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# PUTTING $K$ IN ITS PLACE

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**1** The *Dugundji axioms*[1]

$$A_n \qquad \bigvee_{1 \leqslant i \neq j \leqslant 2^{n+1}} \Box (p_i \leftrightarrow p_j) \qquad (n \in \omega)$$

are of particular interest in the study of extensions of S4 having the finite model property, since for each $n$ the $n$th axiom has the attractive feature of being validated by a reflexive transitive generated frame $\mathfrak{F}$ if and only if $\mathfrak{F}$ contains at most $n$ worlds. Where $\mathsf{L}$ is any extension of S4 and $\mathsf{L}_n$ is the smallest normal extension of $\mathsf{L}$ to contain $A_n$, it is natural to make the following

Conjecture *If* $\mathsf{L}$ *is determined by any class of frames, then* $\mathsf{L}_n$ *is determined by the class of frames for* $\mathsf{L}$ *which contain at most n worlds.*

Some partial results along these lines were announced in [6] where I first entertained this conjecture. But thanks to the recent work of Fine [2] something even stronger can now be established. Since $A_n$ implies

$$\bigwedge_{1 \leqslant i \leqslant 2^{n+1}} \Diamond p_i \to \bigvee_{1 \leqslant i \neq j \leqslant 2^{n+1}} \Diamond (p_i \wedge (p_j \vee \Diamond p_j))$$

in the field of S4, it follows from Fine's very general completeness theorem that any extension of S4 having the axiom as a theorem is determined by the class of its at-most-$n$-membered frames. This has as an unexpected corollary that there are extensions $\mathsf{L}$ of S4 for which

$$\mathsf{L} \neq \bigcap_{n \in \omega} \mathsf{L}_n.$$

For as Fine has shown elsewhere [3], there exist extensions of S4—indeed a continuum of them—which are determined by *no* class of frames.

1. Made famous by Dugundji [1] and later used by Scroggs [8] to axiomatize all consistent extensions of S5. The axioms, as well as Dugundji's results, were evidently familiar to McKinsey at least a year before the appearance of Dugundji's paper (*cf.* [5]).

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In this paper it is shown that a result analogous to the Conjecture holds also for the *weakened* form $\Diamond A_n$ of the Dugundji axioms. It would be interesting to know whether Fine's argument in [2] can be generalized to cover the weakened form $\Diamond I_n$ of his axioms $I_n$. One could then strengthen Theorem 1 below to obtain a pair of results analogous to those which we have for the Dugundji axioms proper.

2 We shall use the familiar relational semantics for normal modal logics with *frames* $\mathfrak{F} = (W, R)$ and *models* $\mathfrak{A} = (W, R, \varphi)$ defined as usual. *Truth at* $w$ is indicated in the obvious way, letting

$$(\mathfrak{A}, w) \models \Box A \text{ iff } (v)(wRv \Rightarrow (\mathfrak{A}, v) \models A).$$

$\mathfrak{F}$ *validates* a formula $A$ if $(\mathfrak{A}, w) \models A$ for all $w \in W$ and models $\mathfrak{A}$ based upon $\mathfrak{F}$. $\mathfrak{F}$ is a *frame for* a logic $\mathsf{L}$ if $\mathfrak{F}$ validates every theorem of $\mathsf{L}$, and a class $\Gamma$ of frames *determines* $\mathsf{L}$ just in case $A$ is a theorem of $\mathsf{L}$ if and only if $A$ is validated by every $\mathfrak{F}$ in $\Gamma$. A subset $X$ of $W$ is said to be a *cluster* of $\mathfrak{F}$ if $R$ is universal on $X$ but on no proper superset, and $X$ is *final* if in addition $wRu$ and $w \in X$ imply $u \in X$.

Suppose $\mathfrak{A} = (W, R, \varphi)$ is a transitive model. Then $\mathfrak{A}^w = (U, S, \psi)$ is the submodel of $\mathfrak{A}$ *generated* by $w$ if $U = \{u \mid wRu\}$ and $S$ and $\psi$ are the restrictions of $R$ and $\varphi$ to $U$. It is of course easy to prove

Lemma 1 *$(\mathfrak{A}, w) \models A$ iff $(\mathfrak{A}^w, w) \models A$.*

The remainder of this section is given over to the description of a way in which one can alter the structure of a model while at the same time leaving certain important features of the model intact. Just as Lemma 1 shows that for many purposes attention can be restricted to generated frames, Lemma 2 below shows that one usually needs only be concerned with frames having no infinite clusters. Variations on the notion here described have been used elsewhere [7] to help obtain a large number of completeness results closely akin to the ones in the next section.

