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AN EQUATIONAL AXIOMATIZATION AND A SEMI-LATTICE THEORETICAL CHARACTERIZATION OF MIXED ASSOCIATIVE NEWMAN ALGEBRAS

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In [2][1] and [3] M. H. A. Newman constructed and investigated two algebraic systems which he calls the fully complemented non-associative mixed algebra and the fully complemented associative mixed algebra respectively. In [6], [7], [8] and in this paper these systems are called simply: Newman algebras and associative Newman algebras. In [4] Newman constructed and investigated two relatively complemented algebraic systems which in some respect correspond respectively to the systems mentioned above. He calls, *cf.* [4], p. 38, these systems "mixed non-associative algebra" and "mixed (associative) algebra." In this paper only the latter system will be investigated and it will be called "mixed associative Newman algebra."

In [4], p. 40, the following characterization (the meaning of which will be explained in section 1 below)

Theorem 3 (Newman). *In order that a double algebra may be a mixed algebra it is necessary and sufficient that it be distributive and idempotent, that* $a(bb) = (ab)b$, *and that there exist a right* ω *and a left* ω.

of mixed associative Newman algebra has been established. Moreover, in [4], it has been proved that this algebraic system whose two basic binary operations are + and $\times$ is the direct join of an associative Boolean ring (without unity element) and a generalized Boolean algebra in the sense of Stone, *cf.* [9], p. 721, section 3.

In this paper it will be shown that, as in the case of the fully

[1]An acquaintance with the papers [2], [3], [4], [6], [7] and [8] is presupposed. In [2], [3] and [4] "*ab*" is used instead of "$a \times b$". An enumeration of the algebraic tables, *cf.* section 5 below, is a continuation of the enumeration of such tables given in [6] and [8].

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complemented associative Newman algebras, *cf.* [7] and [8], there exists such a formalization of the relatively complemented associative Newman algebras that

(i) These algebras can be axiomatized equationally,

and, moreover, that

(ii) These algebras can be considered as semi-lattices with respect to the primitive binary operation $\times$ to which the additional postulates are added concerning the properties of the other operations which are accepted as primitive in the formalization mentioned above.

It should be remarked that throughout this paper the theoretical foundations of the discussed algebraic systems and the properties of "even" and "odd" elements belonging to the carrier set of the given mixed associative Newman algebra will not be discussed, and that the axioms *A1*-*A9*, *B1*, *B1**, *A10* and *A11*, given below, will be used mostly tacitly in the proofs presented in this paper. Also, it should be noticed that several formulas which are valid in the field of the investigated system and which will be needed for our end are already proven by Newman in [4]. But, unfortunately, in [4] the proofs of these formulas are often given either with the analysis of the mixed non-associative Newman algebras, or verbally, or by a simple remark that a proof is analogous to certain deductions in the field of the fully complemented Newman algebras which are presented in [2]. For this reason and in order to separate completely the mixed associative Newman algebras from the other Newman algebraic systems several proofs presented in [4] will be repeated in this paper, obviously, with the indications that they are due to Newman. Although it will increase the length of this paper considerably, it will allow the reader to understand the deductions without a penetrating study of [2], [3] and [4].

1 Newman's Theorem 3 given above can be expressed in a fully formalized way, as follows:

(A) *Any algebraic system*

$$\mathfrak{A} = \langle A, =, +, \times, 0, O\rangle$$

with one binary relation $=$, *two binary operations* $+$ *and* $\times$, *and two constant elements* 0 *and* O, *is a relatively complemented mixed associative Newman algebra if and only if it satisfies the following postulates*:

A1 $[a]:a \epsilon A . \supset . a = a$
A2 $[ab]: a,b \epsilon A . a = b . \supset . b = a$
A3 $[abc]:a,b,c \epsilon A . a = b . b = c . \supset . a = c$
A4 $[ab]:a,b \epsilon A . \supset . a + b \epsilon A$
A5 $[ab]:a,b \epsilon A . \supset . a \times b \epsilon A$
A6 $[abc]:a,b,c \epsilon A . a = c . \supset . a + b = c + b$
A7 $[abc]:a,b,c \epsilon A . b = c . \supset . a + b = a + c$
A8 $[abc]:a,b,c \epsilon A . a = c . \supset . a \times b = c \times b$
A9 $[abc]:a,b,c \epsilon A . b = c . \supset . a \times b = a \times c$

B1 $0 \epsilon A$

B2 $[ab]:a,b \epsilon A \,.\, a\times b = a \,.\supset.\, [\exists c].\, c \epsilon A \,.\, c + a = b \,.\, c\times a = 0$

*B1** $O \epsilon A$

*B2** $[ab]:a,b \epsilon A \,.\, b \times a = a \,.\supset.\, [\exists c].\, c \epsilon A \,.\, c + a = b \,.\, a \times c = O$

C1 $[abc]:a,b,c \epsilon A \,.\supset.\, a\times (b + c) = (a\times b) + (a \times c)$

C2 $[abc]:a,b,c \epsilon A \,.\supset.\, (a + b)\times c = (a\times c) + (b \times c)$

E1 $[a]:a \epsilon A \,.\supset.\, a = a \times a$

E2 $[ab]:a, b \epsilon A \,.\supset.\, a \times (b \times b) = (a \times b) \times b$

The axioms *A1-A9* are the customary closure algebraic postulates with respect to the operations + and ×, and the assumptions concerning the logical properties of the relation =. The right ω and the left ω of Newman's Theorem 3 are the postulates *B1* and *B2*, and the postulates *B1** and *B2** respectively, *cf.* in [4], p. 34 and p. 40. Obviously, the axioms *C1* and *C2* are the laws of distribution, and *E1* is the law of indempotency with regard to the operation ×. Finally, *E2* is a special case of the law of association with respect to the operation ×.

As in the case of the fully complemented Newman algebras, *cf.* [2] and [6], it should be noticed that for all elements of the carrier set of any relatively complemented Newman algebra the additive operation + is commutative and associative, but not necessarily idempotent of nilpotent, and that the multiplicative operation × is idempotent and commutative, but not necessarily associative.

2 It follows implicitly from the considerations given in [4] that the axiom-system of 𝔄 given above in section **1** can be simplified. Namely:

2.1 Let us assume the discussed axiomatization of 𝔄. Then:

F1 $[ab]:a,b \epsilon A \,.\supset.\, (a \times b) \times b = a \times b$ [*E2*; *E1*]

F2 $[a]:a \epsilon A \,.\supset.\, a = a + 0$

PR $[a]:\mathrm{Hp}(1)\,.\supset.$

$[\exists c].$

2. $c \epsilon A \,.$
3. $c \times a = 0 \,.$ [1; *B1*; *E1*]
4. $0 \times a = (c \times a) \times a = c \times a = 0 \,.$ [1; 2; 3; *F1*]
5. $0 \times a = 0 \,.$ [4]

$[\exists d].$

6. $d \epsilon A \,.$
7. $d + 0 = a \,.$ [1; *B1*; *B2*; 5]
8. $d \times 0 = 0 \,.$
9. $d \times a = d \times (d + 0) = (d \times d) + (d \times 0) = d + 0 = a \,.$ [1; 6; 7; *C1*; *E1*; 8; 7]
10. $a = a \times a = (d + 0) \times a = (d \times a) + (0 \times a) = a + 0 \,.$ [1; *E1*; 6; 7; *C2*; 9; 5]

$a = a + 0$ [10]

