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THE FORM OF *REDUCTIO AD ABSURDUM*

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A recent discussion of this topic by Donald Scherer in [6], pp. 247-252, begins thus:

> *Reductio ad Absurdum* is clearly a valid argument form. Yet logicians tend in their writings either to ignore it or to treat it in a confusing and confused way. The aims of this paper are to expose this confusion as it appears in one of the fullest accounts given (by Copi in his *Symbolic Logic*), and to develop an adequate formulation.

After giving the form of Copi's *reductio ad absurdum* proofs,[1] Scherer argues (1) "that the form presented by Copi fails to manifest the basis upon which *reductio ad absurdum* is informally conceived to rest," (2) "that it is given a form which is . . . less than intuitive," and (3) that it is given a form which is "both *epistemologically and formally*[2] impossible." It seems to me unprofitable to argue about (1) and (2), since one man's informal conception or intuition is all too often another's stumbling-block. Besides, even if Scherer's intuition is better than Copi's, it does not follow that Copi is confused: to show confusion on Copi's part, Scherer must prove (3), which I now discuss.

Consider first what Scherer calls the *epistemological* impossibility. According to him, Copi's typical *reductio* sequence, including the steps[3]

1.	$r \cdot \sim r$	
2.	r	1, Simp.
3.	$\sim r \cdot r$	1, Com.
4.	$\sim r$	3, Simp.
5.	$r \vee q$	2, Add.
6.	q	5, 4, D.S.

is epistemologically impossible because, on the standard tabular interpretations of negation, conjunction and alternation, the conclusion q (line 6) is not acceptably derived from the premise $r \cdot \sim r$ (line 1): "the derivation is unacceptable because it involves the supposition that both conjuncts of the contradiction $r \cdot \sim r$ are true." How then does the derivation involve this supposition?

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> Informally, to say that step 6 is the valid consequence of step 1 is to say that if both r and $\sim r$ are *supposed* to be true, then 6 must be supposed to be true. Thus, to say that q is the valid consequence of $r \cdot \sim r$ is to say that if we suppose $r \cdot \sim r$ to be true, then we must suppose q to be true. Thus, Copi's derivation does involve the supposition of the truth of both r and $\sim r$.

These seem to me very inadequate grounds for condemning this sort of *reductio* sequence. For to be convincing here, Scherer must show that Copi's *derivation* actually involves Copi, or somebody, in simultaneously supposing the truth of both r and $\sim r$. It does not. Using Scherer's criterion of validity, to say that line 6 is the valid consequence of line 1 is to say that *if* we suppose line 1 to be true, then we must suppose line 6 to be true. It would then follow that, on this criterion, the argument form $r \cdot \sim r / \therefore q$ is invalid if we can (consistently) suppose the conclusion to be false and the premise true. But this condition of invalidity cannot be fulfilled, precisely because one cannot suppose that the premise is true: as Scherer says, this supposition is always "necessarily irrational."[4] To be sure, you cannot build knowledge (i.e., prove the *truth* of a conclusion) *solely* on the basis of an irrational supposition, but then neither Copi nor anyone else I know of suggests that you can. On Scherer's own criterion of validity, it is not Copi's *derivation* which involves the supposition that both r and $\sim r$ are true together, but the *assertion of the validity of the argument form* $r \cdot \sim r / \therefore q$, and in this assertion the supposition of the truth of $r \cdot \sim r$ is involved merely in stating a sufficient condition of the truth of q : q is true if $r \cdot \sim r$ is true, which is not to suppose, categorically, that $r \cdot \sim r$ is true.

Nor, I think, does Scherer's *formal* treatment of Copi's analysis of *reductio* fare much better. You can, for one thing, present systems like H.A. so that their primitive operators are given tabular definitions from the start, but this is not what Copi does. For instance, if we are to say that Copi defines '$\sim$' and 'v' for H.A., we must say that the definitions are contained in the recursive rule for wffs, not in a table: that is, they are given syntactically.[5] So too are Copi's definitions of validity for logistic systems.[6] In uninterpreted systems like these, one would hardly want to say that the formal concept of validity was derived in any way from tabular interpretations of primitives. So, for example, in Copi's presentation of H.A., the H.A. sequent $P \cdot \sim P \vdash Q$ is valid because every line of its demonstration is *either* $P \cdot \sim P$ *or* a postulate *or* a ponential of preceding lines. This test can of course be cited and applied without any reference to the intended interpretation of H.A.

