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THREE SUBSTITUTION-INSTANCE INTERPRETATIONS

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1 *Abandoning nominalism* The substitution-instance interpretation of quantifiers is often associated with some form of nominalism. Leśniewski, who developed his logical systems with a substitution-instance interpretation in mind, was a nominalist. (The connection between Leśniewski's philosophical views and his logical systems is explained in [10].) And when Henkin discussed the relation between the substitution-instance interpretation and his completeness proof for quantificational logic, his paper had the title "Some Notes on Nominalism." This association is unfortunate. I know of no view which might be called nominalistic which seems plausible to me. Yet I find myself "taken" with the substitution-instance interpretation of quantifiers. When the substitution-instance interpretation is separated from a nominalistic outlook, we can see that there are different substitution-instance interpretations of quantifiers. In this paper I will discuss three important interpretations. Not all of them are compatible with nominalism (perhaps none of them is), but each *is* ontologically less committing than the usual (referential) interpretation.

The initial motivation that I can offer for accepting substitution-instance interpretations is connected with certain purposes that a formal language can be used to achieve. If a formal language is to be used for studying the logical structure of a natural language, or of some sublanguage of a natural language, it must have certain similarities with the language(s) it will be used to study. But it is also helpful to treat the formal language (in certain respects) as one would treat the natural language. Think of that part of a natural language which consists of sentences composed of names and predicating expressions, and of (nearly truth-functional) compound sentences formed from these. This sublanguage is part of a going concern which exists before we can study it. In this sublanguage, some sentences are true and others are false. And the sentences of the sublanguage exemplify certain forms. Some forms are such that every sentence exemplifying them is true, while others are exemplified by both true and false sentences. In a formal language that corresponds to this natural sublanguage, substitution-instance quantifiers can be used to indicate facts

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about logical forms. And these quantifiers are well suited for treating the formal language as if it too were a going concern.

Initially, I want to justify substitution-instance quantifiers as a useful device for dealing with the logical structure of a realistic language (a useful device that can be incorporated in that very language). But I also want to make a stronger claim about these interpretations. I believe that substitution-instance interpretations of quantifiers are more natural than the ordinary (referential) interpretation. They are more natural in this sense: Quantification as it has been developed is better suited to substitution-instance than referential interpretations. We can see this by considering some important theorems about quantificational languages. Some results that were (and are) surprising for the referential interpretation are not so surprising for substitution-instance interpretations. In general, the theorems about quantificational languages "make more sense" for substitution-instance interpretations than they do for the referential interpretation. Of course, I can support this claim only by appealing to the intuitions of the reader. But I will try to make it convincing.

2 *Leblanc valuations* In order to discuss the different interpretations, we need a formal language to talk about. Let L be the first-order language with the following features:

L has denumerably many individual constants: $a_0, a_1, \ldots$
denumerably many individual variables: $x_0, x_1, \ldots$
denumerably many monadic predicates: $F_0^1, F_1^1, \ldots$
denumerably many n-adic predicates: $F_0^n, F_1^n, \ldots$
quantifiers: $(\forall\ ), (\exists\ )$
sentential connectives: $\sim, \vee, \&, \supset, \equiv$

Overlapping quantifiers with the same quantified variable are allowed, but vacuous quantification is not allowed. A *quasi-well-formed-formula* is either a sentence or a formula like a sentence except for containing free occurrences of individual variables.

I will first define a standard, referential valuation for L. For a given nonempty domain θ, function $\mathcal{V}$ (defined for the individual variables, individual constants, and predicates of L) is a *referential interpreting function of* L *for* θ iff $\mathcal{V}$ assigns an element of θ to each individual variable and constant of L, and $\mathcal{V}$ assigns a set of ordered n-tuples of individuals of θ to each n-adic predicate of L. For a given referential interpreting function $\mathcal{V}$ of L for some nonempty domain θ, a valuation of the quasi-wffs of L is as follows:

(i) If $\langle\alpha_1, \ldots, \alpha_n\rangle$ is an n-tuple of individual expressions of L and φ^n is an n-adic predicate of L, then $\varphi^n(\alpha_1, \ldots, \alpha_n)$ has value $\mathbf{T}$ for the valuation determined by $\mathcal{V}$ if $\langle\mathcal{V}(\alpha_1), \ldots, \mathcal{V}(\alpha_n)\rangle \in \mathcal{V}(\varphi^n)$. Otherwise it has value $\mathbf{F}$ for the valuation determined by $\mathcal{V}$.
(ii) If $(\forall\alpha)A$ is a quasi-wff, and A has value $\mathbf{T}$ for every referential interpreting function $\mathcal{V}'$ of L for θ that is like $\mathcal{V}$ except possibly for the

value assigned α, then $(\forall\alpha)A$ has value **T** for the valuation determined by $\mathscr{V}$. Otherwise it has **F** for this valuation.
(iii)-(viii) are as one would expect.

When $\mathscr{V}$ is an interpreting function, I will sometimes call $\mathscr{V}$ a (referential) valuation of **L** for $\boldsymbol{\theta}$. We shall also have use for the following definitions:

A sentence A of **L** is a *referential logical truth* iff A has value **T** for every referential valuation $\mathscr{V}$ for every nonempty domain $\boldsymbol{\theta}$.

A set X of sentences of **L** has sentence A (of **L**) as a *referential logical consequence* (in symbols: $X \Vdash_{\mathsf{R}} A$) iff, for every nonempty domain $\boldsymbol{\theta}$ and referential valuation $\mathscr{V}$ of **L** for $\boldsymbol{\theta}$ for which all members of X have value **T**, A also has value **T**.

Now let us consider the first sort of substitution-instance interpretation of the quantifiers. The valuations involved are *Leblanc valuations*.[1] We need these definitions:

For A a quasi-wff containing free occurrences of distinct individual variables $\alpha_1, \ldots, \alpha_n$ and no others, A' is a *substitution instance of* A iff A' is a sentence obtained from A by replacing the free occurrences of $\alpha_1, \ldots, \alpha_n$ by occurrences of individual constants.

A function $\mathscr{V}$ (defined for the atomic *sentences* of **L**) is a *Leblanc interpreting function* of **L** iff $\mathscr{V}$ assigns (exactly) one of **T**, **F** to each atomic sentence of **L**.

For a given Leblanc interpreting function $\mathscr{V}$ of **L**, a *Leblanc valuation* of the sentences of **L** is as follows:

(i) An atomic sentence of **L** has the value assigned it by $\mathscr{V}$.
(ii) If $(\forall\alpha)A$ is a sentence of **L**, and every substitution instance A' of A in **L** has value **T** for the valuation determined by $\mathscr{V}$, then $(\forall\alpha)A$ has value **T** for this valuation. Otherwise it has value **F** for the valuation determined by $\mathscr{V}$.
(iii)-(viii) Etc.

