $(\Sigma y)(y = a)$ and $\sim(\Sigma y)(y = a)$ were both in ${}^i\Gamma$, when ${}^i\Gamma$ was being constructed one of the formulae must have occurred before the other in the series of formulae of $\mathsf{QH}^{\underline{i}}$. Assume that there was a set ${}^i\Gamma_n$ such that B was the conjunction of wffs in ${}^i\Gamma_n$. Let $(\Sigma y)(y = a)$ be the $n + 1$'st formula and $\sim(\Sigma y)(y = a)$ be the $n + 2$'nd formula.

Assume that $\dashv_i \alpha(\sim(B \ \&\ (\Sigma y)(y = a)))$ so $(\Sigma y)(y = a) \epsilon\ {}^i\Gamma_{n+1}$. So ${}^i\Gamma_{n+2} = {}^i\Gamma_{n+1} \cup \{\sim(\Sigma y)(y = a)\}$ if $\dashv_i \alpha(\sim(B \ \&\ (\Sigma y)(y = a) \ \&\ \sim(\Sigma y)(y = a)))$ but if $\vdash_i \alpha(\sim(B \ \&\ (\Sigma y)(y = a) \ \&\ \sim(\Sigma y)(y = a)))$ then ${}^i\Gamma_{n+2} = {}^i\Gamma_{n+1}$ and $\sim(\Sigma y)(y = a) \not\epsilon\ {}^i\Gamma_{n+1}$ nor of ${}^i\Gamma$. Now $\vdash_i \alpha(\sim(B \ \&\ (\Sigma y)(y = a) \ \&\ \sim(\Sigma y)(y = a)))$. So if $(\Sigma y)(y = a)$ preceded $\sim(\Sigma y)(y = a)$ in the ordering of $\mathsf{QH}^{\underline{i}}$ formulae not both were in ${}^i\Gamma$. It can easily be shown that the converse is also the case.

9 Beginning with any $\mathsf{QH}^{\underline{i}}$ consistent (A) when $0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13$, or with any $\mathsf{QH}^{\underline{i}}$ α consistent (A) when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15, we construct a single set, ${}^i\Gamma$, of formulae of $\mathsf{QH}^{\underline{i}}$. We also require that ${}^i\Gamma$ shall have what we call the P_i-property.

A set, Λ, is said to have the P_i-property iff for every wff of some given form, α_i, in Λ there is also in Λ some wff of the form β_i:

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6. *Ibid.*, p. 153.

(a) When $i = 0$ or 3 or 11, α_i is $(\Sigma x)(\sim B)$ and β_i is $\sim B(a/x)$ for some a.
(b) When $i = 1$ or 5 or 12 or 13, α_i is $(\Sigma x)(\sim B)$ and β_i is $(\Sigma y)(y = a) \supset \sim B(a/x)$ for some a.
(c) When $i = 2$ or 6, α_i is $(\Sigma x)(\sim B)$ provided x occurs in B and β_i is $(\Sigma y)(y = a) \supset \sim B(a/x)$ for some a.
(d) When $i = 4$, α_i is $(\Sigma x)(\sim B)$ provided x occurs in B and β_i is $\sim B(a/x)$ for some a.
(e) When $i = 7$, α_i is $\alpha(\Sigma x)(\sim B)$ and β_i is $\alpha(\sim B(a/x))$ for some a.
(f) When $i = 9$ or 15, α_i is $\alpha(\Sigma x)(\sim B)$ and β_i is $\alpha((\Sigma y)(y = a) \supset \sim B(a/x))$ for some a.
(g) When $i = 10$, α_i is $\alpha(\Sigma x)(\sim B)$ provided x occurs in B and β_i is as when $i = 9$ or 15.
(h) When $i = 8$ or 14, α_i is $\alpha(\Sigma x)(\sim B)$ provided x occurs in B and β_i is as when $i = 7$.

To ensure that ${}^i\Gamma$ has the P_i-property, we begin with some definitions:

(i) Any wff of the form $\alpha_i \supset \beta_i$ (given the forms α_i and β_i) we shall call a P_i-formula with respect to a, or a P_i^a-formula.
(ii) All P_i-formulae which differ only in that each is a P_i-formula with respect to a different free variable will be said to have the same P_i-form. Clearly, P_i-forms, for each i, are enumerable.
(iii) Let the P_i-forms be enumerated thus: ${}^1P_i, {}^2P_i, {}^3P_i, \ldots, {}^nP_i, \ldots$ and put $\mathsf{P}_i = \{x/(\exists_j)(x = {}^jP_i)\}$.

Then a set of wffs has the P_i^1-property iff it is a superset of a selection set for P_i.

It is easy to show that if a maximal consistent set, Λ, of $\mathsf{QH}\overset{i}{=}$, has the P_i^1-property, it also has the P_i-property. For suppose that α_i is in Λ: Since Λ has the P_i^1-property, there is in Λ, for some free variable a, $\alpha_i \supset \beta_i$, so by Lemma $3i$, β_i is in Λ.

10 For a given $\mathsf{QH}\overset{i}{=}$ consistent (A) when $0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13$, or for a given $\mathsf{QH}\overset{i}{=}$ α consistent (A) when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15, we construct ${}^i\Gamma$ as follows (with provision for constructing either ${}^{Ti}\Gamma$ or ${}^{Fi}\Gamma$): We begin with A.

(a) Taking the P_i-forms as enumerated in **9** (iii) above, we then add, for each one of them in that order, a P_i^a-formula for some a which does not occur anywhere else in that P_i-formula, nor in any preceding P_i-formula, nor in A. Since we have an unlimited supply of free variables at our disposal, and since at each stage only a finite number of formulae are already in the set, there will always be a fresh variable available for this purpose.

