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A NOTE ON TRANSITIVITY

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Prefixing *CCpqCCrpCrq*[1] (**T pre**) and suffixing *CCpqCCqrCpr* (**T suf**) are usually taken to be the two theorem forms of transitivity, chiefly because in the presence of the rules of substitution and detachment, they both yield the derived rule of transitivity "*From Cpq and Cqr infer Cpr.*" Because of this, even though Sobociński [4] reports a result attributed to Łukasiewicz that **T pre** and **T suf** are mutually independent, one might be tempted to suppose that in any reasonable context they are equivalent (interreplaceable) forms of the same basic idea-transitivity. One might suppose this even given Sobociński's proof[2] for matrix 𝔄, which he uses to show the independence of **T pre** from **T suf**, does not even satisfy identity *Cpp* (**T** identity); but such is *not* the case.

To be sure, there are contexts in which they are interreplaceable, e.g., in the presence of permutation "*From CpCqr infer CqCpr*" (**DR perm**), or restricted permutation "*From CpCCqrs infer CCqrCps*" (**DR rest perm**), or as Sobociński shows in the presence of unrestricted assertion *CpCCpqq* (**T** assertion).[3] A closer inspection of **T pre** and **T suf** reveals that there is indeed some permutation already present in **T suf**, i.e., in the consequent *q* precedes *p*, which is not the case in **T pre**. This suggests **T suf** is a more powerful (useful) form of transitivity. An example of this may be taken from Anderson's pure calculus of entailment E_I [1] where one formulation (here denoted by E_I1) has the following axioms together with the rules of substitution and detachment:

E_I1 Ax1. *CCCppqq*
E_I1 Ax2. *CCpqCCqrCpr*
E_I1 Ax3. *CCpCpqCpq*

This formulation is essentially the formulation I_2 of Anderson, Belnap and

1. The symbolism of J. Łukasiewicz is used throughout, *cf.* [3], pp. 77-83. The names of theorems and rules follow Anderson and Belnap, *cf.* [1], p. 42.

2. *Op. cit.*, p. 49.

3. *Ibid.*

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Wallace [2] and it is shown there that **T pre** is derivable from these axioms and rules.[4] However an equivalent system S_I1 is not produced if Ax2 of E_I1 is replaced by **T pre**. S_I1 has the following axioms together with the rules of substitution and detachment:

S_I1 Ax1. *CCCppqq*
S_I1 Ax2. *CCpqCCrpCrq*
S_I1 Ax3. *CCpCpqCpq*

We may use matrix 𝔐1, in which 1 and 2 are designated, to show that *CCpqCCqrCpr* is not a theorem of S_I1.

𝔐1

C	0	1*	2*
0	1	1	2
*1	0	1	2
*2	0	0	2

𝔐1 is sufficient for the axioms and rules of S_I1 but for $p = 0$ or 1 or 2; $q = 2$; $r = 0$ or 1; *CCpqCCqrCpr* takes the undesignated value 0 and so is independent.

We might now be led to suppose that in any reasonable system whenever **T suf** is provable so will **T pre** be provable, but such is not the case. If we weaken E_I1 Ax1 to *Cpp* (**T** identity) the resulting system S_I2 might be taken erroneously for a formulation of Anderson's pure calculus of ticket entailment P_I. S_I2 has the following axioms together with substitution and detachment:

S_I2 Ax1. *Cpp*
S_I2 Ax2. *CCpqCCqrCpr*
S_I2 Ax3. *CCpCpqCpq*

That *CCpqCCrpCrq* is not a theorem of S_I2 may be shown using matrix 𝔐2, in which 1, 2, 3 and 4 are designated.

𝔐2

C	0	1*	2*	3*	4*
0	4	4	4	4	4
*1	0	4	4	4	4
*2	0	0	4	1	4
*3	0	0	0	1	4
*4	0	0	0	0	4

𝔐2 is sufficient for the axioms and rules of S_I2 but for $p = 2$; $q = 3$; $r = 2$; *CCpqCCrpCrq* takes the undesignated value 0 and so is independent. So we may conclude that in the presence of substitution and detachment, **T pre** and **T suf** do have the derived rule of transitivity in common but their claim to being alternative forms of transitivity rests basically just there.

4. *Op. cit.*, p. 94.

REFERENCES

[1] Anderson, Alan Ross, and Nuel D. Belnap, "The pure calculus of entailment," *The Journal of Symbolic Logic*, vol. 27 (1962), pp. 19-52.

[2] Anderson, Alan Ross, Nuel D. Belnap, and John R. Wallace, "Independent axiom schemata for the pure theory of entailment," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik,* vol. 6 [1960], pp. 93-95.

[3] Łukasiewicz, J., *Aristotle's Syllogistic,* Oxford (1951).

[4] Sobociński, B., "Axiomatization of a partial system of three-value calculus of propositions," *The Journal of Computing Systems*, vol. 1 (1952), pp. 23-55.

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