If $\mathscr{V}$ is a Leblanc interpreting function of **L**, I will also call $\mathscr{V}$ a Leblanc valuation of **L**. In the definition of 'Leblanc valuation,' no values are assigned to quasi-wffs which are not sentences. The definition could be changed to include them, but nothing would be gained thereby.

A sentence A of **L** is a *Leblanc logical truth* iff A has value **T** for every Leblanc valuation of **L**. A set X of sentences of **L** has sentence A (of **L**) as a *Leblanc logical consequence* ($X \Vdash_{\mathsf{L}} A$) iff every Leblanc valuation for which all members of X have value **T** is one for which A also has value **T**.

By virtue of the Lowenheim-Skolem Theorem for referential valuations, we have the following results:

(1) If A is a sentence of **L**, then A is a referential logical truth iff A is a Leblanc logical truth.

(2) If X is a finite set of sentences of L and A is a sentence of L, then $X \Vdash_{\mathsf{R}} A$ iff $X \Vdash_{\mathsf{L}} A$.

This second result cannot be extended to infinite sets. For some monadic predicate φ, we might have $X = \{\varphi(a_0), \varphi(a_1), \varphi(a_2), \ldots\}$. Then we would have $X \Vdash_{\mathsf{L}} (\forall x_0)\varphi(x_0)$, but not $X \Vdash_{\mathsf{R}} (\forall x_0)\varphi(x_0)$. (A discussion of this lack of equivalence is found in [2].)

The concept of *Leblanc logical consequence* is not the only sort of logical consequence that can be considered with respect to Leblanc valuations. In fact, Hugues Leblanc himself (in [9]) has given a different definition of 'logical consequence' for Leblanc valuations. And his definition yields a concept which coincides (i.e., it relates the same sets and sentences) with that of referential logical consequence. However, in this paper I am taking substitution-instance interpretations as autonomous, or primary. And I am interested in concepts, and results, that would be natural if these interpretations are taken as primary. I think my definition of 'Leblanc logical consequence' yields such a concept.[2]

3 *Henkin valuations* Leblanc valuations are not well suited for a realistic formal language. These valuations require us to regard L as fixed and final, since quantified statements are construed with respect to (are evaluated in terms of) the constants in L. If L were extended by adding new constants (and sentences which contain them), a Leblanc interpreting function for L would fail to provide for the new sentences. And if the interpreting function were extended to cover the new sentences, statements which had been true might become false, and conversely. But a natural language is never fixed and final; it is open to new expressions, and can always be extended.[3] Because Leblanc valuations treat L as fixed and final, they can prevent us from using quantifiers to make statements with truly universal force. A universally quantified statement is only "concerned" with the names in L; so we can make statements only about those individuals which have names. But something might be true of all named individuals, yet false for some unnamed individual. Of course, by adopting a substitution-instance interpretation, we have separated matters of reference from our valuations. But we have not lost sight of the fact that (at least some) names refer to real individuals. Nor should we lose sight of the desirability of having L suitable for talking about whatever individuals there are. If there is some individual which is not named by a constant of L, that individual will be named in some extension of L.

What is wanted is a substitution-instance interpretation which makes quantifiers "cover" an extension of L. So that a sentence $(\forall\alpha)A$ is not made true just by substitution instances in L, but is instead construed with respect to some suitable extension of L. Given such an interpretation, universally quantified statements function (within L) primarily as inference warrants. They "license" us to assert substitution instances of the quantified formula, but the quantifiers are not used to make claims evaluated in terms of the "fixed totality" of names in L. (Although the user of L does not know what are the new expressions of L's extension, he may still be in a position to make true universally quantified statements.)

Let L^+ be a language obtained by adding zero or more new individual constants to L (at most denumerably many), and zero or more new predicates (at most denumerably many). Then L^+ is an *extension* of L. (For the discussion in the remainder of this section, it would not be necessary to consider extensions with new predicates. But it does not cost anything to allow them, and they will be useful later in the paper.)

Let L^+ be an extension of L, and let $\mathcal{V}$ be a Leblanc valuation of L^+. Let $\mathcal{V}'$ be a function which assigns to the sentences of L the values they receive from $\mathcal{V}$ in L^+. Then $\mathcal{V}'$ is a *Henkin valuation of* L.[4] We define 'Henkin logical truth' and 'Henkin logical consequence' ($\Vdash_{\overline{H}}$) along the lines of earlier definitions. The relation between referential valuations and Henkin valuations is interesting. The Lowenheim-Skolem Theorem for referential valuations can be stated in this form:

(LS) *Let θ be a nonempty domain of individuals, and let $\mathcal{V}$ be a referential interpreting function of* L *for θ. Then there is a nonempty domain θ' which is at most denumerable and a referential interpreting function $\mathcal{V}'$ of* L *for θ' such that the values of the sentences of* L *are exactly the same for $\mathcal{V}'$ as for $\mathcal{V}$.*

This version makes it easy to establish the following theorem.

Theorem 1 *If θ is a nonempty domain, and $\mathcal{V}$ is a referential interpreting function of* L *for θ, then there is a Henkin valuation $\mathcal{V}'$ for which the sentences of* L *have the same values as they have for $\mathcal{V}$. Conversely, we can start with $\mathcal{V}'$ and find a suitable θ and referential valuation $\mathcal{V}$ of* L *for θ.*

The following is an immediate consequence of Theorem 1.

Theorem 2 (a) *If A is a sentence of* L, *then A is a referential logical truth iff A is a Henkin logical truth.*
(b) *If X is a set of sentences of* L *and A is a sentence of* L, *then $X \Vdash_{\overline{R}} A$ iff $X \Vdash_{\overline{H}} A$.*

So the Lowenheim-Skolem Theorem for referential valuations has the effect of establishing the adequacy of Henkin valuations. An analogue of the Lowenheim-Skolem Theorem can be stated for Henkin valuations, but it is completely trivial. To state it we need this definition:

If L^+ is an extension of L and $\mathcal{V}$ is a Henkin valuation of L^+, then the ordered pair $\langle L^+, \mathcal{V}\rangle$ is *instantially complete* iff for every sentence $(\forall\alpha)A$ of L^+ that receives F for $\mathcal{V}$, there is a substitution instance A' of A (in L^+) that receives F for $\mathcal{V}$. And for every sentence $(\exists\alpha)A$ of L^+ that receives T for $\mathcal{V}$ there is a substitution instance A' of A that receives T for $\mathcal{V}$.

