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A NEW PROOF OF COMPLETENESS

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We present a new proof of the completeness of the formalisation $\mathscr{P}$ of sentence logic based on the first four axioms of Russell's *Principia*, with substitution and modus ponens as rules of inference. For the sake of brevity we take for granted various elementary properties of $\mathscr{P}$, for instance that conjunction and disjunction are commutative and associative and that each distributes over the other; that $r \vee \neg r$ is provable in $\mathscr{P}$; that from $A \rightarrow P$ and $B \rightarrow P$ we may infer $(A \vee B) \rightarrow P$, and from $P \rightarrow A$, $P \rightarrow B$ we may infer $P \rightarrow (A \,\&\, B)$. It follows that if $\mathbf{T}$ denotes the provable sentence $r \vee \neg r$, and $\mathbf{F}$ denotes $\neg\mathbf{T}$ then the equivalences

$$p \longleftrightarrow (p \vee \mathbf{F}),\ \mathbf{T} \longleftrightarrow (p \vee \mathbf{T}),\ p \longleftrightarrow (p \,\&\, \mathbf{T})$$

are all provable in $\mathscr{P}$ from which it follows that

$$(*) \qquad p \longleftrightarrow (p \vee \mathbf{F}) \,\&\, (\neg p \vee \mathbf{T})$$

is provable in $\mathscr{P}$.

We start by observing that the negation of any one of the sentences of the set

$$p,\ \neg p,\ \mathbf{T},\ \mathbf{F}$$

and the disjunction of any two, is equivalent to a sentence of the set. It follows (by induction on the number of negations and disjunctions in a sentence) that any sentence $\mathfrak{S}(p)$ in the single variable p is equivalent to one of p, $\neg p$, $\mathbf{T}$, $\mathbf{F}$. Since

$$(p \vee \mathbf{T}) \,\&\, (\neg p \vee \mathbf{T}) \longleftrightarrow \mathbf{T}$$
$$(p \vee \mathbf{F}) \,\&\, (\neg p \vee \mathbf{F}) \longleftrightarrow \mathbf{F}$$
$$(p \vee \mathbf{T}) \,\&\, (\neg p \vee \mathbf{F}) \longleftrightarrow \neg p$$
$$(p \vee \mathbf{F}) \,\&\, (\neg p \vee \mathbf{T}) \longleftrightarrow p$$

are all provable, it follows that to each sentence $\mathfrak{S}(p)$ corresponds α, β such that

$$\mathfrak{S}(p) \longleftrightarrow (p \vee \alpha) \,\&\, (\neg p \vee \beta)$$

where each of α, β is one of $\mathbf{T}$, $\mathbf{F}$ (and so does not contain the variable p).

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Next we prove that to every sentence Σ, in any number of variables, there corresponds A, B (not containing p) such that

$$(**) \qquad \Sigma \longleftrightarrow (p \vee A) \;\&\; (\neg p \vee B) \longleftrightarrow (p \;\&\; B) \vee (\neg p \;\&\; A).$$

It suffices to observe that if (**) holds for sentences Σ_1, Σ_2 (with corresponding A_1, B_1; A_2, B_2) then it holds for $\neg\Sigma_1$ and $\Sigma_1 \vee \Sigma_2$ (with corresponding $\neg A_1$, $\neg B_1$; $A_1 \vee A_2$, $B_1 \vee B_2$), and furthermore that by (*) above it holds when Σ is a single variable p and also when Σ does not contain p (in which case both A and B are just Σ itself). Writing Σ explicitly in the form $\mathfrak{S}(p, q_1, \ldots, q_n)$ and substituting first T, then F, for p in the provable equivalence (**) we obtain

$$\mathfrak{S}(\mathsf{T}, q_1, \ldots, q_n) \longleftrightarrow (\mathsf{T} \vee A) \;\&\; (\mathsf{F} \vee B) \longleftrightarrow B$$
$$\mathfrak{S}(\mathsf{F}, q_1, \ldots, q_n) \longleftrightarrow (\mathsf{F} \vee A) \;\&\; (\mathsf{T} \vee B) \longleftrightarrow A$$

whence it follows that the equivalence

$$\mathfrak{S}(p, q_1, \ldots, q_n) \longleftrightarrow [p \vee \mathfrak{S}(\mathsf{T}, q_1, \ldots, q_n)] \;\&\; [\neg p \vee \mathfrak{S}(\mathsf{F}, q_1, \ldots, q_n)]$$

is provable in $\boldsymbol{P}$.

Let V be the set of variables p, q_1, q_2, ... which does not contain r. We proceed to prove, by induction on the number of variables from V in $\mathfrak{S}$, that if $\mathfrak{S}$ is a tautology then $\mathfrak{S}$ is provable. For if this result holds for all sentences with not more than n variables from V, and if $\mathfrak{S}(p, q_1, \ldots, q_n)$ is a tautology containing $n+1$ variables from V, then $\mathfrak{S}(\mathsf{T}, q_1, \ldots, q_n)$ is a tautology in n variables from V, and is therefore provable by the inductive hypothesis, and likewise $\mathfrak{S}(\mathsf{F}, q_1, \ldots, q_n)$ is provable, whence it follows that

$$[p \vee \mathfrak{S}(\mathsf{T}, q_1, \ldots, q_n)] \;\&\; [\neg p \vee \mathfrak{S}(\mathsf{F}, q_1, \ldots, q_n)]$$

is provable, and finally $\mathfrak{S}(p, q_1, \ldots, q_n)$ is provable.

To complete the inductive proof we observe that if $\mathfrak{S}$ contains but a single variable p then $\mathfrak{S}$ is equivalent to one of T, F, p, $\neg p$; but if $\mathfrak{S} \longleftrightarrow \mathfrak{R}$ is provable, then $\mathfrak{S} \longleftrightarrow \mathfrak{R}$ is a tautology, and so if $\mathfrak{S}$ is a tautology, so too is $\mathfrak{R}$. Since none of F, p, $\neg p$ is a tautology it follows that if $\mathfrak{S}$ is a tautology then $\mathfrak{S} \longleftrightarrow \mathsf{T}$ is provable; but T is provable and so $\mathfrak{S}$ is provable, which shows that $\boldsymbol{P}$ is complete with respect to the truth tables.

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