

PROPOSITIONAL AND PREDICATE CALCULUSES
 BASED ON COMBINATORY LOGIC

M. W. BUNDER

In this article we shall establish various propositional and predicate calculuses based on combinatory logic (see [4]) with suitable restrictions on the variables. These restrictions are needed to avoid Curry's paradox ([4] pp. 258, 259). We require, Rule Ξ , the rule of restricted generality:

$$\Xi xy, xu \vdash yu,$$

and the iterated deduction theorem for Ξ .¹

If $X_0, X_1, \dots, X_m \vdash Y$ where no u_k occurs in any X_j for $j < k$, and if for all $k < m$ $X_0, X_1, \dots, X_k \vdash \mathbf{L}([u_{k+1}]X_{k+1})$, then $X_0 \vdash X_1 \supset_{u_1} \dots X_m \supset_{u_m} Y$.

From this deduction theorem we obtain deduction theorems for implication (\mathbf{P} or \supset) and for universal generality (Π or $\Xi\mathbf{E}$).

If $X_0, X_1, \dots, X_m \vdash Y$ and if for all $k < m$ $X_0, X_1, \dots, X_k \vdash \mathbf{H}(X_{k+1})$ then $X_0 \vdash X_1 \supset: X_2 \supset \dots \supset Y$.

If $X_0, X_1, \dots, X_m \vdash Yu$ for all u , and u does not occur in any X_j , then $X_0, X_1, \dots, X_m \vdash \Pi Y$.

The deduction theorem for Ξ was proved from certain axioms in [2], the other two are derived from it.

If we then take the additional axiom

Axiom PH. $\vdash \mathbf{H}x \supset_x . \mathbf{H}y \supset_y \mathbf{H}(x \supset y)$,

we obtain all of the absolute (or intuitionistic) calculus of pure implication. If we then introduce a slightly more complex axiom connecting Ξ and \mathbf{H} ,

1. In [2] \mathbf{L} was defined as \mathbf{FAH} or $\mathbf{B}(\Xi\mathbf{A}(\mathbf{BH}))$ and \mathbf{HX} was interpreted as "X is a proposition." Here however we take \mathbf{L} as primitive and define \mathbf{H} by \mathbf{BLK} . If we have $\mathbf{L} = \mathbf{FAH}$ and \mathbf{H} primitive, we need either Axiom 2 or Axiom 8 ($\vdash\mathbf{LA}$) of [2] to prove Theorem 3 below. $Xu \supset_u Yu$ stands for ΞXY . Note that this form of the deduction theorem avoids the Kleene-Rosser paradox (See [3]).

which entails Axiom PH, we can define in terms of Ξ all the propositional connectives and quantifiers, in such a way that the complete intuitionistic predicate calculus can be established with suitable restrictions. We then have classical calculus if we adjoin an axiom expressing Peirce's Law.

1. *Pure Implicational Logic* It is well known that pure implicational intuitionistic logic can be axiomatically based on modus ponens and two axiom schemes. Modus ponens follows directly from Rule Ξ and restricted versions of the axiom schemes follow from Axiom PH, the deduction theorem for implication and modus ponens:

Theorem 1. $\mathbf{H}x, \mathbf{H}y \vdash x \supset . y \supset x.$

Theorem 2. $\mathbf{H}x, \mathbf{H}y, \mathbf{H}z \vdash x \supset . y \supset z : \supset : x \supset y . \supset . x \supset z.$

We can therefore state a more general theorem:

Theorem 3. *If $\vdash X$ is a theorem of pure implicational intuitionistic logic and x_1, x_2, \dots, x_n are the free propositional variables in X , then $\mathbf{H}x_1, \mathbf{H}x_2, \dots, \mathbf{H}x_n \vdash X$ is a theorem of the present system.²*

2. *Absolute Propositional Calculus* We shall now consider the full intuitionistic propositional calculus. In such a system, conjunction (\wedge), alternation (\vee) and negation (\neg) are independent of each other and of \mathbf{P} . All of these would therefore normally have to be taken as new primitives. However, it is found that they can be defined in terms of \mathbf{P} and some extraneous (to the propositional calculus) obs namely Ξ, \mathbf{H} and \mathbf{I} .³ Negation will be treated in the next section; here we will treat the absolute or positive intuitionistic system.

