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KRIPKE'S AXIOMATIZATION OF S2

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The purpose of this note is to call attention to a slight inaccuracy in Kripke's axiomatization of S2 in [1]. I shall point out that if a certain unstated restriction on axiom generation is not followed, Kripke's axiomatization admits the provability of $\Box^n(p \supset p)$, where $\Box^n$ indicates a sequence of n necessity signs. And, of course, although $\Box(p \supset p)$ is an S2 theorem, $\Box\Box(p \supset p)$ is not, let alone $\Box^n(p \supset p)$ for all n.

On p. 208 of [1], Kripke presents an axiomatization of S2 in the following way. He first says that there will be axiom schemata sufficient for classical propositional calculus. He then adds two axiom schemata.

(A1) $(\Box A \supset A)$
(A3) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

Then he has a rule for generating axioms, and here I shall quote because this is the rule that must be restricted. The rule reads:

if B is an axiom and A is any formula then $\Box A \supset \Box B$ *is an axiom.*

I shall call this rule **AG** for axiom generator. In this system *Modus ponens* is the sole rule of proof. In a footnote on p. 208 of [1] he sketches a proof that the axiom system obtained so far is equivalent to Lemmon's E2 of [2]. He then claims that S2 is obtained if $\Box(A \supset A)$ is added as an axiom schema. Let us call $\Box(A \supset A)$ axiom schema (2). On p. 218 of [1], Kripke shows that S2 obtained in this way is equivalent to Lemmon's version of S2 in [2].

However, if one is not careful in using **AG**, it is easy to be misled into thinking that $\Box\Box(p \supset p)$ can be proved in the following way. (Here "p" is an object language sign and so is "$\supset$" when used with "p".)

(1)	$\Box(p \supset p)$	An axiom by schema (2)
(2)	$\Box(p \supset p) \supset \Box\Box(p \supset p)$	Rule **AG** applied to (1) to get (2) *as an axiom*
(3)	$\Box\Box(p \supset p)$	Modus ponens on (2) and (1)

We can now go on to prove $\Box^n(p \supset p)$ for all n. For instance, we can prove $\Box^3(p \supset p)$ as follows:

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(4) $\Box(p \supset p) \supset \Box(\Box(p \supset p) \supset \Box\Box(p \supset p))$

Rule **AG** is applied to the axiom of (2) to get a new axiom

(5) $\Box(\Box(p \supset p) \supset \Box\Box(p \supset p))$ Modus ponens on (1) and (4)

(6) $\Box(\Box(p \supset p) \supset \Box\Box(p \supset p)) \supset (\Box\Box(p \supset p) \supset \Box\Box\Box(p \supset p))$

By schema (A3), (6) is an axiom

(7) $\Box\Box(p \supset p) \supset \Box\Box\Box(p \supset p)$ Modus ponens on (5) and (6)

(8) $\Box\Box\Box(p \supset p)$ Modus ponens on (3) and (7)

This type of proof can be continued to get $\Box^n(p \supset p)$ for any n.

Of course, the proof went wrong at line 2 where the **AG** rule was applied to a formula which was an instance of axiom schema (2). Rule **AG** should be rewritten as:

if B is an axiom of the classical propositional calculus, an axiom by schema (A1) *or* (A3), *or an axiom obtained by this rule, then if A is any formula* $\Box A \supset \Box B$ *is an axiom.*

From the order in which he presented his axiomatization of S2, I am sure that Kripke intended the restriction that I have added to **AG**. Still it may be helpful to have it explicitly stated.

REFERENCES

[1] Kripke, Saul A., "Semantical Analysis of Modal Logic II. Non-normal Modal Propositional Calculi," *The Theory of Models* (Ed. by J. W. Addison, Leon Henkin, and A. Tarski), North Holland, Amsterdam (1965), pp. 206-220.

[2] Lemmon, E. J., "New Foundations for Lewis Modal Systems," *The Journal of Symbolic Logic*, vol. 22 (1957), pp. 176-186.

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