(b) We then construct ${}^i\Gamma_0$ thus:

${}_{00}{}^i\Gamma = \{A\}$
${}_{0i}{}^i\Gamma\ (i > 0) = {}_{0(i-1)}{}^i\Gamma \cup \{\alpha_i^j \supset \beta_i^j\}$, where α_i^j is the j'th wff of form α_i, and the a in β_i^j does not occur in α_i^j nor in any member of ${}_{0(i-1)}{}^i\Gamma$.
${}^i\Gamma_0$ is the union of all the ${}_{0j}{}^i\Gamma$'s$(j \geqslant 0)$.

${}^{i}\Gamma$ is then constructed from ${}^{i}\Gamma_0$ and all the wffs of $\mathbf{QH}\underline{\overset{i}{=}}$ in some standard ordering such as is set out in **7B** or **7C** above (ensuring that for pairs of formulae of the forms $(\Sigma y)(y = a)$ and $\sim(\Sigma y)(y = a)$ the former occur before the latter in the ordering of the formulae). Clearly, ${}^{i}\Gamma_0$ will be consistent if each ${}^{i}\Gamma$ is, or will be α consistent if each ${}_{0j}{}^{i}\Gamma$ is, and so ${}^{i}\Gamma$ will be consistent when $0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13$ and ${}^{i}\Gamma$ will be α consistent when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15.

(c) When $3 \leqslant i \leqslant 6$ or $i = 13$, we then construct ${}^{Ti}\Gamma$; and when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15, we construct ${}^{Fi}\Gamma$.

11 To show that the set obtained when all the P_i formulae are added is ${}_i$consistent (where ${}_i$consistent is consistent when $0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13$, and ${}_i$consistent is α consistent when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15): Since $\{A\}$ is ${}_i$consistent by hypothesis, we prove the following lemma:

Lemma 5 *If* ${}^{i}\Lambda$ *is an* ${}_i$*consistent set of wffs, then* ${}^{i}\Lambda \cup \{\alpha_i^j \supset \beta_i^j\}$ *is* ${}_i$*consistent provided* a_j *does not occur in* ${}^{i}\Lambda$ *or* α_i^j, *and when* $3 \leqslant i \leqslant 6$ *or* $i = 13$ *provided that every free variable in* β_i^j *other than* a_j *is in* ${}^{i}\Lambda$, (a) *when i is* 0 *or* 3 *or* 11: *a proof for this case can be obtained from Hughes & Cresswell*;[7] (b) *when i is* 1 *or* 5 *or* 12 *or* 13.

Proof: Let ${}^{i}\Lambda'$ be any finite subset of ${}^{i}\Lambda$, then we prove that ${}^{i}\Lambda' \cup \{(\Sigma x) \sim B \supset. (\Sigma x)(x = a_j) \supset \sim B(a_j/x)\}$ is consistent. Assume that it is not; (let F be the conjunction of wffs in ${}^{i}\Lambda'$) then $\vdash_i \sim(F \;\&\; ((\Sigma x) \sim B \supset. (\Sigma x)(x = a_j) \supset \sim B(a_j/x)))$.

So $\vdash_i F \supset \sim((\Sigma x) \sim B \supset. (\Sigma x)(x = a_j) \supset \sim B(a_j/x))$

So $\vdash_i F \supset (\Pi y) \sim((\Sigma x) \sim B \supset. (\Sigma x)(x = y) \supset \sim B(y/x))$

So $\vdash_i (\Sigma y)((\Sigma x) \sim B \supset. (\Sigma x)(x = y) \supset \sim B(y/x)) \supset \sim F$. Since $\vdash_i (\Sigma y)((\Sigma x) \sim B \supset. (\Sigma x)(x = y) \supset \sim B(y/x))$** by R1 when $i = 1$ or 12 or R1a when $i = 5$ or 13 $\vdash \sim F$

$$\text{so } (F \;\&\; \sim F) \in {}^{i}\Lambda$$

**$(\Pi x)\, B \supset. (\Sigma x)(x = a) \supset B(a/x)$	Axiom 6
$(\Pi x)\, B \supset. (\Pi y)(\Sigma x)(x = y) \supset (\Pi y)B(y/x)$	R2 Axiom 4
$(\Pi y)\, B \;\&\; (\Pi y)(\Sigma x)(x = y) \supset. (\Pi y)B(y/x)$	def of &
since $(\Pi y)(B \;\&\; (\Sigma x)(x = y)) \supset. (\Pi y)\, B \;\&\; (\Pi y)(\Sigma x)(x = y)$	
so $(\Pi y)(B \;\&\; (\Sigma x)(x = y)) \supset (\Pi y)B(y/x)$	
$(\Sigma y) \sim B \supset. (\Sigma y)((\Sigma x)(x = y) \supset \sim B)$	
$(\Sigma z)[(\Sigma y) \sim B \supset. (\Sigma x)(x = z) \supset \sim B(z/y)]$	
so ${}^{i}\Lambda$ is inconsistent, contrary to hypothesis	
so Lemma 5 holds.	

(c) *when* $i = 2$ *or* 6: *by modification of case* (b) *above*.
(d) *when* $i = 4$: *by modification of case* (a) *above*.
(e), (f), (g) *and* (h): *by modification of cases* (a), (b), (c) *and* (d), *respectively*.

7. *Ibid.*, p. 160.

12 For a proof of the Completeness of $\mathsf{QH}^{\underline{i}}_{=}$ $(0 \leqslant i \leqslant 15)$ we show that we can construct a verifying $\mathsf{QH}^{\underline{i}}_{=}$ model (on ^{i}Model) for A, if A is $\mathsf{QH}^{\underline{i}}_{=}$ Consistent (A) $(0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13)$, or if A is $\mathsf{QH}^{\underline{i}}_{=}$ α consistent (A) $(7 \leqslant i \leqslant 10$ or $i = 14$ or $15)$. Consider a Hintikka type model as follows:

$\langle {}^i\Omega, C_i\rangle$ is an ^{i}Model, where ${}^i\Omega$ is a model system such that, when $0 \leqslant i \leqslant 2$ or $11 \leqslant i \leqslant 12$, the system contains one maximal model set ${}^i\mu$, i.e., ${}^i\Omega = \{{}^i\mu\}$, and when $3 \leqslant i \leqslant 10$ or $13 \leqslant i \leqslant 15$, the system is an ordered pair of model sets ${}^i\mu$ and ${}^i\mu'$, i.e., ${}^i\Omega = \langle \mu^i, {}^i\mu'\rangle$; and C_i is a set of consistency rules for deciding which formulae of $\mathsf{QH}^{\underline{i}}_{=}$ can be included (or embedded) in ${}^i\mu$ and ${}^i\mu'$, with or without ${}^i\mu$'s having some given membership, such given membership having been subject to the same set C_i.