Definition \wedge . $\wedge \equiv [x, y] \mathbf{H}z \supset_z . (x \supset . y \supset z) \supset z.$

Definition \vee . $\vee \equiv [x, y] \mathbf{H}z \supset_z : (x \supset z) \supset . (y \supset z) \supset z.$

Using these, and Axiom 7, $\vdash \mathbf{LH}$, of [2] we can prove the basic axioms needed for \wedge and \vee in intuitionistic logic. In order to accomplish this, theorems concerning the propositionality of \wedge and \vee are needed; and since, so far, we only have an axiom to give us $\mathbf{H}(\mathbf{P}xy)$, and \wedge and \vee are defined in terms of Ξ , an axiom for $\mathbf{H}(\Xi xy)$ is needed. For it, we will take a form which will lead to what we had for \mathbf{P} in Axiom A7 of [1].

Axiom 9. $\vdash \mathbf{L}x \supset_x . \mathbf{F}x \mathbf{H}y \supset_y \mathbf{H}(\Xi xy).$

This leads to the following results.

Theorem 4. $\mathbf{L}x, \mathbf{F}x \mathbf{H}y \vdash \mathbf{H}(\Xi xy).$

Corollary 4.1. *If $\vdash \mathbf{H}(xu)$ for all u , then $\vdash \mathbf{H}(\Pi x)$.*

Corollary 4.2. *If $xu \vdash \mathbf{H}(yu)$ for all u then $\mathbf{L}x \vdash \mathbf{H}(\Xi xy)$.*

Theorem 5. $\mathbf{P}x(\mathbf{H}y), \mathbf{H}x \vdash \mathbf{H}(x \supset y).$

2. This is proved fully in Theorem 25.

3. Such definitions of \wedge and \vee occur in Leśniewski's protothetics and related domains, but the definitions given here are suggested in principle by Ono [7].

Proof. Put $\mathbf{K}x$ for x and $\mathbf{K}y$ for y in Corollary 4.2. Now if $x \vdash \mathbf{H}y$, then $\mathbf{L}(\mathbf{K}x) \vdash \mathbf{H}(x \supset y)$ and the result follows by $\mathbf{H} = \mathbf{BLK}$ and the deduction theorem for \mathbf{P} .

Corollary 5.1. $\vdash \mathbf{F}_2 \mathbf{HHHP}$.

An alternative to Axiom 9, which also gives the corollary to Theorem 5, is

Axiom 9'. $\vdash \mathbf{L}x \supset_x . \mathbf{L}y \supset_y \mathbf{H}(\exists xy)$.

However, the other form seems to fit the three-valued tables of [2] more closely. Now we come to the theorem for $\mathbf{H}(\Lambda xy)$.

Theorem 6. $\mathbf{H}x, \mathbf{P}x(\mathbf{H}y) \vdash \mathbf{H}(\Lambda xy)$.

Proof. $x, \mathbf{P}x(\mathbf{H}y) \vdash \mathbf{H}y$, so $\mathbf{H}z, x, \mathbf{P}x(\mathbf{H}y) \vdash \mathbf{H}(y \supset z)$ by Theorem 5; hence by the deduction theorem for \mathbf{P} , $\mathbf{H}z, \mathbf{H}x, \mathbf{P}x(\mathbf{H}y) \vdash \mathbf{P}x(\mathbf{H}(y \supset z))$. Therefore by Theorem 5 again, $\mathbf{H}z, \mathbf{H}x, \mathbf{P}x(\mathbf{H}y) \vdash \mathbf{H}(x \supset . y \supset z)$. Then $\mathbf{H}z, \mathbf{H}x, \mathbf{P}x(\mathbf{H}y) \vdash \mathbf{H}((x \supset . y \supset z) \supset z)$, hence by the deduction theorem and $\vdash \mathbf{LH}$, $\mathbf{H}x, \mathbf{P}x(\mathbf{H}y) \vdash \mathbf{FHH}([z] . (x \supset . y \supset z) \supset z)$. Thus by Theorem 4, $\mathbf{H}x, \mathbf{P}x(\mathbf{H}y) \vdash \mathbf{H}(\Lambda xy)$.

Corollary 6.1. $\mathbf{H}x, \mathbf{H}y \vdash \mathbf{H}(\Lambda xy)$.

Now the remaining results for Λ can be proved.

Theorem 7. $\Lambda xy, \mathbf{H}x, \mathbf{H}y \vdash x$.

Proof. By definition Λ and Rule \exists ,

$$\Lambda xy, \mathbf{H}x, \mathbf{H}z, x \supset . y \supset z \vdash z. \quad (1)$$

Then putting x for z and using Theorem 1, we get $\Lambda xy, \mathbf{H}x, \mathbf{H}y \vdash x$.

Theorem 8. $\Lambda xy, \mathbf{H}x, \mathbf{H}y \vdash y$.