Z1 $[a]:a \epsilon A \,.\supset.\, a = O + a$

PR $[a]:\mathrm{Hp}(1).\supset.$
$[\exists c].$
2. $c \epsilon A.$
3. $c + a = a.$
4. $a \times c = O.$ [1; *B2**; *E1*]
5. $a = a \times a = a \times (c + a) = (a \times c) + (a \times a) = O + a.$
[1; *E1*; 2; 3; *C1*; 4; *E1*]
$a = O + a$ [5]
Z2 $0 = O$ [*B1*; *B1**; *F2*; *Z1*]
B3 $[ab]:a,b \epsilon A.b \times a = a.\supset.[\exists c].c \epsilon A.c + a = b.a \times c = 0$ [*B2**; *Z2*]

2.2 Since, obviously, in the field of the axioms *A1-A9* {*B1*; *B2*; *B3*} → {*B1**; *B2**} and {*E1*; *F1*} → {*E2*}, the proof given above in section **2.1** allows us to reformulate definition (**A**) of the system 𝔄, as follows:

(**B**) *Any algebraic system*

$$\mathfrak{A} = \langle A, =, +, \times, 0\rangle$$

with one binary relation =, *two binary operations* + *and* ×, *and one constant element* 0, *is a relatively complemented mixed associative Newman algebra if and only if it satisfies the postulates A1-A9, B1, B2, B3, C1, C2, E1 and F1.*

Since the definitions (**A**) and (**B**) of the system 𝔄 are inferentially equivalent and the set of postulates given in (**B**) is much simpler than the set accepted in (**A**), our further investigations will be based on formulation (**B**) of the system 𝔄.

3 For our end we need to show that several formulas are valid in the field of the axiom-system given in (**B**). Hence, if we assume the mentioned set of postulates, then we have *F2*, *cf.*, its proof in section **2.1**, and, moreover:

F3 $[a]:a \epsilon A.\supset.a = 0 + a$ [*E1*; *B2*; *C2*; *cf.* proof of **Q1** in [4], p. 35]
F4 $[abc]:a,b,c \epsilon A.c + a = b.c \times a = 0.\supset.b \times a = a$
[*C2*; *F3*; *E1*; *cf.* proof of **Q2** in [4], p. 35]
F5 $[abc]:a,b,c \epsilon A.c + a = b.a \times c = 0.\supset.a \times b = a$ [*C1*; *F3*; *E1*]
F6 $[ab]\therefore a,b \epsilon A.\supset:a \times b = a.\equiv.b \times a = a$
[*B2*; *F4*; *B3*; *F5*; *cf.* **Q17** in [4], p. 40]
F7 $[ab]:a,b \epsilon A.\supset.b \times (a \times b) = a \times b$ [*F1*; *F6*; *cf.* **Q17** in [4], p. 40]
F8 $[abcd]:a,b,c,d \epsilon A.a + b = d.a \times b = 0.a + c = d.a \times c = 0.\supset.b = c$
PR $[abcd]:\mathrm{Hp}(5).\supset.$
6. $b = (a \times b) + (b \times b) = (a + b) \times b = (a + c) \times b = (a \times b) + (c \times b) = c \times b.$
[1; *F3*; 3; *E1*; *C2*; 2; 4; *C2*; 3; *F3*]
$b = b \times c = (a \times c) + (b \times c) = (a + b) \times c = (a \times c) \times c = c$
[1; 6; *F6*; *F3*; 5; *C2*; 2; 4; *C2*; 5; *E1*; *F3*]
F9 $[ab]:a,b \epsilon A.a + a = a.\supset.(a \times b) + (a \times b) = a \times b$ [*C2*]
F10 $[abc]:a,b,c \epsilon A.c + a = b.a \times c = 0.\supset.b \times c = c$
[*C2*; *E1*; *F2*; *cf.* **Q2** in [4], p. 35]
F11 $[abc]:a,b,c \epsilon A.c + a = b.c \times a = 0.\supset.c \times b = c$ [*C1*; *E1*; *F2*]

F12 $[abc]: a,b,c \in A . c + a = b . c \times a = 0 . \supset . a \times c = 0$

PR $[abc]: \mathrm{Hp}(3) . \supset .$

4. $c = c \times b .$ [1; 2; 3; *F11*]

5. $b \times a = a .$ [1; 2; 3; *F4*]

$[\exists d] .$

6. $d \in A .$

7. $d + a = b .$ [1; *B3*; 5]

8. $a \times d = 0 .$

9. $b \times d = d .$ [1; 6; *F10*; 7; 8]

10. $c = c \times b = c \times (d + a) = (c \times d) + (c \times a) = (c \times d) + 0$ [1; 4; 6; 7; *C1*; 3]

$= (c \times d) + (a \times d) = (c + a) \times d = b \times d = d .$ [8; *C2*; 2; 9]

$a \times c = 0$ [8; 10]

F13 $[abc]: a, b, c \in A . c + a = b . a \times c = 0 . \supset . c \times a = 0$

[Similar proof: *F10*; *F5*; *B2*; *F11*; *C2*; *C1*]

F14 $[abcd]: a, b, c, d \in A . c + a = b . c \times a = 0 . d + a = b . d \times a = 0 . \supset . c = d$

PR $[abcd]: \mathrm{Hp}(5) . \supset .$

6. $c = c \times b .$ [1; *F11*; 2; 3]

7. $d = d \times b .$ [1; *F11*; 4; 5]

8. $d = b \times d .$ [1; *F6*; 7]

9. $a \times d = 0 .$ [1; *F12*; 4; 5]

$c = c \times b = c \times (d + a) = (c \times d) + (c \times a) = (c \times d) + 0$ [1; 6; 4; *C1*; 3]

$= (c \times d) + (a \times d) = (c + a) \times d = b \times d = d$ [9; *C2*; 2; 8]

F15 $[ab]: a,b \in A . \supset . [\exists c] . c \in A . c + (a \times b) = b . c \times (a \times b) = 0$ [*B2*; *F1*]

Since we have *F14* and *F15*, we can introduce into the system the following definition:

DI $[abx] \therefore a,b,x \in A . \supset : b \div a = x . \equiv . x + (a \times b) = b . x \times (a \times b) = 0$ [*F14*; *F15*]

F16 $[ab]: a,b \in A . \supset . b \div a \in A$ [*F1*; *B2*; *DI*]

F17 $[ab]: a,b \in A . \supset . (b \div a) + (a \times b) = b$ [*F16*; *DI*]

F18 $[ab]: a,b \in A . \supset . (b \div a) \times (a \times b) = 0$ [*F16*; *DI*]

F19 $[a]: a \in A . \supset . 0 = (a \div a)$

PR $[a]: \mathrm{Hp}(1) . \supset .$

$0 = (a \div a) \times a = (a \div a) \times ((a \div a) + a)$ [1; *F18*; *E1*; *F17*; *E1*]

$= ((a \div a) \times (a \div a)) + ((a \div a) \times a)$ [*C1*]

$= (a \div a) + 0 = a \div a$ [*F18*; *F16*; *E1*; *F2*]

F20 $[abc]: a,b,c \in A . \supset . (b \div a) \times (a \times b) = (c \div c)$ [*F18*; *F19*]

F21 $[abc]: a,b,c \in A . b = c . \supset . b \div a = c \div a$ [1; *F17*; *F18*; *A9*; *F16*; *F14*]

F22 $[abc]: a,b,c \in A . a = c . \supset . b \div a = b \div c$ [1; *F17*; *F18*; *A8*; *F16*; *F14*]