The basic concept is that of satisfiability (or verifiability):

$$A \in {}^i\mu \ .\equiv.\ {}^i\text{Satisfiable } (A)$$

when $0 \leqslant i \leqslant 2$ or $11 \leqslant i \leqslant 12$; or

$$A \in [\,{}^i\mu - ({}^i\mu' \cap {}^i\mu)] \ .\equiv.\ {}^i\text{Satisfiable } (A)$$

when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15 or

$$A \in [\,{}^i\mu \cup {}^i\mu'] \ .\equiv.\ {}^i\text{Satisfiable } (A)$$

when $3 \leqslant i \leqslant 6$ or $i = 13$.

The membership of C_i is drawn from the following conditions:

(C.$\sim$) *If μ contains an atomic formula it does not contain its negation.*

(C.$\supset$) *If $(A \supset B) \in \mu$, then either $\sim A \in \mu$ or, $B \in \mu$, or both.*

(C.$\sim\Sigma$) *If $\sim(\Sigma x)\ A \in \mu$, then $(\Pi x)\ \sim A \in \mu$.*

(C.$\sim\Pi$) *If $\sim(\Pi x)\ A \in \mu$, then $(\Sigma x)\ \sim A \in \mu$.*

(C.*self*$\neq$) *μ does not contain any formula of the form $\sim(a = a)$.*

(C. =) *If $A \in \mu$, $(a = b) \in \mu$, and A is like B except for the interchange of a and b at some (or all) of their occurrences, then $B \in \mu$, provided that A and B are atomic formulae or identities.*

(C.$E\Sigma$) *If $Ea \in \mu$, then $(\Sigma x)(x = a) \in \mu$.*

(C.Σ) *If $(\Sigma x)\ A \in \mu$, then $A(a/x) \in \mu$ for at least one free variable a.*

(C.Π) *If $(\Pi x)\ A \in \mu$, then $A(a/x) \in \mu$.*

(C.Σ') *If $(\Sigma x)\ A \in \mu$ and x occurs in A, then $A(a/x) \in \mu$ for at least one free variable a.*

(C.Π') *If $(\Pi x)\ A \in \mu$ and x occurs in A, then $A(a/x) \in \mu$.*

(C.Σ_0) *If $(\Sigma x)\ A \in \mu$, then $A(a/x) \in \mu$ if $(\Sigma x)(x = a) \in \mu$ for at least one free variable a.*

(C.Π_0) *If $(\Pi x)\ A \in \mu$, then if $(\Sigma x)(x = a) \in \mu$, then $A(a/x) \in \mu$.*

(C.Σ_0') *and* (C.Π_0') *are like* (C.Σ_0) *and* (C.Π_0) *respectively with the proviso that x occur in A.*

(C.$\phi\epsilon$) *If $\sim(\Sigma y)(y = a) \in \mu$ and a occurs in A, then $A \in \mu'$.*

(C.$\supset'$) *If $(A \supset B) \in \mu$, then if every free variable in A is in B, then either $\sim A \in \mu$ or $B \in \mu$, or both.*

(C.ΠV) *If $(\Pi x)\ A \in \mu$ and x does not occur in A, then $A \in \mu$.*

(C.ΣV) *If $(\Sigma x)\ A \in \mu$ and x does not occur in A, then $A \in \mu$.*

($C.\Pi G$) *If* $A \epsilon \mu$ *then* $(\Pi x)\, A \epsilon \mu$, *provided* x *does not occur in* A.
($C.\Sigma G$) *If* $A \epsilon \mu$ *then* $(\Sigma x)\, A \epsilon \mu$, *provided* x *does not occur in* A.
($C.\Pi V'$) *If* $(\Sigma x)(x = a) \epsilon \mu$ *and* $(\Pi x)\, A \epsilon \mu$, *and* x *does not occur in* A, *then* $A \epsilon \mu$.
($C.\Sigma V'$) *If* $(\Sigma x)(x = a) \epsilon \mu$ *and* $(\Sigma x)\, A \epsilon \mu$, *and* x *does not occur in* A, *then* $A \epsilon \mu$.
($C.\Pi G'$) *If* $(\Sigma x)(x = a) \epsilon \mu$ *and* $A \epsilon \mu$ *then* $(\Pi x)\, A \epsilon \mu$, *provided* x *does not occur in* A.
($C.\Pi\Sigma'$) *If* $(\Sigma x)(x = a) \epsilon \mu$ *and* $A \epsilon \mu$ *then* $(\Sigma x)\, A \epsilon \mu$, *provided* x *does not occur in* A.