Proof. We have $\mathbf{H}y \vdash y \supset y$, so by Theorem 1, $\mathbf{H}x, \mathbf{H}y \vdash x \supset . y \supset y$, and putting y for z in (1) gives the result.

Theorem 9. $x, y \vdash \Lambda xy$.

Proof. By modus ponens, $x, y, x \supset . y \supset z \vdash z$, and as $\mathbf{H}x, \mathbf{H}y, \mathbf{H}z \vdash \mathbf{H}(x \supset . y \supset z)$, we have by Axiom 6 and the deduction theorem for \mathbf{P} ,⁴ $x, y, \mathbf{H}z \vdash (x \supset . y \supset z) \supset z$. The result then follows by the deduction theorem and Definition Λ .

It can be seen from the three-valued tables that the conditions for $\mathbf{H}(\mathbf{V}xy)$, should not be the same as those for $\mathbf{H}(\mathbf{P}xy)$ and $\mathbf{H}(\Lambda xy)$, in fact we need $\vdash \mathbf{H}x$ and $\vdash \mathbf{P}(\neg x)(\mathbf{H}y)$. As we have no negation as yet, we can only prove $\mathbf{H}x, \mathbf{H}y \vdash \mathbf{H}(\mathbf{V}xy)$. We can also prove the three other basic rules for \mathbf{V} .

Theorem 10. $\mathbf{H}x, \mathbf{H}y \vdash \mathbf{H}(\mathbf{V}xy)$.

Proof. $\mathbf{H}x, \mathbf{H}y, \mathbf{H}z \vdash \mathbf{H}((x \supset z) \supset . (y \supset z) \supset z)$, so by $\vdash \mathbf{LH}$, Theorem 4, and Definition \mathbf{V} , we get the result.

4. Axiom 6 of [2] is $\vdash \exists \mathbf{IH}$.

Theorem 11. $\mathbf{H}y, x \vdash \mathbf{V}xy$.

Proof. $x, x \supset z \vdash z$, so by Theorem 1 and Axiom 6, $\mathbf{H}z, \mathbf{H}y, x, x \supset z \vdash (y \supset z) \supset z$ and $\mathbf{H}y, x, \mathbf{H}z \vdash x \supset z \supset (y \supset z) \supset z$. Therefore, $\mathbf{H}y, x \vdash \mathbf{H}z \supset z \supset x \supset z \supset (y \supset z) \supset z$, which is the required result.

Theorem 12. $\mathbf{H}x, y \vdash \mathbf{V}xy$.

Proof. $y, y \supset z \vdash z$, so $y, \mathbf{H}z \vdash y \supset z \supset z$, and by Theorem 1, $\mathbf{H}x, y, \mathbf{H}z \vdash x \supset z \supset (y \supset z) \supset z$. Thus, as in Theorem 11, $\mathbf{H}x, y \vdash \mathbf{V}xy$.

Theorem 13. $\mathbf{V}xy, x \supset z, y \supset z, \mathbf{H}z \vdash z$.

Proof. By Definition V we get the result.

3. Negation To this positive system we now add negation, which can also be defined. As a result, as we shall see, we obtain the full intuitionistic calculus. Consider the ob $\Xi\mathbf{H}\mathbf{I}$. As $\vdash \mathbf{L}\mathbf{H}$ and $\mathbf{H}x \vdash \mathbf{H}(\mathbf{I}x)$, and so $\vdash \mathbf{F}\mathbf{H}\mathbf{H}\mathbf{I}$, we get $\vdash \mathbf{H}(\Xi\mathbf{H}\mathbf{I})$. Now for all x , $\Xi\mathbf{H}\mathbf{I}, \mathbf{H}x \vdash x$; so by the deduction theorem for P, $\mathbf{H}x \vdash \Xi\mathbf{H}\mathbf{I} \supset x$.

The proposition $\Xi\mathbf{H}\mathbf{I}$ is therefore one which implies any proposition, and thus $\Xi\mathbf{H}\mathbf{I}$ can be used as a "standard false proposition" in the sense of Wajsberg [8], and so negation can be defined in terms of it. Thus we adopt

Definition \neg . $\neg \equiv \mathbf{C}\mathbf{P}(\Xi\mathbf{H}\mathbf{I})$.

From this, the two intuitionistic axioms for negation can be derived, as well as $\vdash \mathbf{F}\mathbf{H}\mathbf{H} \neg$.

Theorem 14. $\vdash \mathbf{F}\mathbf{H}\mathbf{H} \neg$.

Proof. By Theorem 5, $\vdash \mathbf{H}(\Xi\mathbf{H}\mathbf{I})$, so Definition \neg gives, $\mathbf{H}x \vdash \mathbf{H}(\neg x)$. By the deduction theorem the result follows.