For each set $\Gamma$ of formulas, we let

$$C_\Gamma = \bigwedge_{B \in \mathrm{Sbfl}(A)} B^0$$

where $B^0$ is $B$ or $\sim B$ according as $B \in \Gamma$ or not. Let $\mathfrak{A}$ be the model $(W, R, \varphi)$ and put $\varphi(w) = \{B \in \mathrm{Sbfl}(A) \mid (\mathfrak{A}, w) \models B\}$. Define an equivalence relation on $W$ so that $w \approx u$ if and only if $w$ and $u$ belong to the same cluster of $(W, R)$ and $(\mathfrak{A}, u) \models C_{\varphi(w)}$. Where $[w]$ is the equivalence class of $w$ under $\approx$, let

$$U = \{[w] \mid w \in W\}$$
$$[x]S[u] \text{ iff } wRv \text{ for some } w \in [x] \text{ and } v \in [u]$$
$$\psi(p) = \{[w] \mid w \in \varphi(p)\} \text{ for each variable } p \in \mathrm{Sbfl}(A).$$

We shall then say that $(U, S, \psi)$ is a *contraction* of $\mathfrak{A}$ on $\mathrm{Sbfl}(A)$.

Lemma 2 *If $\mathfrak{B} = (U, S, \psi)$ is a contraction of $\mathfrak{A} = (W, R, \varphi)$ on $\mathrm{Sbfl}(A)$, then*

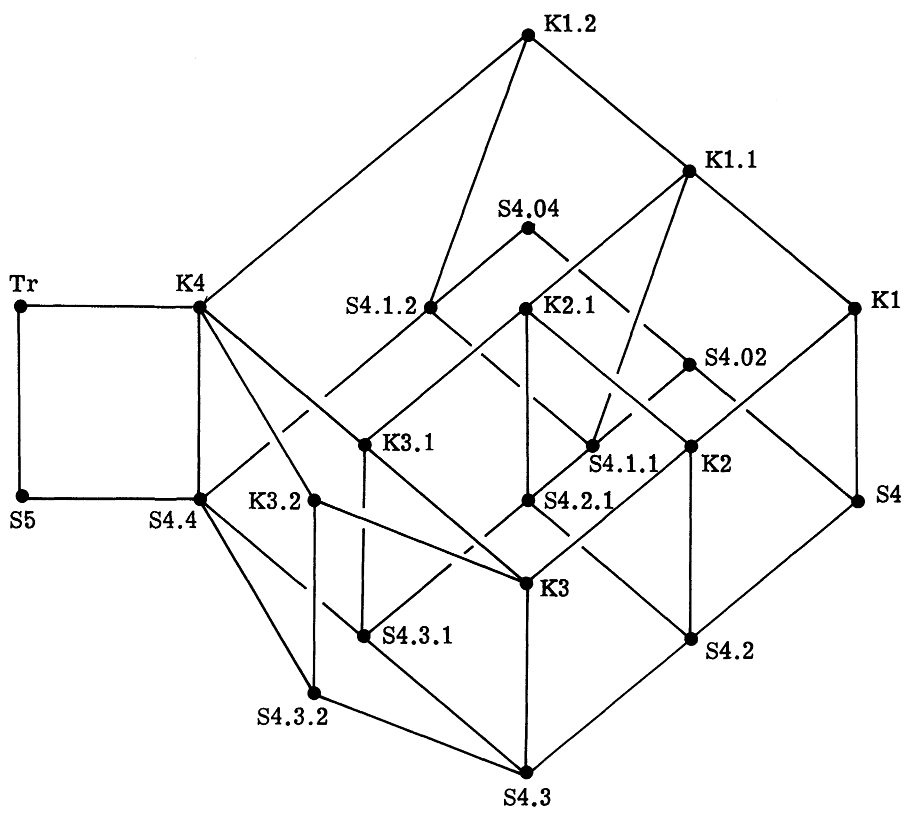
(i) *X is a final cluster of* $(U, S)$ *iff* $X = \{[w] | w \in Y\}$ *where* $Y$ *is a final cluster of* $(W, R)$,
(ii) *each cluster of* $(U, S)$ *is finite,*
(iii) $(\mathfrak{B}, [w]) \models B$ *iff* $(\mathfrak{A}, w) \models B$ *for each* $B \in \mathrm{Sbfl}(A)$ *and* $w \in W$,
(iv) *if* $(W, R)$ *is a frame for the logic* $\mathsf{L}$, *then* $(U, S)$ *is a frame for* $\mathsf{L}$.