$C_0' = \{(C.\sim), (C.\supset), (C.\sim\Sigma), (C.\sim\Pi), (C.self\neq), (C.=), (C.E\Sigma)\}$
$C_0'' = \{(C.\sim), (C.\supset'), (C.\sim\Sigma), (C.\sim\Pi), (C.self\neq), (C.=), (C.E\Sigma)\}$
$C_0 = C_0' \cup \{(C.\Sigma), (C.\Pi), (C.\Pi G), (C.\Sigma G)\}$
$C_1 = C_0' \cup \{(C.\Sigma_0), (C.\Pi_0), (C.\Pi V'), (C.\Sigma V'), (C.\Pi G)\}$
$C_2 = C_0' \cup \{(C.\Sigma_0'), (C.\Pi_0'), (C.\Pi G), (C.\Sigma V)\}$
$C_3 = C_0'' \cup \{(C.\Sigma), (C.\Pi), (C.\Pi G), (C.\Sigma G), (C.\phi\epsilon)\}$
$C_4 = C_0'' \cup \{(C.\Sigma'), (C.\Pi'), (C.\Pi G), (C.\Sigma V), (C.\phi\epsilon)\}$
$C_5 = C_0'' \cup \{(C.\Sigma_0), (C.\Pi_0), (C.\Pi G), (C.\Sigma V), (C.\Pi V'), (C.\Sigma V'), (C.\phi\epsilon)\}$
$C_6 = C_0'' \cup \{(C.\Sigma_0'), (C.\Pi_0'), (C.\Pi G), (C.\Sigma V), (C.\phi\epsilon)\}$
$C_7 = C_0' \cup \{(C.\Sigma), (C.\Pi), (C.\Pi G), (C.\Sigma G), (C.\phi\epsilon)\}$
$C_8 = C_0' \cup \{(C.\Sigma'), (C.\Pi'), (C.\Pi G), (C.\Sigma V), (C.\phi\epsilon)\}$
$C_9 = C_0' \cup \{(C.\Sigma_0), (C.\Pi_0), (C.\Pi G), (C.\Sigma V), (C.\Pi V'), (C.\Sigma V'), (C.\phi\epsilon)\}$
$C_{10} = C_0' \cup \{(C.\Sigma_0'), (C.\Pi_0'), (C.\Pi G), (C.\Sigma V), (C.\phi\epsilon)\}$
$C_{11} = C_0' \cup \{(C.\Sigma), (C.\Pi), (C.\Pi V), (C.\Sigma G)\}$
$C_{12} = C_0' \cup \{(C.\Sigma_0'), (C.\Pi_0'), (C.\Sigma V'), (C.\Pi G')\}$
$C_{13} = C_0'' \cup \{(C.\Sigma_0'), (C.\Pi_0'), (C.\Pi G'), (C.\Sigma V'), (C.\phi\epsilon)\}$
$C_{14} = C_0' \cup \{(C.\Sigma'), (C.\Pi'), (C.\phi\epsilon)\}$
$C_{15} = C_0' \cup \{(C.\Sigma_0'), (C.\Pi_0'), (C.\Pi G'), (C.\Sigma V'), (C.\phi\epsilon)\}$

13 Given $\mathbf{QH}^{\underline{i}}$ Consistent (A) when $0 \leq i \leq 6$ or $11 \leq i \leq 13$ or $\mathbf{QH}^{\underline{i}}$ α Consistent (A) when $7 \leq i \leq 10$ or $i = 14$ or 15, we have constructed ${}^i\Gamma$ such that $A \epsilon\, {}^i\Gamma$, and ${}^{Ti}\Gamma$ such that $A \epsilon\, {}^{Ti}\Gamma$, and ${}^{Fi}\Gamma$ such that $A \epsilon\, {}^{Fi}\Gamma$. So we construct ${}^i\Omega$ as follows:

(A) where $0 \leq i \leq 2$ or $i = 11$ or 12:

(a) each atomic wff B is ^{i}Satisfiable (B) if it is one of the wffs in ${}^i\Gamma$, and is $\sim {}^i$Satisfiable (B) if B is not in ${}^i\Gamma$, i.e., $B \epsilon\, {}^i\Gamma \;.\!\equiv\!.\; B \epsilon\, {}^i\mu$.
(b) each wff of the form $(\Sigma y)(y = a)$ is ^{i}Satisfiable $(\Sigma y)(y = a)$ if it is one of the wffs in ${}^i\Gamma$, and $\sim {}^i$Satisfiable $(\Sigma y)(y = a)$ if it is not in ${}^i\Gamma$.
(c) when $i = 1$ or 2 or 12, and for every $a_j \sim (\Sigma y)(y = a_j) \epsilon\, {}^i\Gamma$ and $B(a_j/x) \epsilon\, {}^i\Gamma$, then if $(\Pi x)\ B \epsilon\, {}^i\Gamma$ (x occurring in B), ^{i}Satisfiable $(\Pi x)B$, and if $(\Pi x)\ B \notin {}^i\Gamma$, $\sim {}^i$Satisfiable $(\Pi x)B$.
(d) when $i = 2$ or 12, and when ${}^i\Gamma$ contains at least one formula of the form $(\Sigma x)(x = a)$, and also $\sim B$, each wff of the form $(\Pi x)B$ (where x does not occur in B) is ^{i}Satisfiable $(\Pi x)B$ if it is one of the wffs in ${}^i\Gamma$.
(e) when $i = 1$ or 2, and when ${}^i\Gamma$ contains no formula of the form $(\Sigma x)(x = a)$, and also $\sim B$, each wff of the form $(\Pi x)B$ (where x does not occur in B) is ^{i}Satisfiable $(\Pi x)B$ if it is one of the wffs in ${}^i\Gamma$.

(f) when $i = 11$, and $B \epsilon\, {}^i\Gamma$, each wff of the form $(\Pi x)B$ (where x does not occur in B) is $\sim {}^i$Satisfiable $(\Pi x)B$ if it is not one of the wffs in ${}^i\Gamma$.
(g) when $i = 12$, and when ${}^i\Gamma$ contains no formulae of the form $(\Sigma x)(x = a)$, then each wff of the form $(\Pi x)B$ is ^{i}Satisfiable $(\Pi x)B$ if it is one of the wffs in ${}^i\Gamma$, and $\sim {}^i$Satisfiable $(\Pi x)B$ if it is not.