Theorem 15. $\mathbf{H}x, \mathbf{H}y \vdash x \supset \neg y \supset y \supset \neg x$.

Proof. $x \supset \neg y, x \vdash \neg y$ so by Definition \neg , $x \supset \neg y, y, x \vdash \Xi\mathbf{H}\mathbf{I}$. Therefore, $\mathbf{H}x, x \supset \neg y, y \vdash \neg x$, and by two applications of the deduction theorem for P, we get the result.

Note that in the above theorem, we obtained the rule:

Theorem 16. $x, \neg x \vdash \Xi\mathbf{H}\mathbf{I}$,

which will be useful in the later work.

Theorem 17. $\mathbf{H}x, \mathbf{H}y \vdash \neg x \supset x \supset y$.

Proof. By Theorem 16 we have $\neg x, x, \mathbf{H}y \vdash y$, so the result follows by the deduction theorem for P.

Now that we have obtained all the axioms of intuitionistic propositional calculus HJ (see [5]) we can state the following general theorem.

Theorem 18. *If T is an assertable formula of HJ whose variables are x_1, \dots, x_n , then $\mathbf{H}x_1, \dots, \mathbf{H}x_n \vdash \mathbf{T}$ is a theorem of this system.*

4. Intuitionistic Predicate Calculus In order to establish predicate calculus results, we need obs which do the work of quantifiers. It is obvious that $\Pi[x]$ (which is not an ob, but a word in the language which has no meaning),

can serve as a universal quantifier in the sense that, $\Pi([x]Xx)$ or ΠX can be interpreted in the same way as $(\forall x)Xx$, i.e., for all x , Xx holds. However, the range of such a quantifier may be too wide, so it is preferable to use $\Xi a[x]$ as a quantifier, where the ob a is an indeterminate representing the fundamental domain. It may be specialized to **E**, **L**, **A** or some other domain. Some of the usual properties of universal quantification can now be proved for $\Xi a[x]$.

Theorem 19. $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, au \vdash \Xi ax \supset xu$.

Proof. $\Xi ax, au \vdash xu$, so as $\mathbf{L}a, \mathbf{F}a\mathbf{H}x \vdash \mathbf{H}(\Xi ax)$, the result follows by the deduction theorem for **P**.

Theorem 20. $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, \mathbf{F}a\mathbf{H}y \vdash \Xi a(\Phi \mathbf{P}xy) \supset \Xi ax \supset \Xi ay$.

Proof. $au, \Xi a(\Phi \mathbf{P}xy), \vdash \mathbf{P}(xu)(yu)$ so

$$\Xi a(\Phi \mathbf{P}xy), \Xi ax, au \vdash yu. \quad (2)$$

Now by Theorem 4,

$$\mathbf{L}a, \mathbf{F}a\mathbf{H}x \vdash \mathbf{H}(\Xi ax). \quad (3)$$

Also $\mathbf{F}a\mathbf{H}x, au \vdash \mathbf{H}(xu)$, and $\mathbf{F}a\mathbf{H}y, au \vdash \mathbf{H}(yu)$; so $\mathbf{F}a\mathbf{H}x, \mathbf{F}a\mathbf{H}y, au \vdash \mathbf{H}(\mathbf{P}(xu)(yu))$, and by Theorem 4,

$$\mathbf{L}a, \mathbf{F}a\mathbf{H}x, \mathbf{F}a\mathbf{H}y \vdash \mathbf{H}(\Xi a(\Phi \mathbf{P}xy)). \quad (4)$$

Then (2), (3) and (4), one application of the deduction theorem for Ξ and two of that for **P** give the result.

Theorem 21. $\mathbf{L}a, \mathbf{H}X \vdash X \supset \Xi a([u]X)$, where u is not involved in X .

Proof. $au, X \vdash \mathbf{K}Xu$, so $\mathbf{L}a, X \vdash \Xi a([u]X)$ if u is not involved in X . The result then follows.

It can be seen that given the interpretation of “ $(\forall x)$ ” for “ $\Xi a[x]$,” in “classical” notation the following theorems have been obtained with various sections restricted to being propositions.

$$\Pi_0 \vdash (\forall x) X(x) \supset X(t).$$

$$\Pi \mathbf{P} \vdash (\forall x) . X(x) \supset Y(x) : \supset : (\forall x) X(x) \supset (\forall x) Y(x).$$

$$\Pi_2 \vdash (\forall x) . Z \supset (\forall x) Z \text{ where } x \text{ is not in } Z.$$

These are simply the axioms for universal quantification as given in [6] or [5] for an intuitionistic system.