*Proof:* Parts (i) and (ii) are obvious and (iii) is by induction on the construction of $B$. For (iv), suppose $(U, S)$ is not a frame for $\mathsf{L}$. Then for some theorem $B$ of $\mathsf{L}$ and $[w] \in U$, $(\mathfrak{D}, [w]) \not\models B$ for some model $\mathfrak{D} = (U, S, \sigma)$. Let $\mathfrak{M} = (W, R, \tau)$ be the model such that $(\mathfrak{M}, u) \models p$ iff $(\mathfrak{D}, [u]) \models p$ for each variable $p$ and $u \in W$. A straightforward induction reveals that $(\mathfrak{M}, w) \not\models B$, whence it follows that $(W, R)$ is not a frame for $\mathsf{L}$.

Lemma 3 *Suppose* $\mathrm{S4} \subsetneq \mathsf{L} \subseteq \mathrm{S5}$. *Then for each* $n \in \omega$ *there is a frame* $\mathfrak{F}$ *for* $\mathsf{L}$ *whose final clusters each contain exactly* $n$ *worlds.*

*Proof:* Let $\mathfrak{F}$ be any $n$-membered universal frame.

**3** Until recently, few extensions of S4 had appeared in the literature which did not belong to one of the two families of logics represented in the Figure. Each logic shown between S4 and S5 is known to be determined by the class




Figure

of its finite frames, while those belonging to Sobociński's family $\mathcal{K}$ are each determined by classes of finite frames whose final clusters each contain only one world. This suggests that there should be an infinite hierarchy of logics *between* those two families with $\mathcal{K}$ as a limiting case. There should exist a family of logics which are slightly weaker than the $\mathcal{K}$ systems and are determined by classes of frames whose final clusters each contain at most two worlds, a family of even weaker logics the final clusters of whose frames each contain at most three worlds, and so forth. And indeed this suspicion is correct.

Let $\mathsf{L}(n)$ be the smallest normal logic containing $\mathsf{L}$ and the $n$th weakened Dugundji axiom, where $\mathsf{L}$ is any logic between S4 and S5. It is readily seen that a finite frame for $\mathsf{L}$ is a frame for $\mathsf{L}(n)$ if and only if each final cluster of that frame contains at most $n$ worlds. Therefore, in light of Lemma 3, we have here a hierarchy of distinct new logics. What remains to be proven is that $\mathsf{L}(n)$ is determined by the frames in question, and hence that this hierarchy is in fact the one we are after.

Theorem 1 *If* $\mathsf{L}$ *is determined by a class of finite frames, then* $\mathsf{L}(n)$ *is determined by the class of finite frames for* $\mathsf{L}$ *whose final clusters each contain at most n worlds.*

*Proof:* It will suffice to show that every nontheorem of $\mathsf{L}(n)$ is invalidated by a frame for $\mathsf{L}$ of the sort in question. Thus suppose $A$ is a nontheorem of $\mathsf{L}(n)$. Then obviously $\Box B \rightarrow A$ is a nontheorem of $\mathsf{L}$, where $B$ is the formula

$$\bigwedge_{\substack{\Sigma \subseteq \wp(\mathrm{Sbfl}(\Box^n A)) \\ \mathrm{card}\,\Sigma = n+1}} \Diamond \bigvee_{\substack{\Gamma, \Theta \subseteq \Sigma \\ \Gamma \neq \Theta}} \Box \left( \bigvee_{\Phi \in \Gamma} C_\Phi \leftrightarrow \bigvee_{\Phi \in \Theta} C_\Phi \right).$$

Since $\mathsf{L}$ is determined by a class of finite frames, it follows from Lemma 1 that there is a finite model $\mathfrak{A} = (W, R, \varphi)$ generated by $w$ and such that