F23 $[ab]: a,b \in A . \supset . (a \times b) \times (b \div a) = 0$ [*F16*; *F17*; *F18*; *F12*]

F24 $[ab]: a,b \in A . \supset . (b \div a) \times b = b \div a$

PR $[ab]: \mathrm{Hp}(1) . \supset .$

$(b \div a) \times b = (b \div a) \times ((b \div a) + (a \times b))$ [1; *F16*; *F17*]

$= ((b \div a) \times (b \div a)) + ((b \div a) \times (a \times b))$ [*C1*]

$= (b \div a) + 0 = b \div a$ [*E1*; *F18*; *F2*]

$F25$ $[ab]:a,b\epsilon A.\supset.b\times(b\div a)=b\div a$

PR $[ab]:\mathrm{Hp}(1).\supset.$

$b\times(b\div a)=((b\div a)+(a\times b))\times(b\div a)$ [1; $F16$; $F17$]

$=((b\div a)\times(b\div a))+((a\times b)\times(b\div a))$ [$C2$]

$=(b\div a)+0=b\div a$ [$E1$; $F23$; $F2$]

$F26$ $[ab]:a,b\epsilon A.a\times b=a.\supset.b\div(b\div a)=a$

PR $[ab]:\mathrm{Hp}(2).\supset.$

3. $b\times a=a.$ [1; 2; $F6$]

$b\div(b\div a)=(b\div(b\div a))\times b=(b\div(b\div a))\times((b\div a)+a)$ [1; $F16$; $F24$; $F17$; 2]

$=((b\div(b\div a))\times((b\div a)\times b))+((b\div(b\div a))\times a)$ [$C1$; $F24$]

$=0+((b\div(b\div a))\times a)=((b\div(b\div a))\times a)+0$ [$F18$; $F3$; $F2$]

$=((b\div(b\div a))\times a)+(((b\div a)\times b)\times a)$ [$F18$; $F24$; 2]

$=((b\div(b\div a))+((b\div a)\times b))\times a=b\times a=a$ [$C2$; $F17$; 3]

$F27$ $[ab]:a,b\epsilon A.b+b=b.\supset.(b\div a)+(b\div a)=b\div a$ [$F16$; $F24$; $C1$]

$F28$ $[\exists a].a\epsilon A$ [$B1$]

$F29$ $[a]:a\epsilon A.\supset.0\times a=0$ [$F19$; $E1$; $F18$; *cf.* **Q10** in [4], p. 36]

$F30$ $[a]:a\epsilon A.\supset.a\times 0=0$ [$B1$; $F29$; $F6$; *cf.* **Q10** in [4], p. 36]

$F31$ $[a]:a\epsilon A.\supset.(a+a)\times a=a+a$ [$C2$; $E1$]

$F32$ $[a]:a\epsilon A.\supset.(a+a)+(a+a)=a+a$ [$E1$; $C1$; $C2$; $E1$; *cf.* proof of $F27$ in [6], p. 261]

$F33$ $[a]:a\epsilon A.\supset.((a\div(a+a))\times a)+((a\div(a+a))\times a)=0$ [$A4$; $F16$; $C1$; $F31$; $F18$]

$F34$ $[a]:a\epsilon A.\supset.[\exists bc].b,c\epsilon A.b+b=0.c+c=c.a=b+c$

PR $[a]:\mathrm{Hp}(1).\supset.$

2. $a+a\epsilon A.$ [1; $A4$]

3. $(a\div(a+a))\epsilon A.$ [1; 2; $F16$]

4. $a=a\times a=((a\div(a+a))+(a+a))\times a$ [1; 2; 3; $E1$; $F17$; $F31$]

$=((a\div(a+a))\times a)+(a+a).$ [$C2$; $F31$]

$[\exists bc].b,c\epsilon A.b+b=0.c+c=c.a=b+c$ [3; 2; $F33$; $F32$; 4]

$F35$ $[abcd]:a,b,c,d\epsilon A.a+a=0.b+b=b.c+c=0.d+d=d.\supset.$

$(a+b)\times(c+d)=(a\times c)+(b\times d)$

[$C1$; $C2$; $F30$; $F29$; $F2$; $F3$; *cf.* proof of $F32$ in [6], p. 261]

Concerning $F34$ and $F35$ see a remark of Newman in [4], p. 37.

$F36$ $[ab]:a,b\epsilon A.a+b=0.\supset.a=b$

PR $[ab]:\mathrm{Hp}(2).\supset.$

3. $0=a+(b\times a).$ [1; $F29$; $C2$; $E1$]

4. $0=(a\times b)+b.$ [1; $F29$; $C2$; $E1$]

5. $a=a\times a=a\times((a\div(b\times a))+(b\times a))$ [1; $A5$; $E1$; $F16$; $F17$; $F1$]

$=((a\times(a\div(b\times a)))+0)+(b\times a)$ [$C1$; $F2$; $F7$]

$=((a\times(a\div(b\times a)))+((b\times a)\times(a\div(b\times a))))+(b\times a)$ [$F23$; $F1$]

$=((a+(b\times a))\times(a\div(b\times a)))+(b\times a)$ [$C2$]

$=0+(b\times a)=b\times a.$ [3; $F29$; $F3$]

$a=a\times b=0+(a\times b)$ [1; $A5$; $F16$; $F6$; 5; $F3$]

$=(((a\times b)+b)\times(b\div(a\times b)))+(a\times b)$ [4; $F29$]

$= (0 + (b \times (b \div (a \times b)))) + (b \times (a \times b))$ [$C2$; $F23$; $F1$; $F7$]
$= b \times ((b \div (a \times b)) + (a \times b)) = b \times b = b$
[$F3$; $C1$; $F17$; $F1$; $E1$; *cf.* **Q18** in [4], p. 41]

F37 $[ab] : a, b \epsilon A . a \times b = a . \supset . a + b = b + a$
PR $[ab] : \mathrm{Hp}(2) . \supset .$
3. $b \times a = a .$ [1; $F6$; 2]
$a + b = (a + b) \times b = (a + b) \times ((b \div a) + a)$ [1; $F16$; 2; $E1$; $C2$; $F17$; 2]
$= ((b \times (b \div a)) + (a \times (b \div a))) + ((b \times a) + (a \times a))$
[$C1$; $C2$; $F23$; 2; $F3$; $F2$; $F23$; 2; $E1$; 3; $E1$]
$= (b + a) \times ((b \div a) + a) = (b + a) \times b = b + a$
[$C2$; $C1$; $F17$; 2; $C2$; $E1$; 2]

F38 $[abc] : a, b, c \epsilon A . a \times c = a . b \times c = b . \supset . a + b = b + a$
PR $[abc] : \mathrm{Hp}(3) . \supset .$
4. $c \times a = a .$ [1; 2; $F6$]
5. $c \times b = b .$ [1; 3; $F6$]
6. $a + b = (a + b) \times (a + b)$ [1; $E1$]
$= ((c \times a) + (b \times a)) + ((a \times b) + (c \times b))$ [$C1$; $C2$; $E1$; 4; 5]
$= ((c + b) \times a) + ((a + c) \times b)$ [$C2$]
$= ((b + c) \times a) + ((c + a) \times b)$ [$F37$; 3; 2]
$= ((b + a) \times a) + ((b + a) \times b) = (b + a) \times (a + b) .$
[$C2$; 4; 5; $E1$; $C1$]
7. $b + a = (a + b) \times (b + a) .$
[Similar proof: 1; 2; 3; 4; 5; $E1$; $C1$; $C2$; $F37$]
8. $b + a = (b + a) \times (a + b) .$ [1; $A4$; 7; $F6$]
$a + b = b + a$ [1; 6; 8]