The Lowenheim-Skolem analogue states that for every Henkin valuation $\mathcal{V}$ of L, there is an extension L^+ of L and a Henkin valuation $\mathcal{V}'$ of L^+ that agrees with $\mathcal{V}$ for sentences of L and is such that $\langle L^+, \mathcal{V}'\rangle$ is instantially complete. This is trivial because every Leblanc valuation is a Henkin valuation, so any Leblanc valuation $\mathcal{V}'$ of an L^+ which $\mathcal{V}'$ "generates" $\mathcal{V}$ will give an instantially complete $\langle \mathsf{L}^+, \mathcal{V}' \rangle$.

Although the Lowenheim-Skolem analogue is trivial for Henkin valuations, we might ask if there is some nontrivial result for Henkin valuations that is in some respects a counterpart to the Lowenheim-Skolem Theorem for referential valuations. To obtain one such result we need a formal system. Let $\mathfrak{F}$ be some standard formal system for L that is complete with respect to logical truth and logical consequence for referential valuations. But suppose that we do not know this about $\mathfrak{F}$, that $\mathfrak{F}$ was developed with Henkin valuations in mind. In that case, it would be easy to establish that $\mathfrak{F}$ is sound for Henkin valuations. But we might think (or just suspect) that $\mathfrak{F}$ could be used to generate a new kind of valuation, unlike Henkin valuations. For X a set of sentences of L and A a sentence of L, let '$X \vdash A$' indicate that A is deducible from X by means of $\mathfrak{F}$. Then we need the following definitions:

Let X be a set of sentences of L. A *partial* $\mathfrak{F}$-*valuation of* L *induced by* X is as follows: If $X \vdash A$, then A is assigned T; if $X \vdash \sim A$, then A is assigned F.

A set X of sentences of L is *consistent* iff in the partial $\mathfrak{F}$-valuation of L induced by X there is no sentence A which is assigned both T, F.

Let X be a consistent set of sentences of L. In some enumeration of sentences of L, let A be the first sentence which does not receive a value in the partial $\mathfrak{F}$-valuation induced by X. If A is consistent with X, take the partial $\mathfrak{F}$-valuation of L induced by $X \cup \{A\}$, and consider the first sentence B in the enumeration which does not receive a value in the partial $\mathfrak{F}$-valuation induced by $X \cup \{A\}$. If A is not consistent with X, take the next sentence in the enumeration which has no value. And so on. Ultimately, every sentence of L will have a value, and the whole valuation is an $\mathfrak{F}$-*valuation of* L.

We might think that an $\mathfrak{F}$-valuation could be different from a Henkin valuation. And that there could be $\mathfrak{F}$-valuations for which it is not possible to extend L and extend the valuation to produce an instantially complete ordered pair. The construction in Henkin's Completeness Theorem (given in [3]) shows that this conjecture is not true. That construction shows that $\mathfrak{F}$-valuations coincide with Henkin valuations; this coincidence is what makes Henkin's proof a completeness result. Note that, given Henkin valuations, and given $\mathfrak{F}$, his approach is the most natural one to take in order to establish the completeness of $\mathfrak{F}$. But from the standpoint of referential valuations, Henkin's procedure has a somewhat artificial character.

4 *Gödel valuations* Henkin valuations of L provide a good "fit" for customary first-order formal systems. But these valuations are not entirely satisfactory for treating the formal language in a realistic manner.

For with a natural language in use at a given time, many extensions are possible. But only some of these will actually be realized. A Henkin valuation of L is determined by a particular extension L^+ of L and a particular valuation of L^+. (Even though the same Henkin valuation can be generated by lots of different extensions of L, and by lots of different valuations for each extension, in a given case we think of it as determined by a specific extension and valuation.) Dealing with such valuations is like considering only the extensions of a natural language that will actually be realized. But we may very well want our universal statements to be true for every (possible) extension of the language. If universally quantified statements are to have a truly universal force, they must be evaluated with respect to more than one extension of L.

Since quantified sentences in a Henkin valuation are evaluated in terms of a particular Leblanc valuation of a particular extension L^+, the quantifiers "cover" only denumerably many constants. Such quantified sentences are not adequate for talking about "situations" where there are nondenumerably many individuals. We could avoid this shortcoming if we construed our quantifiers with respect to every evaluated extension of L. For there are nondenumerably many of these.

Let $\mathcal{V}$ be a Leblanc interpreting function for L. Let L^+ be an extension of L, and $\mathcal{V}'$ be a Leblanc interpreting function for L^+ that agrees with $\mathcal{V}$ on the (atomic) sentences of L. Then $\langle \mathsf{L}^+, \mathcal{V}' \rangle$ is an *extension pair of* $\langle \mathsf{L}, \mathcal{V} \rangle$. If L^+ is different from L, then $\langle \mathsf{L}^+, \mathcal{V}' \rangle$ is a *proper* extension pair.

If $\langle \mathsf{L}^{+1}, \mathcal{V}_1 \rangle$, $\langle \mathsf{L}^{+2}, \mathcal{V}_2 \rangle$ are distinct extension pairs of $\langle \mathsf{L}, \mathcal{V} \rangle$, I will adopt the convention that any new symbols common to L^{+1}, L^{+2} (but not found in L) need not be regarded as having the "same meaning." This is because each extension pair is considered as one possible direction in which $\langle \mathsf{L}, \mathcal{V} \rangle$ can be extended. From the standpoint of $\langle \mathsf{L}, \mathcal{V} \rangle$, a certain symbol not found in L does not need to have any particular meaning attached to it. Choosing one meaning rather than another is a purely conventional matter, which does not affect the truths that were waiting to be expressed. The provision that the same new name (or predicate) can have different meanings in different extension pairs is also necessary if we are to use denumerably many names (or predicates) to talk about non-denumerably many things.

To define 'Gödel valuation,' I will first stipulate what is a *Gödel Tree*. In doing this, I will frequently speak of positions in such a tree rather than elements of the tree, for the same element can occupy more than one position. The top node of a Gödel Tree is of *level* 0, and its *position* is 1. The immediate successors of the top node are members of a sequence. Their positions are *first-level* positions. If this sequence has a σ'th element (an index σ), then the σ'th element of this sequence has position 1, σ in the Gödel Tree. The *second-level* positions are the immediate successors of first-level positions. Each first-level position has a sequence of immediate successors. The sequence of immediate successors of a position 1, σ has the order type of the sequence which results from the first-level sequence by deleting the σ'th element. Each immediate successor of position 1, σ corresponds to an index of the first-level sequence (but nothing corresponds to the σ'th position), and this index is used to give its position in the Gödel Tree. For example, the immediate successor of position 1, σ that occupies a place in its sequence corresponding to the ρ'th position of the first-level sequence has position 1, σ, ρ in the Gödel Tree. So that in the sequence of immediate successors of position 1, σ, the σ'th element (if there is one) occupies position 1, σ, $\sigma + 1$ in the Gödel Tree.