(1) $$(\mathfrak{A}, w) \models \Box B$$

(2) $$(\mathfrak{A}, w) \not\models A.$$

Let $X$ be any final cluster of $(W, R)$ and suppose for *reductio* that $X$ contains a subset $Q$ of $n + 1$ worlds pairwise nonequivalent under $\approx$. By (1), $(\mathfrak{A}, x) \models B$ for all $x \in X$. Hence

$$(\mathfrak{A}, x) \models \Diamond \bigvee_{\substack{G, H \subseteq Q \\ G \neq H}} \Box (D_G \leftrightarrow D_H)$$

where

$$D_\Gamma = \begin{cases} \bigvee_{x \in \Gamma} C_{\varphi(x)} & \text{if } \Gamma \text{ is nonempty} \\ \perp \text{ otherwise.} \end{cases}$$

But then for some $v \in X$,

$$(\mathfrak{A}, v) \models \bigvee_{\substack{G, H \subseteq Q \\ G \neq H}} \Box (D_G \leftrightarrow D_H).$$

Hence, for some subsets $G \neq H$ of $Q$, $(\mathfrak{A}, v) \models \Box(D_G \leftrightarrow D_H)$. With no loss of generality, we can assume $z \in G - H$. Then $vRz$ and so $(\mathfrak{A}, z) \models D_G \leftrightarrow D_H$. But $(\mathfrak{A}, z) \models D_G$ since $(\mathfrak{A}, z) \models C_{\varphi(z)}$ and $z \in G$. Hence $(\mathfrak{A}, z) \models D_H$. Since $(\mathfrak{A}, z) \not\models \perp$, we know $H$ is nonempty. Hence, for some $u \in H$, $(\mathfrak{A}, z) \models C_{\varphi(u)}$. But $u$ and $z$ belong to the same cluster, so $u \approx z$ contrary to the fact that $u$ and $z$ are distinct members of $Q$.

We have thus shown that each final cluster of $(W, R)$ contains at most $n$ worlds pairwise nonequivalent under $\approx$. Hence, if $\mathfrak{B} = (U, S, \psi)$ is a contraction of $\mathfrak{A}$ on $\mathsf{Sbfl}(A)$, then by Lemma 2(i), (iv) we know that $(U, S)$ is a finite frame for $\mathbf{L}(n)$ each final cluster of which contains at most $n$ worlds. Moreover, by (2) and Lemma 2(iii), $(\mathfrak{B}, [w]) \not\models A$. This completes the proof.

**4** Contractions could not have been used to settle our original Conjecture, but a look at their use in the proof of Theorem 1 does suggest a line of attack worth recording here.

Let $\mathfrak{B} = (U, S, \psi)$ and $\mathfrak{A} = (W, R, \varphi)$ be two models and suppose $(\mathfrak{A}, w) \not\models A$ for some $w$. Define an equivalence relation on $W$ such that $v \approx u$ if and only if for each $B \in \Gamma$, $(\mathfrak{A}, v) \models B$ iff $(\mathfrak{A}, u) \models B$, where $\Gamma$ is a logically finite superset of $\mathsf{Sbfl}(A)$ closed under subformulas. We call $\mathfrak{B}$ an *$A$-filtration* of $\mathfrak{A}$ for $\mathbf{L}$ if the following three conditions are met:

$$U = \{[u] \mid u \in W\}$$
$$(\mathfrak{B}, [u]) \models B \text{ iff } (\mathfrak{A}, u) \models B \text{ for each } B \in \Gamma$$
$$(U, S) \text{ is a frame for } \mathbf{L}.$$

Borrowing a handy bit of terminology from Dov Gabbay [5], we can then say that $\mathbf{L}$ *admits weakly of standard filtration* if for each nontheorem $A$ of $\mathbf{L}$ there is an $A$-filtration for $\mathbf{L}$.

Now an argument very similar to that used to prove Theorem 1, but employing $A$-filtrations rather than contractions, yields

Theorem 2 *If* $\mathbf{L}$ *is determined by a class of frames for* S4 *and admits weakly of standard filtration, then* $\mathbf{L}_n$ *is determined by the class of frames for* $\mathbf{L}$ *which contain at most n worlds.*

A great many modal systems, including most of the extensions of S4 to have appeared to date other than Fine's incomplete logics, are known to admit weakly of standard filtration. Unfortunately, there are complete counterexamples as well—e.g., the logic determined by the unit class of the frame $(W, R)$ constructed in [3]. So Theorem 2 would not fully resolve the Conjecture. But the theorem does cover a large and semantically very interesting class of logics, and it does so without appeal to the results of Fine.

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