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NECESSITY AND SOME NON-MODAL PROPOSITIONAL CALCULI

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Sometimes in a non-modal propositional calculus (**PC**) containing a connective (C) for implication a satisfactory definition of 'it is necessary that p'(Lp)' is available. Thus, in the well-known system E of entailment, Lp may be defined as $CCppp$, where 'C' denotes the non-truth-functional implication taken as a primitive connective. A non-modal **PC** may fail to permit an intuitively satisfactory definition of necessity either because it is too weak or because it is too strong. A non-trivial example of the former case is provided in [5], where the authors use the following four-valued model $\mathcal{N}$ (with starred elements as designated)

C	0	1	2	3
0	3	3	3	3
1	0	2	0	3
*2	0	3	2	3
*3	0	0	0	3

of the pure implicational calculus (**PIC**) P_I of ticket entailment defined in [1], to show that there is no pure implicational (**PI**) wff $\alpha(p)$ in the single variable p satisfying the following conditions:

(1) $C\alpha(p)p$ is a theorem of P_I,
(2) $Cp\alpha(p)$ is not a theorem of P_I,
(3) if β is a theorem of P_I, then $\alpha(p/\beta)$ is a theorem of P_I,

and

(4) for any δ, θ, $CC\delta\theta C\alpha(p/\delta)\alpha(p/\theta)$ is a theorem of P_I.

Corresponding to the modal axiom $CLCqrCLqLr$ consider now the condition

(4*) $C\alpha(p/C\delta\theta)C\alpha(p/\delta)\alpha(p/\theta)$ is a theorem of P_I.

Since transitivity of implication and modus ponens are available in P_I, if $\alpha(p)$ satisfies (4), in view of (1), it will also satisfy (4*). The authors of [5] are entitled to the following:

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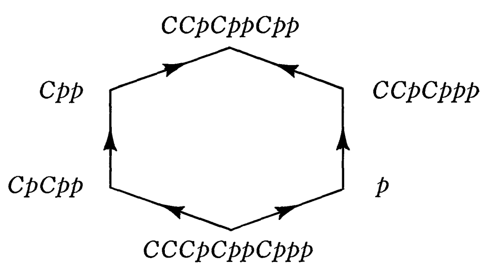
Theorem. *There is no* **PI** *wff* $\alpha(p)$ *in a single variable* p *satisfying conditions* (1), (2) *and* (4*).

Proof. Assume that there is a **PI** wff $\alpha(p)$ satisfying (1), (2) and (4*). Since Cpp is an axiom of P_I, $\alpha(p)$, in view of (2), contains at least one occurrence of C. Consider now the unary operation in the model $\mathcal{N}$ defined by $\alpha(p)$. Since (1) holds and $C10 = C20 = C30 = 0$, $\alpha(p/0) = 0$. Since $\alpha(p)$ is a **PI** wff and $C22 = 2$ and $C33 = 3$, $\alpha(p/2) = 2$ and $\alpha(p/3) = 3$. Since $\alpha(p)$ is different from p and $Cab \neq 1$ for any truth-values a, b in the model $\mathcal{N}$, it follows that $\alpha(p/1) \neq 1$. Since $C31 = 0$, in view of (1), $\alpha(p/1) \in \{0, 2\}$. Consider now the value of $C\alpha(p/Cqr)\, C\alpha(p/q)\, \alpha(p/r)$, for $q = 2$, $r = 1$. It reduces to $C\alpha(p/C21)\, C\alpha(p/2)\, \alpha(p/1) = C\alpha(p/3)\, C2a = C3C2a$, where $a = 0$ or $a = 2$. But $C3C20 = C30 = 0$ and $C3C22 = C32 = 0$. Thus, $\alpha(p)$ fails to satisfy (4*). This completes the proof.

We make some preliminary remarks concerning Church's system W_I of weak implication (see [3]) and another system containing it, before taking up the problem of the definability of necessity in these systems. Consider the following four-valued (with designated elements starred) model $\mathcal{M}$ of W_I given in [7].

C	0	1	2	3
0	1	1	1	1
*1	0	1	0	0
2	0	1	3	0
*3	0	1	2	3

It is proved by Meyer in [4] that W_I has six mutually non-equivalent wffs in one variable that may be conveniently presented in the following hexagonal graph.



Our choice of wffs in the graph is somewhat different from that of Meyer [4] and is more suitable for our present purpose. Arrows indicate the directions in which provable implications hold in W_I. Of the six wffs in the graph, three are classical tautologies and of these three Cpp and $CCpCppCpp$ are theorems of W_I. It follows that any **PI** classical tautology in the variable p that is not a theorem of W_I is equivalent to $CpCpp$ in W_I and hence its addition as an axiom to W_I will give a system in which all one-variable **PI** tautologies are provable.

