Notre Dame Journal of Formal Logic
Volume XIV, Number 3, July 1973
NDJFAM

WHAT RUSSELL LEARNED FROM PEANO

H. C. KENNEDY

On a quiz in a "Study of the History of Mathematics" course, the question was asked: "When Russell met Peano, what language did they speak to one another?" One student answered "symbolic logic." The student was clever, but wrong—and not just wrong: he had missed completely the historical importance of that meeting, for it was precisely symbolic logic that Russell learned as a result of meeting Peano, and he learned it from Peano. The evidence comes from Russell himself in, among other places, the description of the International Congress of Philosophy in Paris, 1900, in his *Autobiography* ([21], p. 217-219):

> The Congress was a turning point in my intellectual life, because I there met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussions at the Congress I observed that he was more precise than anyone else, and that he invariably got the better of any argument upon which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. I therefore got him to give me all his works, and as soon as the Congress was over I retired to Fernhurst to study quietly every word written by him and his disciples. It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years, and that by studying him I was acquiring a new powerful technique for the work that I had long wanted to do.

But what, specifically, did Russell learn from Peano? According to Russell, the enlightment he received came mainly from two purely technical advances. (He notes, by the way, that: "Both these advances had been made at an earlier date by Frege, but I doubt whether Peano knew this, and I did not know it until somewhat later." ([20], p. 66). "The first advance consisted in separating propositions of the form 'Socrates is mortal' from propositions of the form 'All Greeks are mortal'." [20, p. 66]. In the symbolism of Peano, adopted by Russell, this distinction is between $s \varepsilon M$ and $x \varepsilon G \supset_x x \varepsilon M$. "The second important advance I learnt from Peano was that a class consisting of one member is not identical with that one member." [20, p. 67]. That is, $s \varepsilon M$ is not the same as $s \subset M$.

Received June 10, 1971

Both the Greek epsilon for set membership and the subscript for universal quantification were introduced by Peano in 1889 [8]. The letter ε is the initial of ἐστί, *is*. The symbol $\supset$ was substituted in 1898 for the upside-down letter C of 1889. The symbol C, the initial of the Latin *consequentia*, was introduced in 1889, but was used only to define its inverse, symbolized by the same letter upside-down. (Inverted symbols for inverse relations and operations are typical of Peano's notation.) Peano also used $A \supset B$ to symbolize "A is a subset of B," noting that this gave the same symbolic statement of theorems in the calculus of classes and the calculus propositions. He considered this a great practical advantage. It was Russell who, feeling it would be advantageous to have distinct symbols, re-introduced $\subset$ for set inclusion. "It is convenient in mathematics to think of 'classes,' and for a long time I thought it necessary to distinguish between classes and propositional functions." ([20], p. 69). This may be seen as an example of something Russell did *not* learn from Peano, for Peano considered it of the greatest advantage to memory and ease of manipulation to have parallel symbols. (For Russell's criticism, see ([18], §13).) He likewise considered it advantageous to have symbols printed on a single line, something that Russell, in the article he submitted in the fall of 1900 for publication in Peano's journal, admitted that he had not succeeded in doing ([17], p. 116). (Frege remarked in this connection that the convenience of the typesetter was not the highest good! ([3], p. 364).)

Apart from the things specifically mentioned by Russell, it is difficult to know just what of Peano's work he did learn from him and what he discovered for himself. He mentioned that the older theories of number always got into difficulties over 0 and 1, and it was Peano's capacity of dealing with these difficulties that first impressed him. Russell was probably influenced, also, by Peano's stress on the distinction between real and apparent variables (see, for example [9] and [11].) Along with the symbolism mentioned above, Russell adopted Peano's symbol $\exists$ for existential quantification, which had been introduced in 1897 [11], as well as the decimal ordering of propositions, which was introduced by Peano in 1898 [12]. A copy of the "*Formules de logique mathématique*" (July 20, 1899) [13] would certainly have been furnished Russell by Peano, and should be consulted for a systematic exposition of what Russell could have learned from Peano.

Last, but certainly not least, Russell learned about Frege from Peano. Indeed, as Peter Nidditch has pointed out [7], Peano was one of the few before 1900 who took note of Frege's work, and in answer to a direct question, Russell replied: "I know, quite definitely, that it was through Peano that I first became aware of Frege's existence." Nidditch also called attention to the passage in Russell's *Portraits from Memory* ([19], p. 22), which says he learned of Frege from Peano's review of *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet* [10]. Ironically, Russell had been given a copy of Frege's *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* by his philosophy teacher, James Ward, who had not read it, and Russell did not read it until 1901.