The immediate successors of each n'th-level position are $n+1$'*st-level* positions. Each n'th-level position has a sequence of immediate successors. The sequence of immediate successors of an n'th-level position 1, $\sigma_1, \ldots, \sigma_n$ has the order type of the sequence which results from the first-level sequence by deleting the σ_1'th, . . ., σ_n'th elements of that sequence. Each immediate successor of the position 1, $\sigma_1, \ldots, \sigma_n$ corresponds to an index ρ (a nondeleted index) of the first-level sequence, so that the successor has position 1, $\sigma_1, \ldots, \sigma_n$, ρ in the Gödel Tree. A single item is a degenerate case of a Gödel Tree, having no immediate successors. If a Gödel Tree has any first-level positions, then it "continues" until all the first-level indices are "used up." A Gödel Tree with n (n finite) first-level positions has the n'th-level as its lowest level. A Gödel Tree with infinitely many first-level positions has denumerably many levels.

A *Gödel Structure* is a Gödel Tree whose elements are ordered pairs of languages and (Leblanc) valuations. To construct a Gödel Structure, a stock of predicates and individual constants will be required. Let τ be a predicate or individual constant of L. Then ${}_n\tau$ ($n \geqslant 1$) is a *new expression of level n.* And ${}^1_n\tau$, ${}^2_n\tau$, ${}^3_n\tau$, . . . are *old expressions of level n.*

The 0'th-level of a Gödel Structure contains an ordered pair $\langle \mathsf{L}, \mathcal{V} \rangle$, where $\mathcal{V}$ is a Leblanc valuation of L. The first-level elements of the Gödel Structure are proper extension pairs $\langle \mathsf{L}^{+\alpha}, \mathcal{V}_\alpha \rangle$ of $\langle \mathsf{L}, \mathcal{V} \rangle$. The expressions that are new to the first-level pairs are new expressions of level one.

We must be able to establish a connection between terms (both predicates and individual constants) occurring in pairs in different positions of the Gödel Structure. I will speak of terms as "expressing the same (predicative and individual) concepts." For every first-level pair $\langle \mathsf{L}^{+\alpha}, \mathcal{V}_\alpha \rangle$ at a position 1, σ in the Gödel Structure, each predicate or individual constant *in* $\langle \mathsf{L}, \mathcal{V} \rangle$ *at position* 1 *expresses the same concept as the same expression in* $\langle \mathsf{L}^{+\sigma}, \mathcal{V}_\alpha \rangle$ *at position* 1, σ. Any predicate or individual constant *in a pair at a position* in a Gödel Structure *expresses the same concept as itself in that pair at that position.* Expressing the same concept is a symmetric and transitive relation. The new terms in each first-level extension pair at a position express new concepts, and there is no provision for identifying a new concept in one first-level pair at a certain position with a new concept in a first-level pair at a different position.

Not only must we connect predicates and individual constants occurring in pairs at different positions in the Gödel Structure, we must also connect atomic sentences. Let $\varphi(\alpha_1, \ldots, \alpha_n)$ be an atomic sentence of $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at position 1, $\sigma_1, \ldots, \sigma_s$ of the Gödel Structure, and let $\psi(\beta_1, \ldots, \beta_n)$ be an atomic sentence of $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at position 1, $\rho_1, \ldots, \rho_r$ of that same Structure. Let φ in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, $\sigma_1, \ldots, \sigma_s$ express the same concept as ψ in $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at 1, $\rho_1, \ldots, \rho_r$. Similarly, let each α_i in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, $\sigma_1, \ldots, \sigma_s$ express the same concept as β_i in $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at 1, $\rho_1, \ldots, \rho_r$. Then the sentence $\varphi(\alpha_1, \ldots, \alpha_n)$ *in* $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ *at position* 1, $\sigma_1, \ldots, \sigma_s$ is *synonymous with* $\psi(\beta_1, \ldots, \beta_n)$ *in* $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ *at position* 1, $\rho_1, \ldots, \rho_r$.

The immediate successors of a first-level pair $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at position 1, σ are proper extension pairs $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ of $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$. The new terms of these successors are new and old second-level expressions. The new second-level expressions express new concepts, not expressed at the first level; the old second-level expressions express concepts expressed previously, but not expressed in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, σ. There is a function Φ_2 defined for second-level pairs (at their positions) which does the following for each pair $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at a position 1, σ, ρ:

(i) Φ_2 associates (some) predicates and individual constants in $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ in 1, σ, ρ with predicates and constants in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, σ, which express the same (respective) concepts. If τ is a predicate or constant of $L^{+\alpha}$, then Φ_2 associates τ in $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at 1, σ, ρ with τ in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, σ.
(ii) Φ_2 associates the old second-level predicates and constants of $L^{+\beta}$ with those (new first-level) expressions of $\langle L^{+\gamma}, \mathscr{V}_\gamma\rangle$ at 1, ρ which do not express the same concepts as terms in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, σ. These old second-level expressions in $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at 1, σ, ρ express the same concepts as the corresponding expressions in $\langle L^{+\gamma}, \mathscr{V}_\gamma\rangle$ at 1, ρ.
(iii) Φ_2 associates the new second-level expressions of $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at 1, σ, ρ with the same second-level expressions in $\langle L^{+\delta}, \mathscr{V}_\delta\rangle$ at 1, ρ, σ. (So the same new second-level expressions must occur in both $L^{+\beta}$ and $L^{+\delta}$.) These new second-level expressions in $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at 1, σ, ρ express the same concepts as the same new expressions in $\langle L^{+\delta}, \mathscr{V}_\delta\rangle$ at 1, ρ, σ.

Any atomic sentence in $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at 1, σ, ρ is assigned the same value by $\mathscr{V}_\beta$ as synonymous sentences in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, σ, or in $\langle L^{+\gamma}, \mathscr{V}_\gamma\rangle$ at 1, ρ, or in $\langle L^{+\delta}, \mathscr{V}_\delta\rangle$ at 1, ρ, σ are assigned by $\mathscr{V}_\alpha$, $\mathscr{V}_\gamma$, or $\mathscr{V}_\delta$, respectively.