But even if H.A. is (like Copi's method of natural deduction[7]) presented from the start as an interpreted system with tabular definitions of primitives, it is still difficult to see how Scherer's notion of cumulativity benefits his argument. I find this notion somewhat opaque, so perhaps it is best simply to quote Scherer's explanation of it:

> To say that an argument, or its conclusion, is valid cannot mean *merely* that if T is assigned to the premises T must be assigned to the conclusion, for the necessity here expressed in the term 'must' is *conditional* upon the preservation of the truth table definitions of the logical primitives throughout the proof.

From this we see that to say

(A) This argument is valid

is to say not merely

(B) If T is assigned to the premises of this argument, then T must be assigned to the conclusion

but rather

(C) If T is assigned to the premises of this argument, then T must, if the truth table definitions of the logical primitives are preserved throughout the proof, be assigned to the conclusion.

This is all very reasonable, and from it a very reasonable inference is drawn:

> Thus, on formal grounds we reach the conclusion that either F must be assigned to [the premise] $r \cdot \sim r$ or that no inference dependent on the use of both r and $\sim r$ is valid.

But after this, it seems to me, things go badly wrong. First we are told that the method proposed by Scherer

> assigns F to $r \cdot \sim r$, whereas, contrarily, the method Copi employs does not reject the self contradictory supposition but purports to draw valid inferences dependent upon the use of both r and $\sim r$.

Then we are told that the standard Copi derivation of q from $r \cdot \sim r$ is not valid.

It is difficult to know what to make of this. If you follow Copi's recipe for testing the argument form $r \cdot \sim r / \therefore q$,[8] you get the table

q	r	$\sim r$	$r \cdot \sim r$
T	T	F	F
T	F	T	F
F	T	F	F
F	F	T	F

On the table, $r \cdot \sim r$ interprets as F in every row, so what *does* Scherer mean when he implies, as he clearly does, that Copi's method does not assign F to $r \cdot \sim r$? Secondly, just what does "reject the self contradictory supposition $r \cdot \sim r$" mean: to assign F to $r \cdot \sim r$ or to negate $r \cdot \sim r$? On the standard interpretation of Copi's method, one *cannot but* assign F to $r \cdot \sim r$, and while Copi can easily establish $\sim(r \cdot \sim r)$ if he wants to (one supposes that this *might* be the sort of rejection of $r \cdot \sim r$ that Scherer has in mind), I cannot see why, *on purely formal grounds*, Copi should *need* to do this. Finally, even on the reformulation (C) of the definition of validity, Copi's derivation of q from $r \cdot \sim r$ surely satisfies at every step the requirements of this new definition. Indeed, it is only if (C) is replaced by

(D) *An argument is valid if and only if—*

(i) T can be assigned [consistently] to all its premises;
(ii) the truth table definitions of the logical primitives are preserved throughout the proof; *and*
(iii) [in all cases where (i) and (ii) are satisfied] T must [in accordance with the truth table definitions of the logical primitives] be assigned to the conclusion

that Copi's derivation is faulted, since it cannot then simultaneously satisfy both of conditions (i) and (ii).[9] So far as I am aware, though, neither Copi nor any other reasonable contemporary logician would want (D) as a definition of validity; and it certainly is not the classic test.[10]

It would be unjust, I suspect, to accuse Scherer of secretly believing that $r \cdot \sim r$ is not well-formed; yet just the same he leaves one with a nagging doubt that he thinks that expressions of the form $r \cdot \sim r$ are better left unuttered, at least when they stand in a premise position: it is not so much, perhaps, that they are logically ungrammatical as that they are logically profane. He gives us a neat proof, according to Copi's method, for $(r \cdot \sim r) \supset q$ (a formula which, for some reason, he declares to be well-formed), without once uttering the dreaded profanity. This is all very well for a base-born system of natural deduction like Copi's, but what, one wonders, would he do with a more aristocratic system such as Lemmon's? I suspect that he would be back to square one and all the old nastiness.[11]