F39 $[abcd] : a, b, c, d \epsilon A . a \times d = a . b \times d = b . c \times d = c . \supset .$
$a + (b + c) = (a + b) + c$
PR $[abcd] : \mathrm{Hp}(4) . \supset .$
5. $d \times a = a .$ [1; 2; $F6$]
6. $d \times b = b .$ [1; 3; $F6$]
7. $d \times c = c .$ [1; 4; $F6$]
8. $d + (d + c) = (d + (d + c)) \times ((d \div c) + c)$ [1; $F16$; $E1$; 4; $C2$; $F17$; 4]
$= ((d + d) \times (d \div c)) + ((c + (d + d)) \times c)$
[$C1$; $C2$; $F23$; 4; $F2$; $C2$; 7; $E1$; $C2$]
$= (((d + d) \times (d \div c)) + (c \times (d \div c))) +$
$(((d + d) + c) \times c)$ [$F2$; $F23$; 4; $A4$; $F31$; 4; $F38$]
$= ((d + d) + c) \times ((d \div c) + c) = (d + d) + c .$
[$C2$; $C1$; $F17$; 4; $C2$; $E1$; 4]
9. $d + (b + c) = (d + (b + c)) \times ((d \div b) + b)$ [1; $F16$; $E1$; 3; 4; $C2$; $F17$; 3]
$= ((d \times (d \div b)) + (c \times (d \div b))) +$
$((d + (d + c)) \times b)$ [$C1$; $C2$; $F23$; 3; $F3$; $E1$; 6; $C2$]
$= (((d \times (d \div b)) + (b \times (d \div b))) + (c \times (d \div b))) +$
$(((d + d) + c) \times b)$ [$F2$; $F23$; 3; 8]
$= ((d + b) + c) \times ((d \div b) + b) = (d + b) + c .$
[$C2$; 6; $E1$; $C2$; $C1$; $F17$; 3; $C2$; 2; 3]
$a + (b + c) = (a + (b + c)) \times ((d \div a) + a)$ [1; $F16$; 2; 3; 4; $C2$; $F17$; 2]

$= ((b \times (d \div a)) + (c \times (d \div a))) +$
$((d + (b + c)) \times a)$ [$C1$; $C2$; $F23$; 2; $F3$; $E1$; 5; $C2$]
$= (((a \times (d \div a)) + (b \times (d \div a))) + (c \times (d \div a))) +$
$(((d + b) + c) \times a)$ [$F3$; $F23$; 2; 9]
$= ((a + b) + c) \times ((d \div a) + a) = (a + b) + c$
[$C2$; 2; $E1$; $C2$; $C1$; $F17$; 2; 3; 4; $C2$]

Concerning the proofs of *F38* and *F39 cf.* **Q19** in [4], p. 41 and the proofs of **P17** and **P18** in [2], p. 260.

F40 $[ab] : a, b \epsilon A . a + a = 0 . \supset . (a \times b) + (a \times b) = 0$ [$C2$; $F29$]
F41 $[abc] : a, b, c \epsilon A . b + b = 0 . a + (b \times c) = c + (b \times c) . \supset . a = c$
PR $[abc] : \text{Hp}(3) . \supset .$
4. $((b \times c) \times a) + (b \times c) = (b \times c) \times (c + (b \times c))$ [1; $E1$; $C1$; 3]
$= (b \times c) + (b \times c) = 0 .$ [$C1$; $F1$; $E1$; $F40$; 2]
5. $(b \times c) \times a = (b \times c) .$ [1; 4; $F36$]
$a = a + ((b \times c) + (b \times c)) = (a + (b \times c)) + (b \times c)$
[1; $F2$; $F40$; 2; $F39$; $E1$; 5]
$= (c + (b \times c)) + (b \times c) = c + ((b \times c) + (b \times c)) = c$
[3; $F39$; $E1$; $F1$; $F40$; 2; $F2$]
F42 $[abcd] : a, b, c, d \epsilon A . b + b = 0 . c + c = 0 . d + d = 0 . c \times a = c .$
$d \times b = d . (a + c) + (d + b) = a + b . \supset . c = d$
PR $[abcd] : \text{Hp}(7) . \supset .$
8. $a \times c = c .$ [1; 5; $F6$]
9. $(a \times c) + c = 0 .$ [1; 8; 3]
10. $c + (b \times c) = (a \times c) + (b \times c) = (a + b) \times c$ [1; 8; $C2$]
$= ((a + c) + (d + b)) \times c$ [7]
$= 0 + ((d \times c) + (b \times c)) = (d \times c) + (b \times c) .$ [$C2$; $E1$; 9; $F3$]
11. $c = d \times c .$ [1; $F41$; 2; 10]
12. $(d \times d) + (d \times b) = 0 .$ [1; 4; $E1$; 6]
13. $(d \times a) + c = ((d \times a) + (d \times c)) + ((d \times d) + (d \times b))$ [1; 11; $F2$; 12]
$= d \times ((a + c) + (d + b)) = d \times (a + b)$ [$C1$; 7]
$= (d \times a) + d .$ [$C1$; 6]
14. $(d \times a) + c = ((d \times a) + c) \times a = ((d \times a) + d) \times a = 0 .$
[1; $F1$; 5; $C2$; 13; $C2$; $F1$; $F40$; 4]
15. $d \times a = c .$ [1; $F36$; 14]
$c = d$ [1; 13; 15; 3; $F36$]
F43 $[ab] : a, b \epsilon A . b + b = 0 . \supset . (a \times b) + (a \times b) = 0$ [$C1$; $F30$]
F44 $[ab] : a, b \epsilon A . b + b = 0 . \supset . a \times b = b \times a$
PR $[ab] : \text{Hp}(2) . \supset .$
3. $a + b = (a + b) \times (a + b) = (a + (b \times a)) + ((a \times b) + b) .$ [1; $E1$; $C1$; $C2$]
$a \times b = b \times a$ [1; $F42$; 2; $F40$; $F43$; $F1$; 3]
F45 $[abc] : a, b, c \epsilon A . b + b = 0 . \supset . a \times (b \times c) = (a \times b) \times c$
PR $[abc] : \text{Hp}(2) . \supset .$
3. $a \times b = b \times a .$ [1; 2; $F44$]
4. $(b \times a) + (b \times a) = 0 .$ [1; 2; $F40$]
5. $(b \times c) + (b \times c) = 0 .$ [1; 2; $F40$]
6. $(b \times c) \times a = a \times (b \times c) .$ [1; 5; $F44$]

7. $((b \times a) \times c) + ((b \times a) \times c) = 0$. [1; 4; *F40*]
8. $((b \times c) \times a) + ((b \times c) \times a) = 0$. [1; 5; *F40*]
9. $((b \times a) \times c) \times (b \times a) = (c \times (b \times a)) \times (b \times a)$ [1; 4; *F44*]
$= (b \times a) \times c$. [*F1*; 4; *F44*]
10. $((b \times c) \times a) \times (b \times c) = (a \times (b \times c)) \times (b \times c)$ [1; 6]
$= (b \times c) \times a$. [*F1*; 4; *F44*]
11. $(b \times a) + (b \times c) = b \times (a + c) = (b \times (a + c)) \times (a + c)$ [1; *C1*; *F1*]
$= ((b \times a) + ((b \times c) \times a)) +$
$(((b \times a) \times c) + (b \times c))$. [*C1*; *C2*; *F1*]
12. $(b \times c) \times a = (b \times a) \times c$. [1; *F42*; 5; 7; 8; 9; 10; 11]
$a \times (b \times c) = (a \times b) \times c$ [1; 12; 6; 3]

The theses *F41*, *F42*, *F44* and *F45* are stronger than the corresponding formulas **Q20**, **Q21**, **Q22** and **Q23** proven in [4], pp. 41-42.