A new ob by means of which existential quantification can be represented is now defined.

$$\text{Definition } \Sigma. \Sigma \equiv [x, y] . \mathbf{H}z \supset_z . (xu \supset_u . yu \supset z) \supset z.$$

Taking the range of quantification as a , as before, $\Sigma a[x]$ can be taken as the existential quantifier, in the sense that, $\Sigma a([x]Xx)$ can be interpreted in the same way as $(\exists x) Xx$, i.e., as “There exists an x such that Xx .” Now some properties of this form of existential quantification are proved.

Theorem 22. $\mathbf{L}a, \mathbf{F}a\mathbf{H}x \vdash \mathbf{H}(\Sigma ax)$.

Proof. $\mathbf{F}a\mathbf{H}x, au, \mathbf{H}z \vdash \mathbf{H}(xu \supset z)$, so by the deduction theorem, $\mathbf{F}a\mathbf{H}x, \mathbf{L}a, \mathbf{H}z \vdash \mathbf{F}a\mathbf{H}([u].xu \supset z)$; so by Theorem 4 with a for x and $[u].xu \supset z$ for y ,

$$\mathbf{L}a, \mathbf{F}a\mathbf{H}x, \mathbf{H}z \vdash \mathbf{H}(au \supset_u . xu \supset z), \quad (1)$$

and then $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, \mathbf{H}z \vdash \mathbf{H}((au \supset_u . xu \supset z) \supset z)$. Then as $\vdash \mathbf{LH}$, we have as above $\mathbf{L}a, \mathbf{F}a\mathbf{H}x \vdash \mathbf{H}(\mathbf{H}z \supset_z . (au \supset_u . xu \supset z) \supset z)$. We have then $\mathbf{L}a, \mathbf{F}a\mathbf{H}x \vdash \mathbf{H}(\Sigma ax)$. **Q.E.D.**

Theorem 23. $\mathbf{L}a, av, \mathbf{F}a\mathbf{H}x \vdash xv \supset \Sigma ax$.

Proof. By (1) in the proof of Theorem 22, $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, \mathbf{H}z \vdash \mathbf{H}(au \supset_u . xu \supset z)$. Therefore, $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, \mathbf{H}z, av, xv \vdash (au \supset_u . xu \supset z) \supset z$, by the deduction theorem for **P**. By the deduction theorem for \exists and $\vdash \mathbf{LH}$, $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, av, xv \vdash \Sigma ax$; and as $\mathbf{F}a\mathbf{H}x, av \vdash \mathbf{H}(xv)$, we have $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, av, \vdash xv \supset \Sigma ax$.

Theorem 24. $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, \mathbf{H}y \vdash (\exists a[u].xu \supset y) \supset . \Sigma ax \supset y$.

Proof. $\Sigma ax, \mathbf{H}y \vdash (au \supset_u . xu \supset y) \supset y$, so $(\exists a[u].xu \supset y), \Sigma ax, \mathbf{H}y \vdash y$. Hence by Theorem 22, step (1) in the proof of that theorem, and the deduction theorem for **P**, $\mathbf{L}a, \mathbf{F}a\mathbf{H}x, \mathbf{H}y \vdash (\exists a[u].xu \supset y) \supset . \Sigma ax \supset y$.

Theorems 23 and 24 correspond to the following predicate axioms from [5]:

$$\begin{aligned} \Sigma_0 \vdash A(t) \supset . (\exists x) A(x), \\ \Sigma_1 \vdash (\forall x) . A(x) \supset C \supset : \supset : (\exists x) A(x) \supset . C. \end{aligned}$$

As we have no restrictions on a in the above theorems, they all hold in the case where a is an "empty universe." Suppose we consider for the moment that a is **B-(WQ)**. Then in Theorems 19 and 23 the premises au and av will not hold so these theorems become vacuous.

5. A General Theorem on Predicate Calculus It is shown in [5] that Π_0 , $\Pi\mathbf{P}$, Π_2 , Σ_0 , and Σ_1 , together with Rule **P** and a rule for generalization, plus the propositional calculus axioms, are sufficient for all of the intuitionistic predicate calculus. Here we have under certain restrictions propositional calculus, Π_0 , $\Pi\mathbf{P}$, Π_2 , Σ_0 , Σ_1 , Rule **P** and also a rule of generalization:

Rule $\exists a$. If $au \vdash xu$ then $\mathbf{L}a \vdash \exists ax$.