The following lemma which shows that Sobociński's four-valued model $\mathcal{M}$, given above, characterizes the class of all one-variable theorems of W_I may have some interest for computational purposes.

Lemma. *A* **PI** *wff in a single variable is a theorem of Church's system of weak implication if and only if it is valid in the model* $\mathcal{M}$.

Proof. Since $\mathcal{M}$ is a model of W_I, the 'only if' part is trivial. Since $\mathcal{M}$ is a model of W_I, there are no more than six mutually non-equivalent wffs in the variable p available in $\mathcal{M}$. It is now sufficient to show that the six wffs of Meyer's graph are mutually non-equivalent in $\mathcal{M}$. We note that for any truth-values a, b of the model $\mathcal{M}$, Cab and Cba are both designated if and only if $a = b$. Therefore, if $C\alpha\beta$ and $C\beta\alpha$ are both valid in $\mathcal{M}$, then for any assignment f of truth-values in $\mathcal{M}$ to the variables of α, β, $f(\alpha) = f(\beta)$. But each of the six wffs in Meyer's graph defines a distinct unary operation in $\mathcal{M}$ as the following table shows.

p	Cpp	$CpCpp$	$CCpCppp$	$CCpCppCpp$	$CCCpCppCppp$
0	1	1	0	1	0
*1	1	1	1	1	1
2	3	0	1	1	0
*3	3	3	3	3	3

Therefore the six wffs are mutually non-equivalent in $\mathcal{M}$. This completes the proof.

Remark. Consider the model $\mathcal{M}^*$ obtained from $\mathcal{M}$ by deleting the row and the column for 2. $\mathcal{M}^*$ is isomorphic to the implicational part of the three-valued model axiomatized by Sobociński in [6]. Since all classical **PI** tautologies in a single variable are available in $\mathcal{M}^*$ it follows that for any such wff $\alpha(p)$, $\alpha(p)$ is a theorem of W_I if and only if $\alpha(p/2) = 1$ or $\alpha(p/2) = 3$ holds in $\mathcal{M}$. Since 2 generates the model $\mathcal{M}$, it follows that every **PI** theorem of Sobociński's three-valued logic studied in [6] which is invalid in $\mathcal{M}$ has a substitution instance in one variable that is a non-theorem of W_I.

Consider now the system W_I. Let $\alpha(p)$ be $CCCpCppCppp$. Then up to equivalence $\alpha(p)$ is the only wff that provably implies p in W_I without being implied by it. By using the table given in the proof of the lemma it is easily verified that $\alpha(p)$ satisfies conditions (1)-(3) with 'P_I' replaced by 'W_I'. However, it fails to satisfy (4) because $CCqrC\alpha(p/q)\,\alpha(p/r)$ takes the value 0 in the model $\mathcal{M}$ of W_I when q and r take respectively the values 3 and 2. It seems that (4*) also fails for the given $\alpha(p)$ in W_I.

Let $\alpha(p)$ be as in the preceding paragraph. This $\alpha(p)$ continues to satisfy conditions (1)-(3) in $\mathcal{M}$ as in W_I. But $C\alpha(p/Cqr)\,C\alpha(p/q)\,\alpha(p/r)$ is valid in the model $\mathcal{M}$ as can be ascertained by elimination of cases. Thus, $\alpha(p)$ satisfies (4*) in $\mathcal{M}$. Thus, one can claim that a reasonable definition of necessity is available in the **PIC** defined by $\mathcal{M}$.

On the other hand, Sobociński's three-valued logic (see [6]) is too strong a system to permit a definition of necessity. It is easily verified that any *CN* wff $\beta(p)$ which satisfies conditions (1) and (3) in the three-valued logic of Sobociński must define the identity operation in the three-valued model of [6] and hence must fail to satisfy condition (2).

REFERENCES

[1] Anderson, Alan Ross, "Entailment shorn of modality" (abstract), *The Journal of Symbolic Logic*, vol. 25 (1960), p. 388.

[2] Anderson, Alan Ross, "A problem concerning entailment" (abstract), *The Journal of Symbolic Logic*, vol. 27 (1962), p. 382.

[3] Church, Alonzo, "The weak theory of implication," in *Kontrolliertes Denken* (1951), pp. 22-37.

[4] Meyer, Robert K., "R_I—The bounds of finitude," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 16 (1970), pp. 385-387.

[5] Parks, R. Zane, and John R. Chidgey, "Necessity and ticket entailment," *Notre Dame Journal of Formal Logic*, vol. XIII (1972), pp. 224-226.

[6] Sobociński, Bolesław, "Axiomatization of a partial system of three-valued calculus of propositions," *The Journal of Computing Systems*, vol. 1 (1952), pp. 23-55.

[7] Sobociński, Bolesław, "A note concerning the many-valued propositional calculi," *Notre Dame Journal of Formal Logic*, vol. II (1961), pp. 127-128.

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