F46 $[ab] : a, b \in A . b + b = b . \supset . a + (b \times a) = a$
PR $[ab] : \mathrm{Hp}(2) . \supset .$
$a + (b \times a) = (a + (b \times a)) \times a = (a + (b \times a)) \times$
$((a \div b) + (b \times a))$ [1; *F16*; *E1*; *F1*; *C2*; *F17*]
$= (a \div b) + ((a \times (b \times a)) + (b \times a))$
[*C1*; *C2*; *F25*; *F23*; *F2*; *E1*]
$= (a \div b) + (b \times a) = a$ [*F7*; *C2*; 2; *F17*]

F47 $[ab] : a, b \in A . b + b = b . \supset . (b \times a) + a = a$
[Similar proof: *F7*; *E1*; *C1*; *F16*; *F17*; *C2*; *C1*; *F18*; *F3*; *F24*; *E1*; *F1*; *C2*; *F17*]

Concerning *F46* and *F47* *cf.* **Q26** in [4], p. 42.

F48 $[ab] : a, b \in A . a + a = a . b + b = b . \supset . (a + b) + (a + b) = a + b$
PR $[ab] : \mathrm{Hp}(3) . \supset .$
$[\exists cd]$.
4. $c, d \in A$.
5. $c + c = 0$.
6. $d + d = d$.
7. $a + b = c + d$. } [1; *A4*; *F34*]
8. $a + b = (a + b) \times (c + d) = ((a + b) \times c) + ((c + d) \times d)$
[1; 4; *E1*; 7; *C1*; 7]
$= ((a \times 0) + (b \times 0)) + ((0 \times d) + d) = d$.
[2; 3; 5; 6; *C2*; *C1*; *E1*; *F30*; *F29*; *F3*]
$(a + b) + (a + b) = a + b$ [1; 4; 8; 6; 8; *cf.* **Q13** in [4], p. 37]

F49 $[abcdefg] : a, b, c, d, e, f, g \in A . a + a = a . b + b = b . c + c = c . d + d = d .$
$e + e = e . f + f = f . g + g = g . \supset . [\exists m] . m \in A . m + m = m . a \times m = a .$
$b \times m = b . c \times m = c . d \times m = d . e \times m = e . f \times m = f . g \times m = g$
PR $[abcdefg] : \mathrm{Hp}(8) . \supset .$
9. $\mathfrak{A} = a + (b + (c + (d + (e + (f + g)))))$. [Abbreviation]
10. $\mathfrak{A} \in A$. [9; 1; *A4*]
11. $\mathfrak{A} + \mathfrak{A} = \mathfrak{A}$. [9; 1-8; *F48*]
12. $\mathfrak{A} \times a = a + ((b + (c + (d + (e + (f + g))))) \times a) = a$.
[9; 1; 10; *C2*; *E1*; 3-8; *A4*; *F48*; *F46*]

13. $a \times \mathfrak{A} = a$. [9; 1; 10; 12; *F6*]
14. $b \times \mathfrak{A} = b$.
[Similar proof: 9; 1; 10; *C2*; *E1*; 4-8; *A4*; *F48*; *F46*; *F47*; 2; *F6*]
15. $c \times \mathfrak{A} = c$.
[Similar proof: 9; 1; 10; *C2*; *E1*; 5-8; *A4*; *F48*; *F46*; *F47*; 3; 2; *F6*]
16. $d \times \mathfrak{A} = d$.
[Similar proof: 9; 1; 10; *C2*; *E1*; 6-8; *A4*; *F48*; *F46*; *F47*; 4; 3; 2; *F6*]
17. $e \times \mathfrak{A} = e$.
[Similar proof: 9; 1; 10; *C2*; *E1*; 7; 8; *A4*; *F48*; *F46*; *F47*; 5; 4; 3; 2; *F6*]
18. $f \times \mathfrak{A} = f$.
[Similar proof: 9; 1; 10; *C2*; *E1*; 8; *F46*; *F47*; 6; 5; 4; 3; 2; *F6*]
19. $g \times \mathfrak{A} = g$. [Similar proof: 9; 1; 10; *C2*; *E1*; *F47*; 7; 6; 5; 4; 3; 2; *E6*]
$[\exists m] . m \epsilon A . m + m = m . a \times m = a . b \times m = b . c \times m = c . d \times m = d .$
$e \times m = e . f \times m = f . g \times m = g$
[9; 10; 11; 13; 14; 15; 16; 17; 18; 19; *cf.* **Q29** in [4], p. 43]

F50 $[abc] : a, b, c \epsilon A . b + b = b . a \times c = a . b \times c = b . \supset .$
$c \div ((c \div a) \times (c \div b)) = a + b$

PR $[abc] : \mathrm{Hp}(4) . \supset .$
5. $0 = (b \times c) \times (c \div b) = (b \times ((c \div a) + a)) \times (c \div b)$ [1; *F16*; *F23*; *F17*; 3]
$= ((b \times (c \div a)) \times (c \div b)) + ((b \times a) \times (c \div b))$.
[*C1*; *C2*]
6. $(b \times (c \div a)) \times (c \div b) = (b \times a) \times (c \div b)$. [1; *F16*; *F36*; 5]
7. $0 = ((b \times (c \div a)) \times (c \div b)) + ((b \times (c \div a)) \times (c \div b))$ [1; *F16*; 5; 6]
$= (b \times (c \div a)) \times (c \div b)$. [*C2*; 2]
8. $(a + b) \times c = a + b$. [1; *C2*; 3; 4]
9. $0 = (c \div (a + b)) \times (a + b) = ((c \div (a + b)) \times a) +$
$((c \div (a + b)) \times b)$. [1; *A4*; *F16*; *F18*; 8; *C1*]
10. $(c \div (a + b)) \times a = (c \div (a + b)) \times b$. [1; *A4*; *F16*; *F36*; 9]
11. $0 = ((c \div (a + b)) \times b) + ((c \div (a + b)) \times b)$ [1; *A4*; *F16*; 9; 10]
$= (c \div (a + b)) \times (b + b) = (c \div (a + b)) \times b$. [*C1*; 2]
12. $0 = (c \div (a + b)) \times a$. [1; *A4*; *F16*; 11; 10]
13. $(c \div (a + b)) \times (c \div b) = ((c \div (a + b)) \times (c \div b)) +$
$((c \div (a + b)) \times b)$ [1; *A4*; *F16*; *F2*; 11]
$= (c \div (a + b)) \times ((c \div b) + b)$ [*C1*]
$= (c \div (a + b)) \times c = c \div (a + b)$. [*F17*; 4; *F24*]
14. $(c \div (a + b)) \times (c \div a) = ((c \div (a + b)) \times (c \div a)) +$
$((c \div (a + b)) \times a)$ [1; *A4*; *F16*; *F2*; 12]
$= (c \div (a + b)) \times ((c \div a) + a)$ [*C1*]
$= (c \div (a + b)) \times c = c \div (a + b)$. [*F17*; 3; *F24*]
15. $b \times (c \div a) = (a \times (c \div a)) + (b \times (c \div a))$ [1; *F16*; *F3*; *F23*; 3]
$= (a + b) \times (c \div a)$. [*C2*]
16. $(c \div (a + b)) + (b \times (c \div a)) = ((c \div (a + b)) \times (c \div a)) +$
$((a + b) \times (c \div a)$ [1; *A4*; *F16*; 14; 15]
$= ((c \div (a + b)) + (a + b)) \times (c \div a)$ [*C2*]
$= c \times (c \div a) = c \div a$. [*F17*; 8; *F25*]
17. $(c \div a) \times (c \div b) = ((c \div (a + b)) + (b \times (c \div a))) \times (c \div b)$
[1; *A4*; *F16*; 16]