Thus any theorem of intuitionistic predicate calculus should be obtainable here, with certain restrictions on the variables. This we shall prove below. Also if we add Peirce's law, which does not alter the proof, we have the corresponding theorem for classical predicate calculus (see Theorem 26).

Before stating this theorem we shall specify what we mean by predicate calculus. It is a formal system formulated as follows:

The primitive notions are individual variables, propositional variables, functional variables of any degree and predicate variables of any degree.⁵

We define a class of terms inductively as follows: Individual variables

5. The degree is the number of arguments a predicate ranges over or the number of arguments a function has.

are terms; if f is an n -adic functional variable and t_1, \dots, t_n are terms then $f(t_1, \dots, t_n)$ is a term.

Formulas are defined inductively as follows: Propositional variables are formulas; if g is an n -adic predicate variable and t_1, \dots, t_n are terms then $g(t_1, \dots, t_n)$ is a formula. If A and B are formulas then $\neg A, A \supset B, A \wedge B, A \vee B, (\forall x)A$ and $(\exists x)A$ are formulas.

Certain of these formulas we will designate as axiom schemes. We will take the ones for the propositional calculus (in [5]) as well as the five mentioned above. Also we will have modus ponens and the rule of generalization.

We now formulate a translation from the predicate logic to the combinatory system.

If t is a term, its translation is an ob t' such that; if t is an individual variable, t' is t ; if t is $f(t_1, \dots, t_n)$, t' is $ft'_1 \dots t'_n$.

If X is a formula, its translation X' is defined thus: if X is a propositional variable, X' is X ; if X is $g(t_1, \dots, t_n)$, X' is $gt'_1 \dots t'_n$; if X is $\neg Y$, X' is $\neg Y'$; if X is $Y \vee Z$, X' is $\mathbf{V}Y'Z'$; if X is $Y \wedge Z$, X' is $\mathbf{\Lambda}Y'Z'$; if X is $Y \supset Z$, X' is $\mathbf{P}Y'Z'$; if X is $(\forall x)A(x)$, X' is $\mathbf{\Xi}a[x](A(x))'$; if X is $(\exists x)A(x)$, X' is $\mathbf{\Sigma}a[x](A(x))'$.

With each ob of the combinatory system obtained in this way, we associate a sequence of obs called the set of grammatical conditions for it. Given an ob T this set contains the following:

(i) $\mathbf{L}a$, (ii) ax for each individual variable x in T , (iii) $\mathbf{H}y$ for each propositional variable y in T , (iv) $F_n a \dots a \mathbf{H}z^6$ (with n a 's) for each predicate variable z of degree n in T , and (v) $F_m a \dots at$ (with $m + 1$ a 's) for each functional variable t of degree m in T . Now we prove three lemmas which will be used in the proof of the theorem.

Lemma 1. *If t is a term and N is the set of grammatical conditions for t' , then, $N \vdash at'$.*

Proof. The proof is by induction on the structure of t . If t is a primitive term it is an individual variable and so N will be at' and therefore $N \vdash at'$.

Now we assume that the lemma holds for terms t_1, \dots, t_k involved in forming $t' = wt'_1 \dots t'_k$ where w is a functional variable of degree k . Now by the hypothesis of the induction $N_i \vdash at'_i$ ($1 \leq i \leq k$) where N_i is the set of grammatical conditions for t'_i . Therefore $\mathbf{F}_k a \dots aw, N_1, \dots, N_k \vdash a(w'_1 \dots t'_k)$ and as $\mathbf{F}_k a \dots aw, N_1, \dots, N_k$ must make up N , $N \vdash at'$.

Lemma 2. *If U is a formula and N is the set of grammatical conditions for U , then $N \vdash \mathbf{H}U'$.*

Proof. The proof is by induction on the structure of U .

If U is a prime formula U' takes the form $\mathbf{p}t'_1 \dots t'_m$. We then have in the N appropriate to U' , $\mathbf{F}_m a \dots a \mathbf{H}p$, and N_1, \dots, N_m , the sets of grammatical conditions for the terms t_1, \dots, t_m . But by Lemma 1, $N_i \vdash at'_i$.

6. $\mathbf{F}_1 x_1 x_2 x_3 = \mathbf{F} x_1 x_2 x_3$ and $\mathbf{F}_{n+1} x_1 x_2 \dots x_{n+2} x_{n+3} = \mathbf{F}_n x_1 \dots x_n (\mathbf{F} x_{n+1} x_{n+2}) x_{n+3}$.

Therefore $N \vdash \mathbf{H}(pt'_1, \dots, t'_m)$. That is, $N \vdash \mathbf{H}U'$. Alternatively U can be a propositional variable. Then also $N \vdash \mathbf{H}U'$. This can be regarded as the case $m = 0$ of the above case.