(B) where $3 \leqslant i \leqslant 6$ or $i = 13$:

(a) each atomic wff B is in ${}^i\mu$ iff it is in ${}^i\Gamma$, i.e., $B \epsilon\, {}^i\Gamma$.≡. $B \epsilon\, {}^i\mu$ and each atomic wff B is in ${}^i\mu'$ iff it is in ${}^i\gamma$, i.e., $B \epsilon\, {}^i\mu'$.≡. $B \epsilon\, {}^i\gamma$.
(b) each wff of the form $(\Sigma x)(x = a)$ is in ${}^i\mu$ iff it is in ${}^i\Gamma$, and each wff of the form $\sim(\Sigma y)(y = a)$ is in ${}^i\mu'$ iff it is in ${}^i\gamma$.
(c) is as for A (c) except that $i = 5$ or 6 or 13 and for ${}^i\Gamma$ we have ${}^{\mathrm{T}i}\Gamma$.
(d) is as for A (d) except that $i = 4$ or 6 or 13 and for ${}^i\Gamma$ we have ${}^{\mathrm{T}i}\Gamma$.
(e) is as for A (e) except that $i = 5$ or 6 and for ${}^i\Gamma$ we have ${}^{\mathrm{T}i}\Gamma$.
(f) is as for A (g) except that $i = 13$ and for ${}^i\Gamma$ we have ${}^{\mathrm{T}i}\Gamma$.

(C) where $7 \leqslant i \leqslant 10$ or $i = 14$ or 15:

(a) each atomic wff B is in ${}^i\mu$ iff it is in ${}^i\Gamma$, i.e., $B \epsilon\, {}^i\mu$.≡. $B \epsilon\, {}^i\Gamma$ and each atomic wff B is in ${}^i\mu'$ iff it is in ${}^i\gamma$, i.e., $B \epsilon\, {}^i\mu'$.≡. $B \epsilon\, {}^i\gamma$.
(b) each wff of the form $(\Sigma y)(y = a)$ is in ${}^i\mu$ iff it is in ${}^i\Gamma$, and each wff of the form $\sim(\Sigma y)(y = a)$ is in ${}^i\mu'$ iff it is in ${}^i\gamma$.
(c) is as for A (d) except that $i = 8$ or 10 or 14 or 15 and for ${}^i\Gamma$ we have ${}^{\mathrm{F}i}\Gamma$.
(d) is as for A (e) except that $i = 9$ or 10 and for ${}^i\Gamma$ we have ${}^{\mathrm{F}i}\Gamma$.
(e) is as for A (g) except that $i = 15$ and for ${}^i\Gamma$ we have ${}^{\mathrm{F}i}\Gamma$.
(f) when $i = 14$, each wff of the form $(\Pi x)B$ (where x does not occur in B) is ^{i}Satisfiable $(\Pi x)B$ if $(\Pi x)B$ is in ${}^{\mathrm{F}i}\Gamma$ and $\sim {}^i$Satisfiable $(\Pi x)B$ if not.

14 Completeness Theorem: Given ^{i}Satisfaction as defined above, for every wff, B, there are three cases:

(a) when $0 \leqslant i \leqslant 2$ or $i = 11$ or 12, ^{i}Satisfiable (B) or $\sim$Satisfiable (B) according as $B \epsilon\, {}^i\Gamma$ or $B \notin {}^i\Gamma$, respectively.
(b) when $3 \leqslant i \leqslant 6$ or $i = 13$, ^{i}Satisfiable (B) or $\sim$Satisfiable (B) according as $B \epsilon\, {}^{\mathrm{T}i}\Gamma$ or $B \notin {}^{\mathrm{T}i}\Gamma$, respectively.
(c) when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15, ^{i}Satisfiable (B) or $\sim$Satisfiable (B) according as $B \epsilon\, {}^{\mathrm{F}i}\Gamma$ or $B \notin {}^{\mathrm{F}i}\Gamma$, respectively.

Since by hypothesis, when $0 \leqslant i \leqslant 6$ or $11 \leqslant i \leqslant 13$ our original $\mathbf{QH}^{\underline{i}}$ Consistent (A) is in ${}^i\Gamma$ then A is in ${}^{\mathrm{T}i}\Gamma$ when $3 \leqslant i \leqslant 6$ or $i = 13$ and so ^{i}Satisfiable (A). Also by hypothesis, when $7 \leqslant i \leqslant 10$ or $i = 14$ or 15 our original $\mathbf{QH}^{\underline{i}}\ \mathfrak{x}$ Consistent (A) is in ${}^i\Gamma$ and by proof **7C** is in ${}^{\mathrm{F}i}\Gamma$ and so ^{i}Satisfiable (A).

Proof by induction over the construction of $\mathbf{QH}^{\underline{i}}$ $(0 \leqslant i \leqslant 15)$ formulae:

Case (a): where $0 \leqslant i \leqslant 2$ or $i = 11$ or 12.

(1) If B is an atomic wff, the theorem holds for B by **13 A** (a).
(2) If the theorem holds for a wff B, then it also holds for $\sim B$: If $\sim B \epsilon\, {}^i\Gamma$,

then by Lemma $1i$ $B \notin {}^i\Gamma$, hence $\sim$Satisfiable (B), hence $\sim B \epsilon\, {}^i\mu$ since ${}^i\mu$ is maximal, so Satisfiable $(\sim B)$. If $\sim B \notin {}^i\Gamma$, then by Lemma $2i$ $B \epsilon\, {}^i\Gamma$, hence Satisfiable (B), and by $(C.\sim)$ $\sim$Satisfiable $(\sim B)$.

(3) If the theorem holds for wffs B and C, then it also holds for $(B \supset C)$. Suppose that $(B \supset C) \epsilon\, {}^i\Gamma$ then either $\sim B \epsilon\, {}^i\Gamma$ or $C \epsilon\, {}^i\Gamma$ for the following reason: if neither $\sim B \epsilon\, {}^i\Gamma$ nor $C \epsilon\, {}^i\Gamma$, then $B \epsilon\, {}^i\Gamma$ and $\sim C \epsilon\, {}^i\Gamma$ (by Lemma $2i$ and the construction of ${}^i\Gamma$), and so by Lemma $3i$, since $\vdash_i B \supset (\sim C \supset \sim(B \supset C))$, thus $\sim(B \supset C)$; but then $(B \supset C) \notin {}^i\Gamma$ (by Lemma $1i$) which contradicts the supposition. So either $\sim B \epsilon\, {}^i\Gamma$ or $C \epsilon\, {}^i\Gamma$. Hence either $\sim B \epsilon\, {}^i\mu$ or $C \epsilon\, {}^i\mu$. $(B \supset C)$ can be not Satisfiable only if $B \epsilon\, {}^i\mu$ and $\sim C \epsilon\, {}^i\mu$ $(C.\supset)$, and since $\sim B \epsilon\, {}^i\mu$ or $C \epsilon\, {}^i\mu$ it follows that Satisfiable $(B \supset C)$. Suppose that $(B \supset C) \notin {}^i\Gamma$, hence $B \epsilon\, {}^i\Gamma$ and $\sim C \epsilon\, {}^i\Gamma$. So $B \epsilon\, {}^i\mu$ and $\sim C \epsilon\, {}^i\mu$. Thus, by $(C.\supset)$ and $(C.\sim)$, $\sim$Satisfiable $(B \supset C)$.