$= ((c \div (a + b)) \times (c \div b)) + ((b \times (c \div a)) \times (c \div b))$ [C2]
$= (c \div (a + b)) + 0 = c \div (a + b)$. [13; 7; F2]
$c \div ((c \div a) \times (c \div b)) = c \div (c \div (a + b)) = a + b$
[1; A4; F16; F26; 17; 8]

F51 $[abc]: a, b, c \epsilon A . a \times c = a . b \times c = b . (a \times b) \times c = a \times b . (b \times a) \times c = b \times a . c + c = c . \supset . a \times b = b \times a$

PR $[abc]: \mathrm{Hp}(6) . \supset .$

7. $c \div (a \times b) = c \div ((c \div (c \div a)) \times (c \div (c \div b)))$ [1; A4; F16; F26; 2; 3]
$= (c \div a) + (c \div b) = (c \div b) + (c \div a)$ [F50; F27; 6; F24; F38]
$= c \div ((c \div (c \div b)) \times (c \div (c \div a)))$ [F50; F27; 6; F24]
$= c \div (b \times a)$. [F26; 3; 2; 1]

$a \times b = b \times a$ [1; A4; F16; F8; F17; 4; F18; 4; F17; 5; F18; 5; 7]

F52 $[ab]: a, b \epsilon A . a + a = a . b + b = b . \supset . a \times b = b \times a$ [F9; F49; F51]

F53 $[ab]: a, b \epsilon A . \supset . a \times b = b \times a$
[F44; F52; F34; F35; *cf.* proof of F33 in [6], p. 261]

F54 $[abcd]: a, b, c, d \epsilon A . d + d = d . a \times d = a . b \times d = b . c \times d = c . (a \times b) \times d = a \times b . (b \times c) \times d = b \times c . (a \times (b \times c)) \times d = a \times (b \times c) . ((a \times b) \times c) \times d = (a \times b) \times c . \supset . a \times (b \times c) = (a \times b) \times c$

PR $[abcd]: \mathrm{Hp}(9) . \supset .$

10. $d \div (a \times (b \times c)) = d \div ((d \div (d \div a)) \times (d \div (d \div (b \times c))))$ [1; A5; F16; F26; 3; F26; 7]
$= (d \div a) + (d \div (b \times c))$ [F50; F27; 2]
$= (d \div a) + (d \div ((d \div (d \div b)) \times (d \div (d \div c))))$ [F26; 4; F26; 5]
$= (d \div a) + ((d \div b) + (d \div c))$ [F50; F27; 2]
$= ((d \div a) + (d \div b)) + (d \div c)$ [F39; F27; 2]
$= (d \div ((d \div (d \div a)) \times (d \div (d \div b)))) + (d \div c)$ [F50; F27; 2]
$= (d \div (a \times b)) + (d \div c)$ [F26; 3; F26; 4]
$= d \div ((d \div (d \div (a \times b))) \times (d \div (d \div c)))$ [F50; F27; 2]
$= d \div ((a \times b) \times c)$. [F26; 6; F26; 5]

$a \times (b \times c) = (a \times b) \times c$
[1; A5; F16; F8; F17; 8; F18; 8; F17; 9; F18; 9; 10]

F55 $[abc]: a, b, c \epsilon A . a + a = a . b + b = b . c + c = c . \supset . a \times (b \times c) = (a \times b) \times c$ [A5; F9; F49; F54]

F56 $[abc]: a, b, c \epsilon A . \supset . a \times (b \times c) = (a \times b) \times c$
[F45; F55; F34; F35; *cf.* proof of L1 in [7], p. 267]

Since system $\mathfrak{A}$ contains *E1* as one of its postulates, and in its field the formulas *F53* and *F56* are provable, we know that the multiplicative operation $\times$ of the mixed associative Newman algebras is idempotent, commutative and associative. Concerning the proofs of *F53* and *F56* it should be noticed that:

(a) Entirely the same modes of reasoning which were used above in order to obtain *F53* and *F56* allow us to prove these formulas in the field of $\mathfrak{A}$ without the applications of the defined operation $\div$. However, in such cases the proofs would be longer and less transparent due to the necessity of the constant use of particular quantifiers;

and that:

(b) In order to show that *F53* and *F56* are valid in 𝔄 we have to prove before hand that the formulas *F44*, *F45*, *F52* and *F55* are the consequences of the axiom-system of 𝔄. Although, as it is mentioned above, *F44* and *F45* are stronger than the corresponding formulas **Q22** and **Q23**, *cf.* [4], p. 42, their proofs are essentially the same as Newman's proofs of **Q22** and **Q23**. On the other hand, in this paper *F52* and *F55* are obtained in a completely different way than the corresponding formulas established by Newman in [4], pp. 42-43. The proofs presented here of *F52* and *F55* are rather similar, in some respects, to the deductions which allowed Newman to prove the formulas **P31** and **P32** in [2], p. 263.

4 In this section we shall prove the validity of the following formalization of mixed associative Newman algebras:

(**C**) *Any algebraic system*

$$\mathfrak{B} = \langle B, =, +, \times, \div \rangle$$

with one binary relation = *and three binary operations* +, × *and* ÷, *is a relatively complemented associative Newman algebra if and only if it satisfies the postulates A1-A9 given in* (**A**), *adjusted to the carrier set B of* 𝔅, *and, additionally, the following axioms:*

$A10 \quad [ab]:a, b \in B . \supset . b \div a \in B$
$A11 \quad [\exists a] . a \in B$

$G1 \quad [abc]:a, b, c \in B . \supset . a \times (b + c) = (a \times b) + (a \times c)$
$G2 \quad [ab]:a, b \in B . \supset . a \times b = b \times a$
$G3 \quad [ab]:a, b \in B . \supset . (b \div a) + (a \times b) = b$
$G4 \quad [abc]:a, b, c \in B . \supset . (b \div a) \times (a \times b) = c \div c$

The proof presented below that system 𝔅 is a correct formalization of the algebras under consideration will imply at once the desired theorems. Namely:

Theorem 1. *The mixed associative Newman algebras can be axiomatized equationally*,

and

Theorem 2. *The mixed associative Newman algebras can be considered as semi-lattices with respect to the primitive binary operation* × *to which the additional postulates are added concerning the properties of the primitive operations* + *and* ÷.

4.1 Concerning the primitive binary operation ÷ of the system 𝔅 it should be remarked that this operation is not a pseudo-difference (such terminology is used, e.g., in [5], p. 57) which is a familiar primitive operation in Brouwerian algebras. The following algebraic table, *cf.* [7], p. 266:

$+$	O	α
O	O	α
α	α	O

$\times$	O	α
O	O	O
α	O	α

$\div$	O	α
O	O	O
α	α	O

verifies all postulates of system $\mathfrak{B}$, but falsifies a formula

(α) $\quad [abc]:a,b,c\,\epsilon\,B\,.\supset.\,(a \div c)\times((a+b)\div c) = a \div c$

which corresponds to one of the proper axioms of Brouwerian algebra, *cf.* [1], p. 125:

$$a \div c \leq (a+b) \div c$$

Namely, formula (α) fails for a/α, b/α and c/O: (i) $(\alpha \div O)\times((\alpha+\alpha)\div O) = \alpha\times(O\div O) = \alpha\times O = O$, and (ii) $\alpha \div O = \alpha$.

