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A DEDUCTION THEOREM FOR RESTRICTED GENERALITY

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In this paper, a deduction theorem for restricted generality (Ξ) will be proved on the basis of a finite number of axioms which do not contain variables. The theorem is in such a form as to avoid both Curry's paradox[1] and the Kleene Rosser paradox.[2] In fact it can be shown that nothing inconsistent can be proved using this form of the deduction theorem and the basic rules given below.[3]

An iterated form of the theorem can also be derived, as well as deduction theorems for **P** (implication) and Π (universal generality).

1. *The combinatory system* The notation we use in this paper is as in [4], in addition we take $\mathsf{H}x$ to stand for "x is a proposition." The system in which we prove the deduction theorem will contain at least two rules, others are expressible in terms of them. The first is the basic rule for restricted generality Ξ:

Rule Ξ. $\Xi xy, xu \vdash yu$.

Note that xu may be interpreted as "u has the property x" or as "u is an element of the class x" and Ξxy may be interpreted as "for all u for which xu holds, yu also holds" or as "x is a subclass of y." Ξxy will also be written as $xu \supset_u yu$. The second rule is one for equality (**Q**).

Rule Eq. $\mathsf{Q}xy, x \vdash y$.

The system may also include any set of axioms without variables. These can include axioms for equality such as $\vdash \mathsf{Q}XX$, for every primitive ob X, (there are a finite number of these: Ξ, **Q**, **K**, **S** so far) and

1. See [4] Chapter 5. It is obtained when implication **P** is defined by $\mathsf{P} \equiv [x,y]\ \Xi(\mathsf{K}x)(\mathsf{K}y)$.

2. See Kleene and Rosser [5]. This form of the theorem also avoids a generalized version of the Kleene Rosser paradox. This version of the paradox will appear in a later paper.

3. This is done in an as yet unpublished paper by H. B. Curry and the author.

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Axiom 1. $\vdash \mathsf{Q}xx \supset_x . \mathsf{Q}yy \supset_y \mathsf{Q}(xy)(xy)$,

to give equality for composite obs. If variables are required in the system a single axiom scheme $\vdash \mathsf{Q}uu$, where u is any variable in the system is sufficient. Any other axiom containing variables can then also be included in the system. Take an axiom of the form $\vdash \mathsf{T}(u_1, \ldots, u_n)$ for all $u_1, \ldots, u_n$, this can be expressed as:

$$\vdash \mathsf{Q}u_1u_1 \supset_{u_1}: \mathsf{Q}u_2u_2 \supset_{u_2} \ldots \mathsf{Q}u_nu_n \supset_{u_n} \mathsf{T}(u_1, \ldots, u_n) \ldots .$$

$\vdash \mathsf{T}(u_1, \ldots, u_n)$ is then derivable using $\vdash \mathsf{Q}u_1u_1, \ldots, \vdash \mathsf{Q}u_nu_n$ and Rule Ξ.

2. *The basic theorems and axioms* The axioms we use are stated in terms of a new ob L, which is such that "$\mathsf{L}X$" is interpreted as "for all u in A, Xu is a proposition." This L is defined in terms of the primitive ob H and a new (unspecified) primitive category A, thus:[4]

Definition L. $\mathsf{L} \equiv \mathsf{FAH}$.

In addition to $\vdash \mathsf{WQK}$, $\vdash \mathsf{WQS}$, $\vdash \mathsf{WQH}$, etc. and Axiom 1 which must be in the system if it contains equality, we need three further axioms, viz:

Axiom 2. $\vdash \mathsf{L}x \supset_x \Xi xx$.
Axiom 3.[5] $\vdash \mathsf{L}x \supset_{x,y}: xu \supset_u . yuv \supset_v xu$.
Axiom 4. $\vdash \mathsf{L}x \supset_{x,t}: xu \supset_u yu(tu) \supset_y . (xu \supset_u (yuv \supset_v zuv)) \supset_x (xu \supset_u zu(tu))$.

Of these, Axioms 2 and 3 are (ΞI) and (ΞK) (see [4]) with the restriction $\mathsf{L}x$ on x. Axiom 4 also has x restricted to $\mathsf{L}x$, but it is not exactly (ΞS) as it has the order of the expressions altered as well as the order of $\supset_z$, $\supset_y$ and $\supset_t$. Also the γx in Curry's [3] form (which is also Cogan's [2] form) is replaced by a single symbol. We now prove a theorem from each of the above axioms.

Theorem 1. $\mathsf{L}x \vdash \Xi xx$.

Proof. The theorem follows by Axiom 2 and Rule Ξ. Similarly we get the following two.