(4) If the theorem holds for any wff B then it holds for $(\Pi x)B$. There are several cases where $(\Pi x)B \epsilon\, {}^i\Gamma$ or $(\Pi x)B \notin {}^i\Gamma$.

First, there are cases where x occurs in B in $(\Pi x)B$:

(i) where for every a_j such that $(\Sigma y)(y = a_j) \epsilon\, {}^i\Gamma$ then iff $B(a_j/x) \epsilon\, {}^i\Gamma$ will $(\Pi x)B \epsilon\, {}^i\Gamma$;

Proof: 1. Assume $(\Sigma y)(y = a_j) \epsilon\, {}^i\Gamma$ and $B(a_j/x) \epsilon\, {}^i\Gamma$ for every a_j, but $(\Pi x)B \notin {}^i\Gamma$ (with the proviso fulfilled for the relevant ${}^i\Gamma$). So $\sim(\Pi x)B \epsilon\, {}^i\Gamma$ by Lemma $2i$, now by the construction of ${}^i\Gamma$, since ${}^i\Gamma$ has the relevant P^i-property, and by Lemma $3i$ $\sim B(a_j/x) \epsilon\, {}^i\Gamma$ contrary to hypothesis. So $(\Pi x)B \epsilon\, {}^i\Gamma$.

2. Assume $(\Sigma y)(y = a_j) \epsilon\, {}^i\Gamma$ and $(\Pi x)B \epsilon\, {}^i\Gamma$, then by axioms 6 or 6a or 7 or 7a, and Lemma $3i$ $B(a_j/x) \epsilon\, {}^i\Gamma$.

(ii) 1. where there is an a_j such that $(\Sigma y)(y = a_j) \epsilon\, {}^i\Gamma$ and $\sim B(a_j/x) \epsilon\, {}^i\Gamma$, then $(\Pi x)B \notin {}^i\Gamma$;

Proof: Assume that there is an a_j such that $(\Sigma y)(y = a_j) \epsilon\, {}^i\Gamma$ and $\sim B(a_j/x) \epsilon\, {}^i\Gamma$, but $(\Pi x)B \epsilon\, {}^i\Gamma$. By axioms 6 or 6a or 7 or 7a, and Lemma $3i$ $B(a_j/x) \epsilon\, {}^i\Gamma$ contrary to hypothesis, so $(\Pi x)B \notin {}^i\Gamma$.

2. where there is an a_j such that $(\Sigma y)(y = a_j) \epsilon\, {}^i\Gamma$ and $(\Pi x)B \notin {}^i\Gamma$, then $\sim B(a_j/x) \epsilon\, {}^i\Gamma$: Proof follows at once from the construction of ${}^i\Gamma$ with the P^i-property and Lemma $3i$.

(iii) where for every a_j such that $\sim(\Sigma y)(y = a_j) \epsilon\, {}^i\Gamma$ and $B(a_j/x) \epsilon\, {}^i\Gamma$, then $(\Pi x)B(x/a_j)$ is arbitrarily in ${}^i\Gamma$ or not (except that from Lemma $4i$, $i \neq 0$ or 11).

In these three cases, (i) to (iii), we show that if $(\Pi x)B \epsilon\, {}^i\Gamma$ then Satisfiable $(\Pi x)B$, and if $(\Pi x)B \notin {}^i\Gamma$ then $\sim$Satisfiable $(\Pi x)B$.

In case (i): If $(\Pi x)B \epsilon\, {}^i\Gamma$ under (i) then Satisfiable $(\Pi x)B$, i.e., $(\Pi x)B \epsilon\, {}^i\mu$.

Proof: Assume that for every a_j $(\Sigma y)(y = a_j)$ and $B(a_j/x) \epsilon\, {}^i\Gamma$ and $(\Pi x)B \notin {}^i\mu$, i.e., since ${}^i\mu$ is maximal, $\sim(\Pi x)B(x/a_j) \epsilon\, {}^i\mu$. So by $(C.\Sigma)$ when i is 0 or 11 or $(C.\Sigma_0)$ when i is 1 or 12 or $(C.\Sigma_0')$ when i is 2 $\sim B(a_j/x) \epsilon\, {}^i\mu$. But since $B(a_j/x) \epsilon\, {}^i\Gamma$, Satisfiable $B(a_j/x)$. So $B(a_j/x) \epsilon\, {}^i\mu$ which is absurd. So under case (i), if $(\Pi x)B \epsilon\, {}^i\Gamma$ then Satisfiable $(\Pi x)B$.

If $(\Pi x)B \notin {}^i\Gamma$ under (i) then $\sim$Satisfiable $(\Pi x)B$, i.e., $(\Pi x)B \notin {}^i\mu$.

Proof: Assume that for every a_j $(\Sigma y)(y = a_j)$ and $\sim B(a_j/x) \in {}^i\Gamma$ and $(\Pi x)B \in {}^i\mu$. So by assignment 13 A (b) and by $(C.\Pi)$ when i is 0 or 11 or $(C.\Pi_0)$ when i is 1 or 12 or $(C.\Pi_0')$ when i is 2 $B(a_j/x) \in {}^i\mu$. But since $\sim B(a_j/x) \in {}^i\Gamma$ then Satisfiable $\sim B(a_j/x)$, so $B(a_j/x) \notin {}^i\mu$ which is absurd. So under case (i), if $(\Pi x)B \notin {}^i\Gamma$ then $\sim$Satisfiable $(\Pi x)B$.