If a Gödel Structure does not terminate at the n'th level, it is fairly easy to see what the n + 1'st level will be like. But, for the record, I will describe it. The immediate successors of an n'th-level pair $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at position 1, $\sigma_1, \ldots, \sigma_n$ are proper extension pairs $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ of $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$. The new terms of these successors are new and old n + 1'st-level expressions. There is a function Φ_{n+1} defined for n + 1'st-level pairs which does the following for each pair $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at a position 1, $\sigma_1, \ldots, \sigma_n$, ρ:
(i) For the predicates and individual constants in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, $\sigma_1, \ldots, \sigma_n$, the function Φ_{n+1} associates these expressions in $\langle L^{+\beta}, \mathscr{V}_\beta\rangle$ at 1, $\sigma_1, \ldots, \sigma_n$, ρ with the same expressions in $\langle L^{+\alpha}, \mathscr{V}_\alpha\rangle$ at 1, $\sigma_1, \ldots, \sigma_n$. These terms in one pair at its position express the same concepts as they do in the other pair at its position.
(ii) Consider each n'th-level pair $\langle L^{+\gamma}, \mathscr{V}_\gamma\rangle$ at a position 1, $\mu_1, \ldots, \mu_n$ in

the Structure where each μ_i is the same as one of $\sigma_1, \ldots, \sigma_n, \rho$. Each expression in such a pair at its location that does not express the same concept as a term in $\langle \mathsf{L}^{+\alpha}, \mathscr{V}_\alpha\rangle$ at $1, \sigma_1, \ldots, \sigma_n$ requires an old $n + 1$'st-level expression in $\langle \mathsf{L}^{+\beta}, \mathscr{V}_\beta\rangle$ at $1, \sigma_1, \ldots, \sigma_n, \rho$ that corresponds to it. Φ_{n+1} relates each old $n + 1$'st-level expression in $\langle \mathsf{L}^{+\beta}, \mathscr{V}_\beta\rangle$ to an expression in some $\langle \mathsf{L}^{+\gamma}, \mathscr{V}_\gamma\rangle$ at $1, \mu_1, \ldots, \mu_n$ in such a way that no term in one of those pairs expresses a concept not expressed by a term in $\langle \mathsf{L}^{+\beta}, \mathscr{V}_\beta\rangle$ at $1, \sigma_1, \ldots, \sigma_n, \rho$. (And, of course, the old $n + 1$'st-level expressions in $\langle \mathsf{L}^{+\beta}, \mathscr{V}_\beta\rangle$ at $1, \sigma_1, \ldots, \sigma_n, \rho$ express the same concepts as the terms to which they are related by Φ_{n+1} in the various pairs at their respective positions.)
(iii) Let $1, \nu_1, \ldots, \nu_{n+1}$ be the next index in alphabetic order after $1, \sigma_1, \ldots, \sigma_n, \rho$ which is such that each ν_i is the same as one of $\sigma_1, \ldots, \sigma_n, \rho$. If $1, \sigma_1, \ldots, \sigma_n, \rho$ is the last such index in alphabetic order, then $1, \nu_1, \ldots, \nu_{n+1}$ is the first such index in alphabetic order (i.e., $\nu_1 < \ldots < \nu_{n+1}$). Then Φ_{n+1} associates the new $n + 1$'st-level expressions of $\langle \mathsf{L}^{+\beta}, \mathscr{V}_\beta\rangle$ at $1, \sigma_1, \ldots, \sigma_n, \rho$ with the same new $n + 1$'st-level expressions of $\langle \mathsf{L}^{+\delta}, \mathscr{V}_\delta\rangle$ at $1, \nu_1, \ldots, \nu_{n+1}$. These new $n + 1$'st-level expressions in $\langle \mathsf{L}^{+\beta}, \mathscr{V}_\beta\rangle$ at $1, \sigma_1, \ldots, \sigma_n, \rho$ express the same concepts as the same new terms in $\langle \mathsf{L}^{+\delta}, \mathscr{V}_\delta\rangle$ $1, \nu_1, \ldots, \nu_{n+1}$. (Since expressing the same concept is a transitive relation, all the $n + 1$'st-level pairs whose position markers contain the same indices will contain the same new $n + 1$'st-level expressions which express the same concepts at each position.)

Any atomic sentence in $\langle \mathsf{L}^{+\beta}, \mathscr{V}_\beta\rangle$ at $1, \sigma_1, \ldots, \sigma_n, \rho$ is assigned the same value by $\mathscr{V}_\beta$ as synonymous sentences in $\langle \mathsf{L}^{+\alpha}, \mathscr{V}_\alpha\rangle$ at $1, \sigma_1, \ldots, \sigma_n$, or in any $\langle \mathsf{L}^{+\gamma}, \mathscr{V}_\gamma\rangle$ at a position $1, \mu_1, \ldots, \mu_n$ where each μ_i is the same as one of $\sigma_1, \ldots, \sigma_n, \rho$, or in $\langle \mathsf{L}^{+\delta}, \mathscr{V}_\delta\rangle$ at position $1, \nu_1, \ldots, \nu_{n+1}$ (as described above in (iii)) are assigned by their (respective) interpreting functions. To refer to elements in a Gödel Structure $\mathscr{S}$, we can attach an index to the ordered pair notation. So that if $\langle \mathsf{L}, \mathscr{V}\rangle$ is the 0'th-level pair of Gödel Structure $\mathscr{S}$, we can write that $\langle \mathsf{L}, \mathscr{V}\rangle_1$ is an element of $\mathscr{S}$. And to say that $\langle \mathsf{L}^{+\alpha}, \mathscr{V}_\alpha\rangle_{1,\sigma_1,\ldots,\sigma_n}$ is an element of $\mathscr{S}$ means that $\langle \mathsf{L}^{+\alpha}, \mathscr{V}_\alpha\rangle$ occurs at position $1, \sigma_1, \ldots, \sigma_n$ in $\mathscr{S}$. If $\langle \mathsf{L}, \mathscr{V}\rangle_1$ is an element of $\mathscr{S}$, then $\mathscr{S}$ is a *Gödel Structure headed by* $\langle \mathsf{L}, \mathscr{V}\rangle$.

A Gödel Structure can be regarded as representing the possible developments of an evaluated language $\langle \mathsf{L}, \mathscr{V}\rangle$. Each first-level pair $\langle \mathsf{L}^{+\alpha}, \mathscr{V}_\alpha\rangle$ at a position $1, \sigma$ represents one possible extension of $\langle \mathsf{L}, \mathscr{V}\rangle$. These possibilities are thought of as determined by an "independent reality" which is unaffected by the choice of an extension of $\langle \mathsf{L}, \mathscr{V}\rangle$, for no matter which first-level pair is chosen, all the other possibilities remain. (Recall that each first-level pair $\langle \mathsf{L}^{+\alpha}, \mathscr{V}_\alpha\rangle_{1,\sigma}$ is extended by every other first-level pair $\langle \mathsf{L}^{+\beta}, \mathscr{V}_\beta\rangle_{1,\rho}$ to obtain the immediate successors of $\langle \mathsf{L}^{+\alpha}, \mathscr{V}_\alpha\rangle_{1,\sigma}$.) A branch through a Gödel Structure represents a possible history of the language. Such a language is incapable of commenting on its own development, for at every stage exactly the same expressive possibilities are open to all languages introduced at that stage. (A Gödel

Structure represents a classical conception of the relation of language (and language users) to the world rather than a "quantum-theoretical" conception. But the Gödel Structure can accommodate entirely new expressions, new concepts, at every stage.)