Reporting this, Russell remarked: "I rather suspect that I was its first reader." ([19], p. 22).

Russell, of course, did not accept everything Peano was doing. To illustrate this, we consider an important instance in an area of most concern to Peano: definitions in mathematics. (This was the title of his paper at the International Congress of Philosophy, Paris, 1900, and he published two more articles with the same title.) Peano accepted definition by postulates and definition by abstraction, as well as nominal definitions. Russell rejected the first two ([18], §108), and it was possibly his rejection of definition by abstraction that led him to the definition of a cardinal number as a class of classes. He published this definition (for the first time, I believe) in Peano's journal ([17], p. 121). This issue is dated July 15, 1901, but the manuscript was submitted the previous year. (The one reference in it to the *Formulaire* of 1901 was probably added by Peano.) We have, in this delayed publication, the explanation of the curious passage in §32 of the *Formulaire* of 1901 in which Peano considers the class of equinumerous classes—and rejects it as a possible definition of cardinal number.

Section 32 of the *Formulaire*, both of 1899 and of 1901, presents the definition, by abstraction, of cardinal number, symbolized Num by Peano. In F1901 ([14], p. 70) we read: "This proposition defines the quality 'Num a = Num b,' which holds if a reciprocal correspondence can be established between a and b. We do not write an equality of the form

Num a = (expression composed of the preceding symbols)."

(In F1899 ([13], p. 61) he had said, "we are unable to write . . .") Peano then adds: "Given a class a, we may consider the class of classes [similar to a (Peano gives a symbolic definition)] . . .; but we cannot identify Num a with the class of classes considered, for these objects have different properties." Russell later commented on this passage ([18], §111): "He does not tell us what these properties are, and for my part I am unable to discover them." (Although the publication of F1901 preceded that of Russell's article by several months, Peano's remark was almost certainly prompted by Russell's definition of cardinal number as a class of classes, for he would have had the manuscript of Russell's article since October 1900.)

J. J. A. Mooij also finds curious, in §32, the statement following his definition of cardinal number: "This definition is expressed only by signs of logic. We can begin arithmetic here: we shall define directly the signs $> 0\ N_0 + \times \wedge$, without going through the primitive ideas of §20" (which contains the postulates for the natural numbers). "This statement," Mooij remarks ([6], p. 46) "seems to be leading up to logicism, which is all the more curious, since earlier in the same edition he had remarked: 'Can number be defined? The answer depends on the set of ideas that we suppose known. If we assume only those represented by the logical signs Cls, ε, Ɔ, ∩, =, of §1, then the answer is negative.'" The contradiction that Mooij finds between this statement of Peano and the fact that he defines Num independently of §20 (as Peano himself admits in §20) disappears if we assume that Peano was asking about the possibility of a nominal definition of number. Since, according to Peano, number cannot be given a nominal definition, the choice was arbitrary as to whether to begin arithmetic with a definition by postulates or a definition by abstraction. Peano chose the former.

That this is the correct interpretation is shown by Peano's constant insistence on the *form* of a definition. In F1901, §1, for example, in paragraph 2 "Definitions," we find: "A *possible definition* is an equality that contains in one member a sign which does not occur in the other, or which occurs there in a different position," and a bit further: "The primitive ideas are explained here by ordinary language, and are determined by the primitive propositions; the latter play the role of definitions with respect to the primitive ideas, but they do not have their form." This last sentence is copied from F1899, §2, where we also find: "Let us suppose that the signs which represent the ideas of a science have been ordered. The symbolic definition of a simple sign x has the form

$$x = \text{(expression composed of preceding signs).''}$$

To say, as Russell has ([18], §108), that there were "three kinds of definitions admitted by Peano" obscures this distinction. Peano "admitted" more than three (e.g., what Peano called "definition by induction"), but the nominal definition was always pre-eminent. The idea of the class of classes considered in F1901, §32, may or may not have occurred to him before it was suggested by Russell (although not mentioned in Peano's review, it was in Frege's *Grundgesetze* ([1], p. 56), but he clearly rejected it. It was only later, after Russell had shown how to reduce definitions by abstraction to nominal definitions, that Peano gave some degree of acceptance to the class of classes definition of number. That was in 1913, in a review of the *Principia Mathematica* [15]. There, Peano seems to accept it, in context, as a valid technical device. That he did not accept it as a final answer to the question, "Can number be defined?" is seen clearly in a statement in 1915 [16]: "If a and b are two classes (sets, groups), we write $\text{Num}\ a = \text{Num}\ b$, and read it 'the number (cardinal number or power of G. Cantor) of the a is equal (or identical) to that of the b' when we can establish a one-to-one correspondence between the a and b. We thus define the equality of two numbers, not number itself; and this because this definition may be placed before arithmetic, and also because the number that results is not the finite number of arithmetic." (For a critique of the views of Peano and Russell regarding definition by abstraction, see [22], Chapter V.)