4.2 Now, let us assume the axioms of system $\mathfrak{B}$. Then:

G5 $\quad [abc]:a,b,c\,\epsilon\,B\,.\supset.\,(a+b)\times c = (a\times c)+(b\times c)$ $\quad$ [*G1*; *G2*]

G6 $\quad [a]:a\,\epsilon\,B\,.\supset.\,a\times a = a\times(a\times a)$

PR $\quad [a]:\text{Hp}(1)\,.\supset.$

$a\times a = ((a\times a)\div(a\times a))+((a\times a)\times(a\times a))$ $\quad$ [1; *A5*; *G3*]

$= ((a\div a)\times(a\times a))+((a\times a)\times(a\times a))$ $\quad$ [*G4*]

$= ((a\div a)+(a\times a))\times(a\times a) = a\times(a\times a)$ $\quad$ [*G5*; *G4*]

G7 $\quad [a]:a\,\epsilon\,B\,.\supset.\,a\times a = ((a\times a)\div a)+(a\times a)$

PR $\quad [a]:\text{Hp}(1)\,.\supset.$

$a\times a = ((a\times a)\div a)+(a\times(a\times a))$ $\quad$ [1; *A5*; *G3*]

$= ((a\times a)\div a)+(a\times a)$ $\quad$ [*G6*]

G8 $\quad [a]:a\,\epsilon\,B\,.\supset.\,a\times a = (a\times a)\times(a\times a)$

PR $\quad [a]:\text{Hp}(1)\,.\supset.$

$a\times a = ((a\times a)\div(a\times a))+((a\times a)\times(a\times a))$ $\quad$ [1; *A5*; *G3*]

$= (((a\times a)\div a)\times(a\times(a\times a)))+((a\times a)\times(a\times a))$ $\quad$ [*G4*]

$= (((a\times a)\div a)\times(a\times a))+((a\times a)\times(a\times a))$ $\quad$ [*A9*; *G6*]

$= (((a\times a)\div a)+(a\times a))\times(a\times a)$ $\quad$ [*G5*]

$= (a\times a)\times(a\times a)$ $\quad$ [*G7*]

G9 $\quad [a]:a\,\epsilon\,B\,.\supset.\,a = a\times a$

PR $\quad [a]:\text{Hp}(1)\,.\supset.$

$a = (a\div a)+(a\times a)$ $\quad$ [1; *G3*]

$= ((a\div a)\times(a\times a))+((a\times a)\times(a\times a))$ $\quad$ [*G4*; *G8*]

$= ((a\div a)+(a\times a))\times(a\times a) = a\times(a\times a) = a\times a$ $\quad$ [*G5*; *G3*; *G6*]

G10 $\quad [a]:a\,\epsilon\,B\,.\supset.\,(a\div a)+a = a$ $\quad$ [*G3*; *G9*]

G11 $\quad [ab]:a,b\,\epsilon\,B\,.\supset.\,(a\times b)\times b = a\times b$

PR $\quad [ab]:\text{Hp}(1)\,.\supset.$

$(a\times b)\times b = b\times(a\times b) = ((b\div a)+(a\times b))\times(a\times b)$ $\quad$ [1; *G2*; *G3*]

$= ((b\div a)\times(a\times b))+((a\times b)\times(a\times b))$ $\quad$ [*G5*]

$= ((a\times b)\div(a\times b))+(a\times b) = a\times b$ $\quad$ [*G4*; *G9*; *G10*]

G12 $\quad [ab]:a,b\,\epsilon\,B\,.\supset.\,a\div a = b\div b$ $\quad$ [*G4*]

Therefore, having *G12* we can introduce into the system the following definition:

D1 $\quad [a]:a\,\epsilon\,B\,.\supset.\,0 = a\div a$ $\quad$ [*G12*]

G13 $0 \epsilon B$ [*A11*; *A10*; *D1*]

G14 $[ab]:a,b \epsilon B \,.\, a \times b = a \,.\supset.\, [\exists c].\, c \epsilon B \,.\, c + a = b \,.\, c \times a = 0$ [*A10*; *G3*; *G4*; *D1*]

G15 $[ab]:a,b \epsilon B \,.\, b \times a = a \,.\supset.\, [\exists c].\, c \epsilon B \,.\, c + a = b \,.\, a \times c = 0$ [*G14*; *G2*]

G16 $[a]:a \epsilon B \,.\supset.\, a = 0 + a$ [*G10*; *D1*]

G17 $[abc]:a,b,c \epsilon B \,.\supset.\, (b \div b) \times a = c \div c$ [*G12*; *G9*; *G4*]

G18 $[a]:a \epsilon B \,.\supset.\, a = a + 0$

PR $[a]:\mathrm{Hp}(1) \,.\supset.$

2. $(a \div (a \div a)) + (a \div a) = a \,.$ [1; *G3*; *G17*]
3. $(a \div (a \div a)) \times (a \div a) = a \div a \,.$ [1; *G4*; *G17*]
4. $(a \div (a \div a)) \times a = (a \div (a \div a)) \times ((a \div (a \div a)) + (a \div a))$ [1; 2]

$= ((a \div (a \div a)) \times (a \div (a \div a))) + ((a \div (a \div a)) \times (a \div a))$ [*G1*]

$= (a \div (a \div a)) + (a \div a) = a \,.$ [*G9*; 3; 2]

$a = a \times a = ((a \div (a \div a)) + (a \div a)) \times a$ [1; *G9*; 2]

$= ((a \div (a \div a)) \times a) + ((a \div a) \times a) = a + 0$ [1; *G5*; 4; *G17*; *D1*]

G19 $[ab]:a,b \epsilon B \,.\supset.\, (b \div a) \times b = b \div a$ [*G3*; *G1*; *G9*; *G4*; *D1*; *G18*]

G20 $[abx]:a,b,x \epsilon B \,.\, x + (a \times b) = b \,.\, x \times (a \times b) = 0 \,.\supset.\, b \div a = x$

PR $[abc]:\mathrm{Hp}(3) \,.\supset.$

$b \div a = (b \div a) \times b = b \times (b \div a) = (x + (a \times b)) \times (b \div a)$ [1; *G19*; *G2*; 2]

$= (x \times (b \div a)) + ((a \times b) \times (b \div a))$ [*G5*]

$= (x \times (b \div a)) + 0 = (x \times (b \div a)) + (x \times (a \times b))$ [*G2*; *G4*; *D1*; 3]

$= x \times ((b \div a) + (a \times b)) = x \times b = x \times (x + (a \times b))$ [*G1*; *G3*; 2]

$= (x \times x) + (x \times (a \times b)) = x + 0 = x$ [*G1*; *G9*; 3; *G18*]

G21 $[abx]\therefore a,b,x \epsilon B \,.\supset:\, b \div a = x \,.\equiv.\, x + (a \times b) = b \,.\, x \times (a \times b) = 0$ [*G3*; *G4*; *D1*; *G20*]