Theorem 2. $\mathsf{L}x, xu \vdash yuv \supset_v xu$.
Theorem 3. $\mathsf{L}x, xu \supset_u . yuv \supset_v zuv, xu \supset_u yu(tu) \vdash xu \supset_u zu(tu)$.

One would expect it to be possible to prove (ΞI) from (ΞK) and (ΞS); but here with the extra condition $\mathsf{L}x$ this is not possible unless we have an ob Y such that $\vdash \mathsf{Y}$ and $\vdash \mathsf{L}(\mathsf{KY})$. In that case we have the next theorem, which provides an alternative to Axiom 2.

Theorem 4. *If there is an ob* Y, *not containing any variables, such that* $\vdash \mathsf{Y}$ *and* $\vdash \mathsf{L}(\mathsf{KY})$, *then* $\mathsf{L}x \vdash \Xi xx$, *follows from Axioms 3 and 4.*

Proof. By putting $\mathsf{B}(\mathsf{KY})$ for y and $\mathsf{BK}x$ for z in Theorem 3 we get,

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4. $\mathsf{F} \equiv [x,y,z]\, \Xi x(\mathsf{B}yz)$.

5. "$X \supset_{x,y} Y$" is an abbreviation for "$\mathsf{E}y \supset_y \cdot X \supset_x Y$."

$\mathsf{L}x, xu \supset_u . \mathsf{Y} \supset_v xu, xu \supset_u \mathsf{Y} \vdash xu \supset_u xu$. Now by Axiom 3, $\mathsf{L}x, \vdash xu \supset_u . \mathsf{Y} \supset_v xu$ and by Theorem 2 $\vdash xu \supset_u \mathsf{Y}$, provided $\vdash \mathsf{Y}$ and $\vdash \mathsf{L}(\mathsf{KY})$. Thus the result follows.

The next axiom that is needed for the deduction theorem gives a property of the "universal class" **WQ**.

Axiom 5. $\vdash \mathsf{L}x \supset_x \Xi x(\mathsf{WQ})$.

We also need two properties of **H**:

Axiom 6. $\vdash \Xi \mathsf{IH}$,

which states that every assertion is a proposition; and

Axiom 7. $\vdash \mathsf{LH}$,

which asserts that **H** is an element of **L**. We can then prove,

Theorem 5. $\mathsf{H}y \vdash \mathsf{L}(\mathsf{K}y)$.
Proof. By Theorem 2 with **H** for x, **KA** for y and y for u we have $\mathsf{LH}, \mathsf{H}y \vdash \mathsf{A}v \supset_v \mathsf{H}y$. Thus by Axiom 7 and the properties of $\mathsf{K}, \mathsf{H}y \vdash \mathsf{A}v \supset_v \mathsf{H}(\mathsf{K}yv)$, which is $\mathsf{H}y \vdash \mathsf{FAH}(\mathsf{K}y)$.
Note that this theorem allows us to remove Axiom 1, as Theorem 1 can be derived using Theorem 4 where $\vdash \mathsf{Y}$ is any axiom.

3. *The deduction theorem: statement and motivation* Now the deduction theorem[6] can be stated.

Theorem 6. (The Deduction Theorem for Ξ). *If $X_0, X \vdash Y$ and $X_0 \vdash \mathsf{L}([u]X)$ where u is not involved in X_0, then $X_0 \vdash X \supset_u Y$.*

The motivation for this form of the deduction theorem is based mainly on its special case for **P**. Because of the paradoxes, the axioms of propositional calculus have to be restricted in some way. This is usually done by requiring all the variables in such a statement to be propositions. Such a procedure has the disadvantage, however, that obs X of which it is not certain whether or not they are propositions can never be used. We are setting up a theory here which can be interpreted in a kind of 3-valued logic in which statements are T (true), F (false) or N ("neither" or "not sure"). Truth tables for the propositional connectives are then as follows:

x	$-x$
T	F
F	T
N	N

	$\vee xy$	y: T	F	N
	T	T	T	T
x	F	T	F	N
	N	T	N	N

	$\wedge xy$	y: T	F	N
	T	T	F	N
x	F	F	F	F
	N	N	F	N

	$\mathsf{P}xy$	y: T	F	N
	T	T	F	N
x	F	T	T	T
	N	T	N	N

6. A similar theorem was proved independently by Seldin [6] using somewhat different assumptions. His methods allowed me to simplify some parts of my proof. The theorem is also similar to one proved by Church in his 1932 paper. Church used instead of $\mathsf{L}([u]X)$ what in classical notation would be $(\exists u)X$, and he requires that u actually appears in X. He used axioms and rules which were proved inconsistent by Kleene and Rosser [5].