In case (ii): If $(\Pi x)B \notin {}^i\Gamma$ under (ii) then $\sim$Satisfiable $(\Pi x)B$, i.e., $(\Pi x)B \notin {}^i\mu$.

Proof: Assume that there is an a_j such that $(\Sigma y)(y = a_j) \in {}^i\Gamma$ and $B(a_j/x) \notin {}^i\Gamma$ but that $(\Pi x)B \in {}^i\mu$.

So by assignment 13 A (b) and by $(C.\Pi)$ when i is 0 or 11 or $(C.\Pi_0)$ when i is 1 or 12 or $(C.\Pi_0')$ when i is 2 $B(a_j/x) \in {}^i\mu$. But if $B(a_j/x) \notin {}^i\Gamma$ then $\sim$Satisfiable $B(a_j/x)$, i.e., $B(a_j/x) \notin {}^i\mu$ which is absurd. So under case (ii), if $(\Pi x)B \notin {}^i\Gamma$ then $\sim$Satisfiable $(\Pi x)B$.

If $(\Pi x)B \in {}^i\Gamma$, then it is as for either case (i) or case (iii).

In case (iii) if $(\Pi x)B \in {}^i\Gamma$ then by 13 A (c) and Lemma 1i Satisfiable $(\Pi x)B$, and conversely.

Secondly, there are the cases where x does not occur in B in $(\Pi x)B$. These cases fall into two groups. First, there are the cases where ${}^i\Gamma$ contains at least one formula of the form $(\Sigma x)(x = a)$. If i is 0 or 11, ${}^i\Gamma$ will always be of this kind:

(iv) when $0 \leqslant i \leqslant 2$ or $i = 12$ if $B \in {}^i\Gamma$ then $(\Pi x)B \in {}^i\Gamma$ because of $\vdash_i B \supset (\Pi x)B (0 \leqslant i \leqslant 12)$ and Lemma 3i or axiom 6 when $i = 12$.
(v) when $i = 0$ or 11 if $(\Pi x)B \in {}^i\Gamma$ then $B \in {}^i\Gamma$ (axiom 6 and Lemma 3i).
(vi) when $i = 0$ iff $(\Pi x)B \in {}^i\Gamma$ then $B \in {}^i\Gamma$.
(vii) (a) when $i = 2$ or 12 if $\sim B \in {}^i\Gamma$ then whether $(\Pi x)B \in {}^i\Gamma$ is arbitrary.
(b) when $i = 11$, and $B \in {}^i\Gamma$, then whether $(\Pi x)B \in {}^i\Gamma$ is arbitrary.

The second group of cases are when ${}^i\Gamma$ does not contain any formula of the form $(\Sigma x)(x = a)$:

(viii) when $i = 1$ or 2 if $B \in {}^i\Gamma$ then $(\Pi x)B \in {}^i\Gamma$.
(ix) (a) when $i = 1$ or 2 or 12 and $\sim B \in {}^i\Gamma$ then whether $(\Pi x)B \in {}^i\Gamma$ is arbitrary.
(b) when $i = 12$ and $B \in {}^i\Gamma$ then whether $(\Pi x)B \in {}^i\Gamma$ is arbitrary.

In these six cases we show that if $(\Pi x)B \in {}^i\Gamma$ then Satisfiable $(\Pi x)B$, and if $(\Pi x)B \notin {}^i\Gamma$ then $\sim$Satisfiable $(\Pi x)B$.

In case (iv) if $(\Pi x)B \notin {}^i\Gamma$ then $\sim$Satisfiable $(\Pi x)B$. Assume $(\Pi x)B \notin {}^i\Gamma$ but $(\Pi x)B \in {}^i\mu$. So by $(C.\Pi)$ or $(C.\Pi V)$ $B \in {}^i\mu$, so $B \in {}^i\Gamma$, but $\sim B \in {}^i\Gamma$ by axiom 2, Lemmas 2i and 3i; which is absurd.

In case (v) if $(\Pi x)B \in {}^i\Gamma$ then Satisfiable $(\Pi x)B$. Assume $(\Pi x)B \in {}^i\Gamma$ and $\sim(\Pi x)B \in {}^i\mu$, so by $(C.\Sigma)$ $\sim B \in {}^i\mu$, i.e., $B \notin {}^i\Gamma$. But by $\vdash_i (\Pi x)B \supset B$ and Lemma 3i $B \in {}^i\Gamma$; which is absurd.

In case (vi) it follows from (iv) and (v).

In case (vii) (a) if $(\Pi x) B \epsilon\, {}^i\Gamma$ then Satisfiable $(\Pi x)B$ by **13A** (d), (g); (b) if $(\Pi x) B \epsilon\, {}^i\Gamma$ then Satisfiable $(\Pi x)B$ by **13A** (f).

In case (viii) if $(\Pi x) B \notin {}^i\Gamma$ then $\sim$Satisfiable $(\Pi x)B$ as in (iv).

In case (ix) (a) if $(\Pi x) B \epsilon\, {}^i\Gamma$ then Satisfiable $(\Pi x)B$ by **13A** (e); (b) if $(\Pi x) B \epsilon\, {}^i\Gamma$ then Satisfiable $(\Pi x)B$ by **13A** (e).

Case (b): where $3 \leq i \leq 6$ or $i = 13$.