Finally, Scherer's thesis that

> the deduction of a contradiction can (and should) be taken, not to prove that anything can thereafter be made to follow from the contradictory conclusion, but, instead, that a supposition previously made is irrational, at least in conjunction with the argument's premises, and can be validly denied

seems to me to miss its mark if it is aimed at Copi. Admittedly one can use Copi's method to construct a formal proof of validity with any conclusion whatever, once an explicit contradiction has been derived from premises. But no such proof will count as an indirect or *reductio* proof unless the contradictory of the final conclusion has already been assumed as a line preceding the contradiction which warrants the inference of that conclusion,[12] and it is this feature of Copi's presentation which brings it into line with standard treatments of *reductio.*

I wonder though whether Scherer's concern here might not be rather for the good name of *reductio*. For while most of us are happy with a straightforward piece of *reductio* like Euclid's proof that two intersecting circles cannot have the same centre,[13] we are a bit upset by tomfoolery like

Each competitor has a different number
There are two competitors with the same number
Therefore Aristotle was not bald

and protest that premises ought to be relevant to conclusions. Yet I suspect that, for most modern logicians, the difference between these two arguments is not that you can have a formally adequate *reductio* proof only of the first: both arguments are equally amenable to *reductio*. The essential difference appears in the *reductio* proofs themselves; for in the first case the assumption that there is at least one pair of intersecting circles with a common centre is necessary for the derivation of the crucial contradiction, while in the second case the assumption that Aristotle was bald plays no part in this derivation, but is there only to ensure that the proof will formally count as a *reductio* proof. We might say, perhaps, that proofs of this latter sort are degenerate *reductio* proofs, and piously hope that economists and politicians will not find out about them; but once we start purging logic of such weaklings, where and at what cost will the slaughter stop?[14]

NOTES

1. *Cf.* [1], pp. 62-65, 66, 88. Scherer cites the more developed form of proof (p. 88) which uses the strengthened rule of conditional proof (pp. 84-88). Briefly, the technique is that where q is the conclusion to be derived, one assumes $\sim q$, derives $r \cdot \sim r$ from $\sim q$ and the premises, and then derives q from $r \cdot \sim r$, as indicated in the next paragraph. One then infers $\sim q \supset q$ by C. P. and reduces this expression to q by Imp., D. N., and Taut.

2. The italics are mine.

3. [1], pp. 63, 88. The six steps cited here closely parallel a celebrated proof of the Pseudo-Scot, quoted by W. and M. Kneale in [3], pp. 281-282.

4. I take this to be another way of saying "logically false": I cannot see what else it might mean.

5. [1], pp. 250-251.

6. [1], pp. 193-194, 212-214, 252. Formally, an argument in H. A. is valid if and only if there is a demonstration of its validity. See also A. Church, s.v. "Valid inference" in [5].

7. [1], Chapter 2.

8. [1], Chapter 2, Section 2.3.

9. Symbolizing 'This argument is valid' as V, 'T is or can be assigned to all the premises of this argument' as P, 'The truth table definitions of the logical primitives are preserved throughout the proof of this argument' as Q, and 'T must be assigned to the conclusion' as R, (D) is seen to be of the form $V \equiv (P \cdot Q \cdot R)$, and (C) to be of the form $V \equiv [P \supset (Q \supset R)]$. Is Scherer perhaps attributing to (C) the form $V \equiv (P \cdot Q \cdot R)$?

10. *Cf.* [3], pp. 277, 286-288.

11. Compare Lemmon's proof in [4], p. 50, for the sequent $\vdash \sim(P \cdot \sim P)$, which requires the assumption of $P \cdot \sim P$.

12. Copi's strengthened rule of conditional proof ([1], pp. 84-89) would further require the discharging of this assumption by C. P.

13. Euclid, *Elements*, III.5, *cf.* [7].

14. There are also, of course, those who, like Anderson and Belnap (in [2], pp. 88-97) cannot wait for the slaughter to start. On their view the second argument above would commit a fallacy of relevance, so that with the sort of calculus that they envisage it would be formally impossible to construct either a *reductio* proof or even an ordinary formal proof for it. This, I should think, is much farther than Scherer would want to go, but I wonder whether it is not the direction in which he ought to be heading.

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