In order to define a Gödel valuation, we must redefine 'substitution instance.' Let A be a quasi-wff of $\langle \mathsf{L}^{+\alpha}, \mathcal{V}_\alpha \rangle$ at position ε in Gödel Structure $\mathcal{S}$. And let A contain free occurrences of distinct individual variables $\delta_1, \ldots, \delta_n$ and no others. Then sentence A' in $\langle \mathsf{L}^{+\beta}, \mathcal{V}_\beta \rangle$ at ε' in $\mathcal{S}$ is a *substitution instance of A in $\langle \mathsf{L}^{+\alpha}, \mathcal{V}_\alpha \rangle$ at* ε iff A' can be obtained from A by replacing $\delta_1, \ldots, \delta_n$ by constants from $\mathsf{L}^{+\beta}$, and ε' is a (possibly null) extension of ε.

Let $\mathcal{S}$ be a Gödel Structure headed by $\langle \mathsf{L}, \mathcal{V} \rangle$. Then a *Gödel valuation for* $\mathcal{S}$ is given by:

(i) The atomic sentences of a pair $\langle \mathsf{L}^{+\alpha}, \mathcal{V}_\alpha \rangle$ located at ε in $\mathcal{S}$ receive the values assigned them by $\mathcal{V}_\alpha$.
(ii) Let $(\forall\alpha)A$ be a sentence of $\langle \mathsf{L}^{+\beta}, \mathcal{V}_\beta \rangle$ located at ε in $\mathcal{S}$. If every substitution instance of A in $\langle \mathsf{L}^{+\beta}, \mathcal{V}_\beta \rangle$ at ε has value **T** (at its proper location), then $(\forall\alpha)A$ has value **T** in $\langle \mathsf{L}^{+\beta}, \mathcal{V}_\beta \rangle$ at ε. Otherwise it has value **F** in $\langle \mathsf{L}^{+\beta}, \mathcal{V}_\beta \rangle$ at ε.
(iii)-(viii) Etc.

It is easy to see that Leblanc and Henkin valuations are Gödel valuations given by very simple Gödel Structures. The definitions of 'Gödel logical truth' and 'Gödel logical consequence' ($\Vdash_{\overline{G}}$) can be constructed along the lines of earlier definitions. We shall now establish the equivalence between Gödel valuations and referential valuations.

Theorem 3 *Let $\mathcal{S}$ be a Gödel Structure headed by $\langle \mathsf{L}, \mathcal{V} \rangle$. Then there is a nonempty domain $\boldsymbol{\theta}$ and a referential valuation $\mathcal{V}'$ of L for $\boldsymbol{\theta}$ which assigns to sentences of L the same values they receive in the Gödel valuation for $\mathcal{S}$.*

Proof: For each individual constant δ of $\langle \mathsf{L}^{+\alpha}, \mathcal{V}_\alpha \rangle$ located at ε, the pair $\langle \delta, \varepsilon \rangle$ is a *located individual constant* of $\mathcal{S}$. Let $\boldsymbol{\theta}$ consist of equivalence classes of located individual constants of $\mathcal{S}$ that express the same concepts. The construction of $\mathcal{V}'$ is straightforward.

Theorem 4 *Let $\boldsymbol{\theta}$ be a nonempty domain and let $\mathcal{V}$ be a referential valuation of L for $\boldsymbol{\theta}$. Then there is a Gödel Structure $\mathcal{S}$ for which there are at least as many distinct individual concepts as there are distinct individuals in $\boldsymbol{\theta}$, and for which the sentences of L have the same values as for $\mathcal{V}$.*

Proof: If only a finite number of individuals of $\boldsymbol{\theta}$ are not named by constants of L (as evaluated by $\mathcal{V}$), this theorem is trivial. Suppose an infinite number of individuals of $\boldsymbol{\theta}$ fail to be named by constants of L. Let $\boldsymbol{\theta}'$ be the set of all unnamed individuals in $\boldsymbol{\theta}$, and let R_i be a well-ordering relation for $\boldsymbol{\theta}'$. Let $\boldsymbol{\eta}$ be the set of all finite, nonempty subsets of $\boldsymbol{\theta}'$ (the set of all unordered n-tuples of individuals of $\boldsymbol{\theta}'$, for $n > 0$), and let R_η be a well-ordering relation for $\boldsymbol{\eta}$. Let $\mathcal{V}'$ be the Leblanc interpreting function for L which assigns to atomic sentences the same values they receive for $\mathcal{V}$. Then $\langle \mathsf{L}, \mathcal{V}' \rangle$ is the head of the Gödel Structure being constructed.

For each $n > 0$, let L^{+n} be obtained from L by adding individual constants ${}_1a_1, \ldots, {}_1a_n$. For a set $\{k_1, \ldots, k_n\}$ of individuals of $\boldsymbol{\theta}'$, ordered by R_i, let $\mathcal{V}_{\langle k_1,\ldots,k_n\rangle}$ be the Leblanc interpreting function of L^{+n} which agrees with $\mathcal{V}'$ on the atomic sentences of L and which assigns to atomic sentences containing ${}_1a_1, \ldots, {}_1a_n$ the values that corresponding quasi-wffs of L would receive from $\mathcal{V}$ for the values $k_1, \ldots, k_n$ of their free variables. The first-level pairs of the Gödel Structure are pairs $\langle \mathsf{L}^{+n}, \mathcal{V}_{\langle k_1,\ldots,k_n\rangle} \rangle$ ordered by the obvious relation obtained from R_η. For each m, n (both >0), let $\mathsf{L}^{+n,m}$ be obtained from L^{+n} by adding individual constants ${}_2^1a_1, \ldots, {}_2^1a_m$. For each two distinct first-level positions 1, ρ and 1, σ of pairs $\langle \mathsf{L}^{+m}, \mathcal{V}_{\langle k_1',\ldots,k_m'\rangle} \rangle$ and $\langle \mathsf{L}^{+n}, \mathcal{V}_{\langle k_1,\ldots,k_n\rangle} \rangle$, let $\mathcal{V}_{\sigma,\rho}$ be the Leblanc interpreting function of $\mathsf{L}^{+n,m}$ which assigns atomic sentences common to L^{+n} the same values they receive from $\mathcal{V}_{\langle k_1,\ldots,k_n\rangle}$, and which "treats" the constants ${}_2^1a_1, \ldots, {}_2^1a_m$ as if they were names of $k_1', \ldots, k_m'$. The immediate successors of a pair $\langle \mathsf{L}^{+n}, \mathcal{V}_{\langle k_1,\ldots,k_n\rangle} \rangle$ at position 1, σ are the pairs $\langle \mathsf{L}^{+n,m}, \mathcal{V}_{\sigma,\rho} \rangle$ at 1, σ, ρ obtained from every other first-level pair $\langle \mathsf{L}^{+m}, \mathcal{V}_{\langle k_1',\ldots,k_m'\rangle} \rangle$ at a position 1, ρ (ρ must be different from σ). The immediate successors of the second-level pairs are obtained in the same manner as the second-level pairs were obtained from the first-level pairs. And so on. This is a relatively simple Gödel Structure, because new predicates are never introduced, and new individual constants are not introduced after the first level.