Proof of the theorem is as in Case (a), except when a formula contains a free variable, a, which occurs in a formula of the form $\sim(\Sigma x)(x = a)$ in ${}^{Ti}\Gamma$. Now if $\sim(\Sigma x)(x = a) \epsilon\, {}^{Ti}\Gamma$, then by the construction of ${}^{Ti}\Gamma$ every formula of $QH^{\underline{i}}$ containing a is in ${}^{Ti}\Gamma$, since every such formula is in ${}^i\gamma$.. Also, by **13B** (b), Lemma $2i$ and the construction of ${}^i\Gamma$ and ${}^i\gamma$, if $\sim(\Sigma x)(x = a) \epsilon\, {}^{Ti}\Gamma$ then $\sim(\Sigma x)(x = a) \epsilon\, {}^i\mu'$, and so, by $(C.\phi\epsilon)$ every formula containing a is in ${}^i\mu'$, and ${}^i\mu' \cup {}^i\mu$. So, in the case of the exception, if any formula B (containing any free variable, a, for which a formula of the form $\sim(\Sigma x)(x = a) \epsilon\, {}^{Ti}\Gamma$) is in ${}^{Ti}\Gamma$ then Satisfiable (B).

Case (c): where $7 \leq i \leq 10$ or $i = 14$ or 15.

For formulae which contain no free variables it can be shown by means of proofs parallel to those in Case (a) that if $B \epsilon\, {}^i\Gamma$ then $B \epsilon\, {}^{Fi}\Gamma$ and $B \epsilon [{}^i\mu - ({}^i\mu' \cap {}^i\mu)]$ or if $B \notin {}^i\Gamma$ then $B \notin {}^{Fi}\Gamma$ and $B \notin [{}^i\mu - ({}^i\mu' \cap {}^i\mu)]$, and so Satisfiable (B) or $\sim$Satisfiable (B), respectively. For formulae which contain free variables: if a wff B contains at least one free variable, a, such that $\sim(\Sigma y)(y = a) \epsilon\, {}^i\Gamma$ then $B \epsilon\, {}^i\gamma$ and $B \notin {}^{Fi}\Gamma$. By **13C** (b) $\sim(\Sigma y)(y = a)\epsilon\, {}^i\mu'$ and so by $(C.\phi\epsilon)\, B \epsilon\, {}^i\mu'$ and so $B \notin [{}^i\mu - ({}^i\mu' \cap {}^i\mu)]$ and so $\sim$Satisfiable (B). Conversely, if a wff B contains no free variable, a, such that $\sim(\Sigma y)(y = a) \epsilon\, {}^i\Gamma$ then $B \epsilon\, {}^{Fi}\Gamma$ and is subject to the same proofs as for formulae with no free variables such that if $B \epsilon\, {}^{Fi}\Gamma$ then Satisfiable (B).

Parallels for Case (a)(4).

(iv) when $0 \leq i \leq 10$ or $i = 12$ or 13 or 15.
(v) when $i = 0$ or 1 or 3 or 5 or 7 or 9 or 11.
(vi) when $i = 0$ or 1 or 3 or 5 or 7 or 9.
(vii) (a) when $i = 2$ or 4 or 6 or 8 or 10 or 12 or 15.
(b) when $i = 11$.
(viii) when $i = 1$ or 2 or 5 or 6 or 9 or 10.
(ix) (a) when $i = 1$ or 2 or 5 or 6 or 9 or 10 or 12 or 13 or 15.
(b) when $i = 12$ or 13 or 15.

15 We have considered the sixteen systems $QH^{\underline{i}}$ $(0 \leq i \leq 15)$, which are parallel to the systems in "Completeness theorems for some presuppositions-free logics."[8] The systems $QH^{\underline{i}}$ $(0 \leq i \leq 10)$ are parallel to the systems $QC^{\underline{i}}$ $(0 \leq i \leq 10)$ which take up the major part of that article.

8. *Op. cit.*

$\mathbf{QH}^{i}_{=}$ ($11 \leqslant i \leqslant 15$) are parallel to the variant systems set out in the article's Appendices. The basis of the variance is to be found in the treatment of the vacuous quantifier.

It must also be pointed out that $\mathbf{QH}^{i}_{=}$ ($0 \leqslant i \leqslant 15$) are *parallel* to $\mathbf{QC}^{i}_{=}$ ($0 \leqslant i \leqslant 10$) and their variants, but are not *extensions* of the systems. In the semantics used in this paper we only take cognisance of the situation where statements about non-existing (possible) objects are consistent or inconsistent in exactly the same way as statements about existing objects. Furthermore, our logic is existence presupposition free only in the sense in which $\mathbf{QC}^{2}_{=}$ is existence presupposition free. So we can qualify our statement that the $\mathbf{QH}^{i}_{=}$ systems are not extensions of the $\mathbf{QC}^{i}_{=}$ systems to the extent that we can take the $\mathbf{QH}^{i}_{=}$ systems to be extensions of $\mathbf{QC}^{2}_{=}$. The question of extending the other $\mathbf{QC}^{i}_{=}$ systems is now considered.

Let us briefly consider the situation in which we modify $\mathbf{QC}^{i}_{=}$, first by excluding from their Primitive symbols free variables, then renaming the individual constants "free variables," and then providing model system semantics for the modified systems. Then we extend the systems, let us call them $\mathbf{MQC}^{i}_{=}$, by the addition of the Π symbol and the axioms in which Π occurs. Instead of Axiom 11 we would have $(Ex)(x = a) \supset (\Sigma x)(x = a)$.

The extensions of $\mathbf{MQC}^{i}_{=}$ (i is 0 or 3 or 4 or 7 or 8 or 14) are not interesting because there will be no formulae of the form $\sim(Ex)(x = a)$ in any model set. So, nothing consistent can be said about non-existing objects, whether possible or impossible. Similarly, the extensions of $\mathbf{MQC}^{i}_{=}$ where i is 9 or 10 or 15 will rule that everything said about non-existent objects is inconsistent. The extensions of $\mathbf{MQC}^{i}_{=}$ where $i = 5$ or 6 or 13 allow one to say anything whatsoever about non-existent objects, whether possible or impossible, and to have any such statement counted consistent. The sum total of these considerations would indicate that the only non-vacuous extensions of the system $\mathrm{MQC}^{i}_{=}$ would be in the cases of $\mathbf{MQC}^{i}_{=}$ where i is either 1 or 2.

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