Similar tables[7] to these can be found in Chapter 12 of Kleene [5]. The table for **P** is basic here, as the others should be derivable from it using suitable definitions of -, $\vee$ and $\wedge$, if it is to be possible to extend the theory to a classical system. This table for **P** could also be expressed (still informally) as follows. $\mathsf{P}XY$ is T if and only if, $\overline{\mathsf{H}}X$ is T or Y is T, and Y is T whenever X is T. $\mathsf{H}(\mathsf{P}XY)$ is T if and only if, $\overline{\mathsf{H}}X$ is T or Y is T, and $\mathsf{H}Y$ is T whenever X is T. To be consistent with the tables for P, therefore, a deduction principle of the form

$$\textit{If } X \vdash Y \textit{ and } \vdash \mathsf{H}X \textit{ or } \vdash Y, \textit{ then } \vdash \mathsf{P}XY$$

can be adopted. It is the Ξ form of this that we now prove.

4. *Proof of the deduction theorem*

Proof. Let there be n steps $Y_1, Y_2, \ldots, Y_n = Y$ in the proof of Y from X_0 and X. We show by induction on k that $X_0 \vdash X \supset_u Y_k$. There will be five cases to consider.

1. Y_k is X.
2. Y_k is a constant (with respect to u), such that $X_0 \vdash Y_k$.
3. Y_k is $\mathsf{WQ}u$ (i.e., the only axiom which can contain u).
4. Y_k is obtained from Y_i by Rule Eq.
5. Y_k is obtained from Y_j and Y_i by Rule Ξ.

Cases 1, 2 and 3 involve no inductive hypotheses, and so take care of the basic step $k = 1$, but they are also applicable when $k > 1$. In the inductive step the theorem is assumed for $Y_i (i < k)$.

Case 1. By Theorem 1, $\mathsf{L}([u]X) \vdash X \supset_u X$, so as, $X_0 \vdash \mathsf{L}([u]X)$ and $Y_k \equiv X$ the result follows.

Case 2. If Y_k is a constant with respect to u such that, $X_0 \vdash Y_k$, then $[u]Y_k = \mathsf{K}Y_k$. Now $Y_k \vdash \mathsf{L}[\mathsf{K}Y_k]$ holds by Theorem 5 and Axiom 6. Therefore Theorem 2, with $\mathsf{K}Y_k$ for x and $\mathsf{K}([u]X)$ for y, gives $Y_k \vdash X' \supset_v Y_k$, where v is a variable not involved in X_0, X, or Y_k, and X' is X with u replaced by v. Hence $Y_k \vdash X \supset_u Y_k$, and as $X_0 \vdash Y_k$, we get $X_0 \vdash X \supset_u Y_k$.

Case 3. By Axiom 5, $\mathsf{L}([u]X) \vdash X \supset_u \mathsf{WQ}u$, so if $\mathsf{WQ}u = Y_k$, the result follows by $X_0 \vdash \mathsf{L}([u]X)$.

Case 4. If $X_0 \vdash Y_k$ follows from $X_0 \vdash Y_i$ by Rule Eq, then, if $\vdash X_0$, $Y_k = Y_i$, and so $(X \supset_u Y_k) = (X \supset_u Y_i)$. By the hypothesis of the induction $X_0 \vdash X \supset_u Y_i$. Therefore $X_0 \vdash X \supset_u Y_k$.

Case 5. Let Y_k be obtained from $Y_j (j \leqslant k - 1)$ and $Y_i (i \leqslant k - 1, i \neq j)$ by Rule Ξ. Y_j must then be of the form $Z_1 v \supset_v Z_2 v$, Y_i of the form $Z_1 Z_3$ and Y_k of the form $Z_2 Z_3$. Then by the hypothesis of the induction, if $X_0 \vdash \mathsf{L}([u]X)$, then $X_0 \vdash X \supset_u . Z_1 v \supset_v Z_2 v$ and $X_0 \vdash X \supset_u Z_1 Z_3$. Now substitute into Theorem 3 $[u]X$ for x, $[u]Z_1$ for y, $[u]Z_2$ for z and $[u]Z_3$ for t. This gives

7. Note that axioms A1, 2, 3, 4 and 7 of my earlier article [1] satisfy these three-valued tables. A7 is in fact based on the table for **P**. Thus in a system based on these tables the assumption $\vdash \mathsf{H}^{k+1}X$ for any X and a $k \geq 0$ cannot hold.

$$\mathsf{L}([u]X),\ X \supset_u .\ Z_1 v \supset_v Z_2 v,\ X \supset_u .\ Z_1 Z_3 \vdash X \supset_u Z_2 Z_3.$$

Thus given $X_0 \vdash \mathsf{L}([u]X)$ and the above hypothesis $X_0 \vdash X \supset_u Y$ follows. The induction has therefore been completed for all cases and the theorem holds.