Now Theorem 4 is a consequence of the following lemma, which is proved by induction on the length of A:

Let A be a quasi-wff of L containing free occurrences of distinct individual variables $\gamma_1, \ldots, \gamma_n$ and no others. Let $k_1, \ldots, k_n$ be individuals of $\boldsymbol{\theta}$ for which, when assigned as values to $\gamma_1, \ldots, \gamma_n$, $\mathcal{V}$ would assign value W to A. Then in every $\langle \mathsf{L}^{+\alpha}, \mathcal{V}_\alpha \rangle$ at position δ which has constants $\beta_1, \ldots, \beta_n$ "corresponding" to $k_1, \ldots, k_n$, $\mathbf{S}^{\gamma_1\ldots\gamma_n}_{\beta_1\ldots\beta_n}A|$ has value W.

(The notation for substitution is from [1]; it means that free occurrences of $\gamma_1, \ldots, \gamma_n$ are replaced by $\beta_1, \ldots, \beta_n$.)

The statement of Theorem 4 says that $\mathcal{S}$ has *at least* as many individual concepts because in $\mathcal{S}$ more than one concept corresponds to the same individual of $\boldsymbol{\theta}$. This is because each new constant in each first-level pair is regarded as expressing a concept distinct from that expressed by the new constants in every other first-level pair. But constants in different first-level pairs correspond to the same individual in $\boldsymbol{\theta}'$. This situation could be remedied if we provided for identifying concepts by some adaptation of Leibniz's criterion. These theorems establish that referential logical truths coincide with Gödel logical truths, and similarly for the two kinds of logical consequence. The analogue of the Lowenheim-Skolem Theorem for Gödel valuations says that for every Gödel valuation for a Gödel Structure headed by $\langle \mathsf{L}, \mathcal{V} \rangle$, there is a Henkin valuation which gives the same values to the sentences of L. This has roughly the same intuitive force for Gödel valuations that the Lowenheim-Skolem Theorem has for referential valuations. But it is less surprising for Gödel valuations. Because as far as the sentences of L go (there are just denumerably many), their values clearly depend on what happens in (at most) denumerably many first-level pairs $\langle \mathsf{L}^{+\alpha}, \mathscr{V}_\alpha\rangle$ or their successors. We can modify the Gödel Structure by dropping all predicates not found in L, and then consider a branch determined by some denumerable set of pairs that "suffices" for the values given sentences of L. This branch can then be consolidated into one Gödel Structure $\{\langle \mathsf{L}, \mathscr{V}\rangle_1, \langle \mathsf{L}^+, \mathscr{V}'\rangle_{1,1}\}$, where L^+ is the union of all the $\mathsf{L}^{+\alpha}$ on the denumerable branch, and $\mathscr{V}'$ is the union of the corresponding functions.

5 *Some incompleteness results* So far we have considered three substitution-instance interpretations of quantifiers; Leblanc interpreting functions are fundamental to each interpretation. Leblanc valuations are least well suited to treating L as if it were a natural language. And for Leblanc valuations, customary first-order formal systems are sound but not complete (they are complete with respect to logical truth but not with respect to logical consequence). Henkin valuations make possible a more realistic treatment of L, and the customary formal systems seem tailor-made for these valuations. Gödel valuations provide the most realistic treatment for L, and they most closely parallel referential valuations. But I find Gödel valuations simpler than referential valuations in this respect: Gödel valuations enable us to make clear sense of quantification without bringing in the "auxiliary" notion of reference. Substitution-instance interpretations have given us a different perspective on completeness results and on the Lowenheim-Skolem Theorem. We must now consider incompleteness results, for these are also easier to make sense of from a substitution-instance standpoint than from a referential one. We shall begin with our first-order L, because the strategy used in Gödel's Incompleteness Theorem does not require that we be dealing with higher-order languages. We need the following definitions:

Let $A(\alpha_1, \ldots, \alpha_n)$ be a quasi-wff of L which contains free occurrences of n distinct individual variables $\alpha_1, \ldots, \alpha_n$ and no others. Then $A(\alpha_1, \ldots, \alpha_n)$ is an *n-adic predicative expression of* L. (This use of 'predicative' is unrelated to Russell's use of the same word for what he called *predicative functions*.)

Let Φ be a 1-1 function mapping a subset of the individual constants of L onto the expressions and (finite) sequences of expressions of L. Then Φ *assigns the expressions and sequences of expressions of* L *to some of the constants of* L. If α is an individual constant of L for which Φ is defined, then α *names* $\Phi(\alpha)$ *with respect to* Φ.

Let $\mathscr{V}$ be a Leblanc, Henkin, or Gödel valuation of the sentences of L (to be entirely precise, we should specify a Gödel Structure if $\mathscr{V}$ is to be a Gödel valuation). Let Φ be an assignment of the expressions and sequences of expressions of L to some of the constants of L. Let ψ be a property (relation) of n-tuples of expressions and sequences of expressions of L. Then $A(\alpha_1, \ldots, \alpha_n)$ *represents* ψ *with respect to* Φ *under* $\mathcal{V}$ if for n-tuples of individual constants $\beta_1, \ldots, \beta_i$ for which $\Phi(\beta\ )$ is defined, $A(\beta_1, \ldots, \beta_n)$ has value **T** for $\mathcal{V}$ iff $\psi(\Phi(\beta_1), \ldots, \Phi(\beta_n))$. And $A(\beta_1, \ldots, \beta_n)$ has value **F** if there is some β_i for which $\Phi(\beta_i)$ is undefined.

Now, for Leblanc valuations, we can obtain the following result.