Corrado Mangione has remarked [5, pp. 66-67]: "The original and fundamental observation of Frege consists basically in having recognized the possibility of expressing the equality of numbers without bringing in the concept of number itself; . . . Now, let us consider the extension of this concept; it is obviously *the class of all classes similar to* . . ." Peano did not find all this so "obvious," but his final agreement with Frege is remarkable, for Frege also distinguished between cardinal numbers and the numbers of arithmetic ([4], p. 155, as translated in [2], p. liv):

Since the Numbers (Anzahlen) are not proportions, we must distinguish them from the positive whole numbers (Zahlen). Hence it is not possible to enlarge the realm of Numbers to that of real numbers; they are wholly disjoint realms. The Numbers give the answer to the question, "How many objects are there of a given kind?", whereas the real numbers may be regarded as numbers giving a measure, stating how great a magnitude is as compared with a unit magnitude.

REFERENCES

[1] Frege, Gottlob, *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, Vol. 1, Jena (1893); partial translation in [2].

[2] Frege, Gottlob, *The Basic Laws of Arithmetic. Exposition of the System*, translated and edited, with an introduction, by Montgomery Furth, University of California Press, Berkeley (1964).

[3] Frege, Gottlob, "Ueber die Begriffsschrift des Herrn Peano und meine eigene," *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-physikalische Klasse*, vol. 48 (1896), pp. 361-378.

[4] Frege, Gottlob, *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, Vol. 2, Jena (1903); partial translation in [2].

[5] Frege, Gottlob, *Logica e Aritmetica*, Scritti raccolti a cura di Corrado Mangione, Editore Boringhieri, Torino (1965).

[6] Mooij, J. J. A., *La philosophie des mathématiques de Henri Poincaré*, Gauthier-Villars, Paris (1966).

[7] Nidditch, Peter, "Peano and the recognition of Frege," *Mind*, vol. 72 (1963), pp. 103-110.

[8] Peano, Giuseppe, *Arithmetices Principia, nova methodo exposita*, Bocca, Torino (1889).

[9] Peano, Giuseppe, *Notations de Logique Mathématique* (*Introduction au Formulaire de Mathématiques*), Guadagnini, Turin (1894).

[10] Peano, Giuseppe, Review of G. Frege, Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet, vol. 1, *Rivista di Matematica*, vol. 5 (1895), pp. 122-128.

[11] Peano, Giuseppe, "Studii di logica matematica," *Atti della Accademia delle Scienze di Torino*, vol. 32 (1896-97), pp. 565-583.

[12] Peano, Giuseppe, *Formulaire de Mathématiques*, vol. 2, Section 2, Bocca, Turin (1898).

[13] Peano, Giuseppe, "Formules de logique mathématique," *Revue de Mathématiques* (*Rivista di Matematica*), vol. 7 (1900-1901), pp. 1-40.

[14] Peano, Giuseppe, *Formulaire de Mathématiques*, vol. 3, Carré & Naud, Paris (1901).

[15] Peano, Giuseppe, Review of A. N. Whitehead and B. Russell, Principia Mathematica, Vols. I, II, *Bollettino di bibliografia e storia delle scienze matematiche,* vol. 15 (1913), pp. 47-53, 75-81. (Only Vol. I is discussed.)

[16] Peano, Giuseppe, "Le definizioni per astrazione," *"Mathesis" Società italiana di matematica. Bollettino*, vol. 7 (1915), pp. 106-120.

[17] Russell, Bertrand, "Sur la logique des relations avec des applications à la théorie des séries," *Rivista di Matematica,* vol. 7 (1901), pp. 115-148.

[18] Russell, Bertrand, *The Principles of Mathematics,* Cambridge University Press, Cambridge, England (1903).

[19] Russell, Bertrand, *Portraits from Memory and Other Essays*, Simon and Schuster, New York (1951).

[20] Russell, Bertrand, *My Philosophical Development*, George Allen & Unwin, London (1959).

[21] Russell, Bertrand, *The Autobiography of Bertrand Russell: 1872-1914*, Little, Brown and Company, Boston (1967).

[22] Vuillemin, Jules, *Leçons sur la première philosophie de Russell,* Librairie Armand Colin, Paris (1968).

Providence College
Providence, Rhode Island