Now we consider in the inductive step formulas made up by means of subformulas, for which the theorem is assumed, and connectives or quantifiers. If $U = U_1 \supset U_2$ where we have $N_1 \vdash \mathbf{H}U'_1$ and $N_2 \vdash \mathbf{H}U'_2$, and where N_1 and N_2 are sets of grammatical conditions for U_1 and U_2 , then $N_1, N_2 \vdash \mathbf{H}(\mathbf{P}U'_1U'_2)$. Thus $N \vdash \mathbf{H}U'$.

A similar argument holds if U is $U_1 \vee U_2$ or $U_1 \wedge U_2$ and the case where U is $\neg U_1$ is trivial.

If $U = (\forall x)U_1(x)$ we have $N_1 \vdash \mathbf{H}(U'_1x)$ where N_1 includes N and ax , and U'_1 is $[x](U_1(x))'$. Thus $N, ax \vdash \mathbf{H}(U'_1x)$ and so as N includes $\mathbf{L}a$, $N \vdash \mathbf{F}a\mathbf{H}U_1$, which by Axiom 9 gives $\vdash \mathbf{H}(\exists aU'_1)$. Similarly we get the case for existential quantification. Thus we have completed the induction and we have proved, $N \vdash \mathbf{H}U'$.

Lemma 3. *If $N \vdash T'$ where N is a set of grammatical conditions which includes M , the set of grammatical conditions for T' , then, provided a is nonempty, $M \vdash T'$.*

Proof. Consider the grammatical conditions in N which are not in M . These will be conditions on variables not in T' . If there is a condition of the form au , replace u by a (constant) member of a (here we assume that a is nonempty). If there is a condition of the form $\mathbf{H}y$ replace y by \mathbf{QKK} .⁷ If there is a condition of the form $\mathbf{F}_na \dots a\mathbf{H}z$ (with n a 's) replace z by $\mathbf{K}(\mathbf{K}(\dots (\mathbf{WQI}) \dots))$ where there are n \mathbf{K} 's. If there is a condition of the form $\mathbf{F}_na \dots aw$ (with $n + 1$ a 's) replace w by $\mathbf{K}(\mathbf{K}(\dots (\mathbf{KU}) \dots))$ where there are n \mathbf{K} 's and where U is a member of a . The extra conditions in N are thus true and can be removed.

Corollary. *If M is the set of grammatical conditions for T' and N is the set of grammatical conditions for S' then if $T' = S'$ and $M \vdash T'$, $N \vdash S'$.*

Proof. We obtain immediately $M \vdash S'$. We then remove conditions in M that are not in N by means of Lemma 3 and add conditions in N that are not in M .

Now we can state the theorem.

Theorem 25. *Let T be an assertible formula of the intuitionistic pure first order predicate calculus with individuals, predicate and functional variables. Let a be nonempty and let M be the set of grammatical conditions for T' , the translation of T into the combinatory system. Then in the present system, $M \vdash T'$.*

Proof. We shall show that substitution can be carried out in the com-

7. \mathbf{QXX} is interpreted as $X = X$. \mathbf{QKK} is an axiom in [2].

binatory system, on the "axiom schemes,"⁸ just as it can be in the predicate calculus, and retaining the correct grammatical conditions.

Consider the case where T is an instance of the first axiom scheme, viz. $\vdash X \supset Y \supset X$. The translation of T is $\mathbf{P}X'(\mathbf{P}Y'X')$, and by Rule Ξ and Theorem 1 we have, $\mathbf{H}Y', \mathbf{H}X' \vdash \mathbf{P}X'(\mathbf{P}Y'X')$. Now as M contains the grammatical conditions for X' and Y' the result follows by Lemma 2.

Similarly we can establish the result for the other propositional calculus axioms, using Theorem 2, the theorems listed after Theorem 13, and Theorems 15 and 17.

Now consider the axiom scheme Π_0 . Let T be an instance of this, viz. $\vdash (\forall v)X(v) \supset X(U)$. In this case T' is $\mathbf{P}(\Xi aX')(X'U')$, where X' is $[v](X(v))'$. Substituting X' for x in Theorem 19 we have $\mathbf{L}a, \mathbf{F}a\mathbf{H}X', au \vdash \Xi aX' \supset X'u$, where we take u to be a variable not involved in T' . As M will contain $\mathbf{L}a$ and the grammatical conditions for X' , we have by Lemma 2,

$$M, au \vdash \Xi aX' \supset X'u. \tag{1}$$

If $X'u$ does not involve u , T' does not involve u and by Lemma 3 we can drop the premise au and by substituting U' for u in the resulting statement we have the required result. If $X'u$ does involve u , T' will involve all the variables in U' . Hence by Lemma 1, $M \vdash au'$. Using this, and (1) with U' substituted for u , we obtain the result.