G22 $[abc]:a,b,c \epsilon B \,.\, b = c \,.\supset.\, b \div a = c \div a$

PR $[abc]:\mathrm{Hp}(2) \,.\supset.$

3. $(b \div a) + (a \times c) = (b \div a) + (a \times b) = b = c \,.$ [1; 2; *A9*; *G3*; 2]
4. $(b \div a) \times (a \times c) = (b \div a) \times (a \times b) = 0 \,.$ [1; 2; *A9*; *G4*; *D1*]

$b \div a = c \div a$ [1; *A10*; *G20*; 3; 4]

G23 $[abcd]:a,b,c,d \epsilon B \,.\, c + a = b \,.\, c \times a = 0 \,.\, d + a = b \,.\, d \times a = 0 \,.\supset.\, c = d$

PR $[abcd]:\mathrm{Hp}(5) \,.\supset.$

6. $a \times b = a \times (c + a) = (a \times c) + (a \times a)$ [1; 2; *G1*]

$= (c \times a) + a = 0 + a = a \,.$ [*G2*; *G9*; 3; *G16*]

$c = d$ [1; 2; 3; 4; 5; 6; *G20*]

G24 $[abc]:a,b,c \epsilon B \,.\, a = c \,.\supset.\, b \div a = b \div c$

PR $[abc]:\mathrm{Hp}(2) \,.\supset.$

3. $(b \div c) + (a \times b) = (b \div c) + (c \times b) = b \,.$ [1; 2; *A8*; *G3*]
4. $(b \div c) \times (a \times b) = (b \div c) \times (c \times b) = 0 \,.$ [1; 2; *A8*; *G4*; *D1*]

$b \div a = b \div c$ [1; *G3*; *G4*; 3; 4; *G23*]

4.3 An inspection of the deductions presented above in **3** and **4.2** shows that:

(i) The theses *A1*-*A9*, *B1*, *B2*, *B3*, *C1*, *C2*, *E1*, *F1*, *F16*, *F28*, *F17*, *F20* and *F53* of $\mathfrak{A}$ correspond synonymously and respectively to the theses *A1*-*A9*, *G13*, *G14*, *G15*, *G1*, *G5*, *G9*, *G11*, *A10*, *A11*, *G3*, *G4* and *G2* of $\mathfrak{B}$;

and, moreover, that:

(ii) The theses *DI* and *F19* of $\mathfrak{A}$ correspond in the same manner to the theses *G21* and *D1* of $\mathfrak{B}$.

Therefore, due to (ii) it follows immediately from (i) that the system $\mathfrak{B}$ is a relatively complemented associative Newman algebra. Thus, Theorem 1 is proved. Furthermore, it should be noticed that:

(iii) Since (i) and (ii) establish that the systems $\mathfrak{A}$ and $\mathfrak{B}$ are inferentially equivalent, if their respective carrier sets are equal, or they are inferentially equivalent up to isomorphism, if their carrier sets have only the same cardinality, any theorem provable in the field of one of these systems, is also provable in the field of the other;

and that:

(iv) Since the theses *F21* and *F22* are provable in $\mathfrak{A}$, and the theses *G22* and *G24* are the consequences of $\mathfrak{B}$, the acceptance of the binary operation ÷, as a primitive notion in mixed associative Newman algebra, does not necessarily require any assumption of special postulates concerning the extensionality of the relation = with respect to this binary operation.

4.4 It is shown in **4.2** that *G9* holds in $\mathfrak{B}$. Moreover, since *F56* is provable in $\mathfrak{A}$, *cf.* section **3**, it follows from point (iii) of **4.3** that *F56* is also a consequence of the axiom-system of $\mathfrak{B}$. Therefore, Theorem 2 is proved. However, if in order to obtain a set of postulates of $\mathfrak{B}$ which could be considered as a semi-lattice system with respect to the operation × we shall accept the following set of formulas: *A1-A11*, *G9*, *G2*, *F56*, *G1*, *G3* and *G4*, as an axiom-system of $\mathfrak{B}$, then, obviously, such set of postulates will not be mutually independent. If for some reason it would be desired to have an axiom-system of $\mathfrak{B}$ such that its axioms would be mutually independent and that it would contain *G9*, *G2* and *F56*, then such axiomatization of $\mathfrak{B}$ can be easily obtained by reconstructing the set of axioms given above in a way analogous to that which was used in [8], p. 285.

5 The mutual independence of the axioms *G1*, *G2*, *G3* and *G4* is established by using the following algebraic tables (matrices):

$\mathfrak{M}9$

+	O	α	β	γ	δ
O	0	α	β	γ	δ
α	α	α	δ	δ	δ
β	β	δ	β	δ	δ
γ	γ	δ	δ	γ	δ
δ	δ	δ	δ	δ	δ

×	O	α	β	γ	δ
O	O	O	O	O	O
α	O	α	O	O	α
β	O	O	β	O	β
γ	O	O	O	γ	γ
δ	O	α	β	γ	δ

÷	O	α	β	γ	δ
O	O	O	O	O	O
α	α	O	α	α	O
β	β	β	O	β	O
γ	γ	γ	γ	O	O
δ	δ	β	γ	α	O

$\mathfrak{M}10$

$+$	α	I	O
α	α	I	α
I	I	I	I
O	α	I	O

$\times$	α	I	O
α	α	I	O
I	α	I	O
O	O	O	O

$\div$	α	I	O
α	O	O	α
I	O	O	I
O	O	O	O

$\mathfrak{M}11$

$+$	O	α
O	O	α
α	α	O

$\times$	O	α
O	O	O
α	O	O

$\div$	O	α
O	O	O
α	O	O

$\mathfrak{M}12$

$+$	O	α	β	γ
O	O	α	β	γ
α	α	O	γ	β
β	β	γ	O	α
γ	γ	β	α	O

$\times$	O	α	β	γ
O	O	O	O	O
α	O	α	γ	β
β	O	γ	β	α
γ	O	β	α	γ

$\div$	O	α	β	γ
O	O	O	O	O
α	α	O	β	γ
β	β	α	O	γ
γ	γ	α	β	O

Concerning the matrices $\mathfrak{M}9$-$\mathfrak{M}12$ *cf.* $\mathfrak{M}7$ in [8], p. 284, $\mathfrak{M}5$ and $\mathfrak{M}4$ in [6], p. 263, and Newman's example **E15** in [4], p. 46, respectively. Since:

(a) matrix $\mathfrak{M}9$ verifies *G2, G3* and *G4*, but falsifies *G1* for a/α, b/β and c/γ:
(i) $\alpha \times (\beta + \gamma) = \alpha \times \delta = \alpha$, and (ii) $(\alpha \times \beta) + (\alpha \times \gamma) = O + O = O$,
(b) matrix $\mathfrak{M}10$ verifies *G1, G3* and *G4*, but falsifies *G2* for a/α and b/I:
(i) $\alpha \times I = I$, and (ii) $I \times \alpha = \alpha$,
(c) matrix $\mathfrak{M}11$ verifies *G1, G2,* and *G4*, but falsifies *G3* for a/α and b/α:
(i) $(\alpha \div \alpha) + (\alpha \times \alpha) = O + O = O$, and (ii) $\alpha = \alpha$,
(d) matrix $\mathfrak{M}12$ verifies *G1, G2, G3,* but falsifies *G4* for a/β, b/α and c/α:
(i) $(\alpha \div \beta) \times (\beta \times \alpha) = \beta \times \gamma = \alpha$, and (ii) $\alpha \div \alpha = O$,

we know that the axioms *G1, G2, G3* and *G4* are mutually independent.

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