Corollary 1. *Any axiom free of variables can be added to the system and the deduction theorem will still hold. If an axiom of the form* $\vdash Z$ *for all* u *is added, where* Z *involves* u, *the theorem holds if* $\vdash \Xi(\mathbf{WQ})([u]Z)$ *is also an axiom.*

Corollary 2. *The theorem still holds if instead of the single condition* X_0 *there is any finite number of them, say* $X_0, X_1, \ldots, X_n$, *replacing* X_0 *throughout.*

5. *The iterated deduction theorem* We shall now consider some examples of the working of deduction theorem, especially of the condition $X_0 \vdash \mathsf{L}([u]X)$. If we have

$$xu,\ yuv \vdash zuv,\ (\text{for all } u,\ v) \tag{1}$$

we can get

$$xu,\ \mathsf{L}(yu) \vdash yuv \supset_v zuv. \tag{2}$$

(This seems intuitively reasonable and this step is allowed by our form of the deduction theorem). It might seem reasonable then, to also have the following step:

$$\mathsf{L}x,\ \mathsf{L}(yu) \vdash xu \supset_u .\ yuv \supset_v zuv,$$

however, this is wrong, (unless yu does not involve u), as the variable u is not removed throughout by the induction. If however we had $xu \vdash \mathsf{L}(yu)$, (2) would reduce to $xu \vdash yuv \supset_v zuv$, and so the step to

$$\mathsf{L}x \vdash xu \supset_u .\ yuv \supset_v zuv \tag{3}$$

can be made in the same way as the step leading to (2).

We could also write what we have concluded from (1) as

$$\mathsf{L}x,\ \mathsf{F}x\mathsf{L}y \vdash xu \supset_u .\ yuv \supset_v zuv.$$

In this example the deduction theorem was applied twice. From it we can see how the deduction theorem can be iterated. All that is necessary is that the condition $X_0, \ldots, X_k \vdash \mathsf{L}([u_{k+1}]X_{k+1})$ holds for all variables involved in $X_0, \ldots, X_{k+1}$ when we are taking the induction over u_{k+1}.

Theorem 7. (Iterated Deduction Theorem for Ξ). *If* $X_0, X_1, \ldots, X_m \vdash Y$, *where no* u_k *occurs in any* X_j *for* $j < k$; *and if for all* $k < m$, $X_0, X_1, \ldots, X_k \vdash \mathsf{L}([u_{k+1}]X_{k+1})$; *then* $X_0 \vdash X_1 \supset_{u_1} \ldots X_m \supset_{u_m} Y$.

6. *Deduction theorems for* P *and* Π In connection with the form of the deduction theorem that we have proved it is necessary to have some basic obs that belong to the category L. The first of these we have taken to be H. Also we require $\vdash \mathsf{L}\mathsf{A}$, and using this we can prove $\vdash \mathsf{L}\mathsf{E}$.

Axiom 8. $\vdash \mathbf{LA}$.

Theorem 8. $\vdash \mathbf{LE}$.

Proof. By Axiom 5, $\mathbf{LA} \vdash \Xi\mathbf{AE}$. Thus by Axiom 8, $\mathbf{A}u \vdash \mathbf{E}u$, so by Axiom 6, $\mathbf{A}u \vdash \mathbf{H}(\mathbf{E}u)$ and by Axiom 8 and the deduction theorem the result follows.

Deduction theorems for $\mathbf{P}$ and Π[8] are easily obtained from that for Ξ. The one for Π contains no auxiliary premises such as $X_0 \vdash \mathbf{L}(\ldots)$, as we already have $\vdash \mathbf{LE}$ by Theorem 8.

Theorem 9. (The Deduction Theorem for Π). *If* $X_0 \vdash Yu$ *all* u *then* $X_0 \vdash \Pi Y$.

Theorem 10. (The Deduction Theorem for $\mathbf{P}$). *If* $X_0, X \vdash Y$ *and if* $X_0 \vdash \mathbf{H}X$, *then* $X_0 \vdash X \supset Y$.

Proof. If in Theorem 6 a u is taken which is not involved in X or Y, then $X_0 \vdash \mathbf{L}([u]X)$ becomes $X_0 \vdash \mathbf{L}(\mathbf{K}X)$, and this follows if $X_0 \vdash \mathbf{H}X$. Thus $X_0 \vdash X \supset_u Y$, holds and this is, $X_0 \vdash \Xi(\mathbf{K}X)(\mathbf{K}Y)$, i.e., $X_0 \vdash \mathbf{P}XY$.

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8. Note that $\Pi \equiv \Xi(\mathbf{WQ})$. The basic rules for $\mathbf{P}$ and Π are: $\mathbf{P}xy, x \vdash y$ and $\mathbf{WQ}\,u, \Pi x \vdash xu$.

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