Similar results follow for the remaining axiom schemes, using Theorems 20, 21, 23 and 24.

Now we proceed to prove by induction that the translation of any theorem of the predicate calculus, with its grammatical conditions is assertible in the combinatory system.

We already have the result if T is (a substitution instance of) an axiom scheme; if T is not that, we will replace every use of an axiom in its proof, by the theorem which is the translation of the axiom and any use of modus ponens and generalization by Rules \mathbf{P} and Ξa .

Assume, now, in a deductive step that all steps in the proof of T up to the k th (these are all theorems of the predicate calculus) obey the theorem. Let the translation of these steps be T_1, T_2, \dots, T_k , and their sets of grammatical conditions M_1, M_2, \dots, M_k . The next step, T_{k+1} , then comes from two previous steps T_i and $T_j (=T_i \supset T_{k+1})$, or from T_i and generalization so that $T_{k+1} = \Xi a[u]T_i$.

In the first case we obtain from $M_j \vdash T_i \supset T_{k+1}$, and $M_i \vdash T_i$, that $M_j \vdash T_{k+1}$ as $M_i \subseteq M_j$.

Also $M_{k+1} \subseteq M_j$, so by Lemma 3 from M_j we can drop all the grammatical conditions for variables not in T_{k+1} . Thus $M_{k+1} \vdash T_{k+1}$.

In the case of generalization, T_i must contain all the free variables in T_{k+1} , plus an extra one say u , over which the generalization is made. So M_i consists of M_{k+1} and au . Now as the theorem holds for $T_i, M_{k+1}, au \vdash T_i$

8. It is also possible to prove this theorem in systems having a primitive substitution rule and axioms rather than axiom schemes.

so by Rule $\exists a$, since $\exists a([u] T_i) = T_{k+1}$, and $\mathbf{L}a$ is part of M_{k+1} , $M_{k+1} \vdash T_{k+1}$. Thus the theorem is proved for all cases.

The theorem can be extended to higher order predicate calculuses, for example if we include a predicate variable z ranging over individuals, propositions, and other predicates we would have to include in the sequence M :

$$\mathbf{F}_k a \dots a \mathbf{H} \dots \mathbf{H}(\mathbf{F}_i a \dots a \mathbf{H} \dots \mathbf{H}(\mathbf{F}_j a \dots) \dots \mathbf{H})(\mathbf{F}_l \dots \mathbf{H}) \dots \mathbf{H}z$$

where the total number of variables is k and there is an a for each individual variable, a \mathbf{H} for each propositional variable and an appropriate $\mathbf{F}_i a \dots a \mathbf{H} \dots \mathbf{H}(\dots) \dots \mathbf{H}$ for each predicate variable that z ranges over.

6. Classical Propositional and Predicate Calculus The system we have developed in the previous sections can be extended to a classical system by simply introducing Peirce's Law.

Axiom 13. $\vdash \mathbf{H}y \supset y :. \mathbf{H}x \supset x : (x \supset y) \supset x . \supset x$.

The addition of the axiom obviously will not affect the proof of Theorem 25, so it can easily be extended to the classical system.

Theorem 26. *If Axiom 13 is adjoined Theorem 25 holds for the classical system.*

REFERENCES

- [1] Bunder, M. W., "A paradox of illative combinatory logic," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 467-470.
- [2] Bunder, M. W., "A deduction theorem for restricted generality," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 341-346.
- [3] Bunder, M. W., "A generalised Kleene-Rosser paradox for a system containing the combinator \mathbf{K} ," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 53-54.
- [4] Curry, H. B., and R. Feys, *Combinatory Logic*, Amsterdam (1958).
- [5] Curry, H. B., *Foundations of Mathematical Logic*, New York (1963).
- [6] Curry, H. B., *A Theory of Formal Deductibility*, Notre Dame Mathematical Lectures 6, Notre Dame, Indiana (1950).
- [7] Ono, K., "On the universal character of the primitive logic," *Nagoya Mathematical Journal*, vol. 27 (1966), pp. 331-353.
- [8] Wajsberg, M., "Untersuchungen über die Grundlagen der Mengenlehre, I," *Mathematische Annalen*, vol. 65 (1908), pp. 261-281.

Wollongong University College
Wollongong, New South Wales, Australia