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SIMULTANEOUS *VERSUS* SUCCESSIVE QUANTIFICATION

LEROY F. MEYERS

In standard predicate calculus, if u and v are distinct variables, then "for all u and v, Puv" is satisfactorily restated as "for all u, for all v, Puv"; symbolically:

$$\forall(u, v)Puv \text{ as } \forall u\forall vPuv \ .$$

Similarly, "there are u and v such that Puv" is satisfactorily restated as "there is a u such that there is a v such that Puv"; symbolically:

$$\exists(u, v)Puv \text{ as } \exists u\exists vPuv \ .$$

On the other hand, in standard predicate calculus with equality, it is *not* correct to restate "there exist unique u and v such that Puv" as "there exists a unique u such that there exists a unique v such that Puv"; symbolically:

$$\exists!(u, v)Puv \textit{ versus } \exists!u\exists!vPuv \ ,$$

where

$$\exists!vPuv \text{ abbreviates } \exists vPuv \wedge \forall v\forall v_1(Puv \wedge Puv_1 \rightarrow v = v_1),$$
$$\exists!(u, v)Puv \text{ abbreviates } \exists u\exists vPuv \wedge \forall u\forall u_1\forall v\forall v_1(Puv \wedge Pu_1v_1 \rightarrow u = u_1 \wedge v = v_1),$$

and u_1 and v_1 are distinct variables not occurring in $\exists u\exists vPuv$. We have the following counterexample. In the theory of real (or complex) numbers,

$$\exists!(x, y)\ (x = y^2)$$

is false (since there are many pairs (x, y) such that $x = y^2$), but

$$\exists!x\exists!y(x = y^2)$$

is true (since only 0 has exactly one square root). Simultaneous unique existence is often used in stating theorems, as in the following special case of the division algorithm in the theory of natural numbers:

$$\exists!(x, y)\ (0 = 1\cdot x + y \wedge y < 1) \ .$$

It is an exercise in predicate calculus with equality to show that

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$$\vdash \exists!(u, v)Puv \rightarrow \exists!u\exists!vPuv\ .$$

A similar situation occurs with simultaneous *versus* successive uniqueness (not requiring existence); symbolically:

$$!(u, v)Puv \textit{ versus } !u!vPuv\ ,$$

where

$$!vPuv \text{ abbreviates } \forall v \forall v_1(Puv \wedge Puv_1 \rightarrow v = v_1)\ ,$$
$$!(u, v)Puv \text{ abbreviates } \forall u \forall u_1 \forall v \forall v_1(Puv \wedge Pu_1v_1 \rightarrow u = u_1 \wedge v = v_1)\ ,$$

and u_1 and v_1 are distinct variables not occurring in $\exists u\exists vPuv$. However, an implication similar to that for unique existence cannot be expected to hold, as the following examples show. In the theory of complex numbers,

$$!(x, y)\ (x = y^2)$$

is false, but

$$!x!y(x = y^2)$$

is true. In the theory of real numbers,

$$!(x, y)\ (x^2 + y^2 < 0)$$

is true (since there are no pairs (x, y) such that $x^2 + y^2 < 0$), but

$$!x!y(x^2 + y^2 < 0)$$

is false (since for all x there is no y such that $x^2 + y^2 < 0$). (A similar situation occurs whenever $\exists(u, v)Puv$ is false when interpreted in a domain with more than one element.)

Obviously, $\exists!(u, v)Puv$ is equivalent to $\exists(u, v)Puv \wedge !(u, v)Puv$, but there is no simple relation between $\exists!u\exists!v$ and $\exists u\exists vPuv \wedge !u!vPuv$. In fact, in the theory of real numbers,

$$\exists!x\exists!y(x = y^2)$$

is true and

$$\exists x\exists y(x = y^2) \wedge !x!y(x = y^2)$$

is false, whereas

$$\exists!x\exists!y(x^2 \cdot y^2 = 1)$$

is false and

$$\exists x\exists y(x^2 \cdot y^2 = 1) \wedge !x!y(x^2 \cdot y^2 = 1)$$

is true.

While $\exists!(u, v)Puv$ is equivalent to $\exists!(v, u)Puv$ (and similarly for $!(u, v)Puv$), there is no simple relation between $\exists!u\exists!vPuv$ and $\exists!v\exists!uPuv$ (and similarly for $!u!vPuv$), as some of the above examples show.

Remark. A syntactically uniform reading of the quantifiers $\forall u$, $\exists u$,

$\exists!\,u$, and $!u$ is "for all (each, every) u," "for some u," "for some unique u," and "for unique u," respectively. These readings avoid the confusion which sometimes occurs between $\exists!u$ and $!u$ in informally stated uniqueness proofs.

The Ohio State University
Columbus, Ohio