Theorem 5 *Let $\mathcal{V}$ be a Leblanc valuation of* L. *Let Φ assign the expressions and sequences of expressions of* L *to some of the constants of* L. *Let $Q(\alpha_1, \alpha_2)$ be a binary predicative expression which represents $\Phi(\alpha_1) = \alpha_2$ with respect to Φ under $\mathcal{V}$.* (I.e., *$Q(\beta_1, \beta_2)$ represents that the constant β_1 names the constant β_2 with respect to Φ.) Let $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a quadratic predicative expression which represents the following relation: $\Phi(\alpha_4)$ is an expression obtained from an expression $\Phi(\alpha_1)$ by substituting $\Phi(\alpha_2)$ for all occurrences of $\Phi(\alpha_3)$ in $\Phi(\alpha_1)$. And let $B(\alpha_1)$ be a monadic predicative expression. Then there is a sentence A of* L, *named by a constant α (with respect to Φ) such that A has value* **T** *for $\mathcal{V}$ iff $B(\alpha)$ has value* **T** *for $\mathcal{V}$.*

Proof: Let constant β_1 name the variable 'x_1' with respect to Φ. Let constant β_2 name '$(\forall x_2)(\forall x_3)[S(x_1, x_2, \beta_1, x_3) \;\&\; Q(x_2, x_1) \supset B(x_3)]$.' Let the constant β_3 name the constant β_2. And let constant β_4 name

$$\text{`}(\forall x_2)(\forall x_3)[S(\beta_2, x_2, \beta_1, x_3) \;\&\; Q(x_2, \beta_2) \supset B(x_3)].\text{'}$$

Now the sentence A is the one named by the constant β_4, and α is the constant β_4. For suppose A has value **T**; i.e., suppose that

1 $(\forall x_2)(\forall x_3)[S(\beta_2, x_2, \beta_1, x_3) \;\&\; Q(x_2, \beta_2) \supset B(x_3)]$

Then we can obtain the following:

2	$S(\beta_2, \beta_3, \beta_1, \beta_4) \;\&\; Q(\beta_3, \beta_2) \supset B(\beta_4)$	From 1
3	$S(\beta_2, \beta_3, \beta_1, \beta_4)$	These were given in the statement of the theorem.
4	$Q(\beta_3, \beta_2)$	
5	$B(\beta_4)$	From 2, 3, 4, by modus ponens

Now suppose $B(\beta_4)$ has value **T** for $\mathcal{V}$. Then for any constants γ, δ, if '$S(\beta_2, \gamma, \beta_1, \delta) \;\&\; Q(\gamma, \beta_2)$' has value **T** for $\mathcal{V}$, then γ is the constant β_3 and δ is the constant β_4 (because Φ is 1-1 and substitution has a unique result). So $B(\delta)$ will have value **T** for $\mathcal{V}$. Hence,

'$(\forall x_2)(\forall x_3)[S(\beta_2, x_2, \beta_1, x_3) \;\&\; Q(x_2, \beta_2) \supset B(x_3)]$' has value **T** for $\mathcal{V}$.

Theorem 5 is not true for Henkin and Gödel valuations. We can understand why it is not if we consider the second half of the proof of Theorem 5. For Henkin and Gödel valuations, we would not have that γ is the constant β_3 and δ is the constant β_4. Both sorts of valuation consider L as subject to being extended. The fact that a Henkin or Gödel valuation of L satisfies the conditions of Theorem 5 is not sufficient to guarantee that $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $Q(\alpha_1, \alpha_2)$ will represent what they are supposed to in extensions of L. In some extension we might have '$S(\beta_2, \gamma, \beta_1, \delta)$ & $Q(\gamma, \beta_2)$' being true for new constants γ, δ that are different from β_3, β_4. But then we cannot argue that if $B(\beta_4)$ has value **T**, so does A.

In order to get Theorem 5 for Henkin and Gödel valuations, we can enrich L with identity to produce L′. Leblanc interpreting functions for L′ must "respect" the values assigned to atomic identity statements. And extension pairs $\langle L'^{+}, \mathcal{V}^{+}\rangle$ of $\langle L', \mathcal{V}\rangle$ must also "respect" the true identities of $\langle L', \mathcal{V}\rangle$ when new predicates are added.[5] We can find valuations of language L′ which place sufficient restrictions on extensions (and their valuations) to give us the following result.

Theorem 5a *Let $\mathcal{V}$ be a valuation of* L′ *($\mathcal{V}$ can be a Leblanc, Henkin, or Gödel valuation). Let Φ, $Q(\alpha_1, \alpha_2)$, $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $B(\alpha_1)$ be as in* Theorem 5. *In addition, let the following sentences of* L′ *have value* **T** *for $\mathcal{V}$:*

$$(\forall x_1)(\forall x_2)(\forall x_3)[Q(x_1, x_2) \;\&\; Q(x_1, x_3) \supset x_2 = x_3]$$
$$(\forall x_1)(\forall x_2)(\forall x_3)[Q(x_1, x_3) \;\&\; Q(x_2, x_3) \supset x_1 = x_2]$$
$$(\forall x_1)(\forall x_2)(\forall x_3)(\forall x_4)(\forall x_5)[S(x_1, x_2, x_3, x_4) \;\&\; S(x_1, x_2, x_3, x_5) \supset x_4 = x_5]$$

Then there is a sentence A, named by a constant α (with respect to Φ), such that A has value **T** *for $\mathcal{V}$ iff $B(\alpha)$ has value* **T** *for $\mathcal{V}$.*[6]

For Theorems 5 and 5a, there are corollaries related to Tarski's results in [11].

Corollary *Let the conditions be as in* Theorem 1 (1a). *Let $T(\alpha_1)$ be a monadic predicative expression such that if constant β names a false sentence (with respect to Φ), then $T(\beta)$ has value* **F** *for $\mathcal{V}$. Then there is a sentence A named by a constant α which has value* **T** *for $\mathcal{V}$ but for which $T(\alpha)$ has value* **F** *for $\mathcal{V}$.*

Let us move back to L, and consider just Leblanc valuations of L. Let Φ be as before, and let $Q(\alpha_1, \alpha_2)$, $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $C_{\mathfrak{M}}(\alpha_1)$ be predicative expressions with the number of free variables indicated. Consider how we would construct a set $\mathfrak{M}$ of sentences of L such that for every Leblanc valuation $\mathcal{V}$ which makes all sentences of $\mathfrak{M}$ true,

$Q(\alpha_1, \alpha_2)$ represents $\Phi(\alpha_1) = \alpha_2$ with respect to Φ under $\mathcal{V}$,
$S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ represents substitution as before,
$C_{\mathfrak{M}}(\alpha_1)$ represents $\mathfrak{M} \vdash \Phi(\alpha_1)$.

To construct this $\mathfrak{M}$ we could begin with Q: If $\Phi(\beta_1) = \beta_2$, then '$Q(\beta_1, \beta_2)$' is in $\mathfrak{M}$; otherwise '$\sim Q(\beta_1, \beta_2)$' is in $\mathfrak{M}$. We could similarly put in the "true" instances of $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and put in the negations of the rest. We would also need other predicative expressions to characterize expressions and sequences of expressions of L, and they could be treated in the same way. Finally we would need a monadic predicative expression $M(\alpha_1)$ to represent membership in $\mathfrak{M}$. And we could again put in the true instances and the negations of all the rest. Given all this apparatus, we could construct a suitable $C_{\mathfrak{M}}$. But then, by Theorem 5, there is a sentence A of L, named by a constant α (with respect to Φ), such that for every Leblanc valuation $\mathcal{V}$

which makes all sentences of $\mathfrak{M}$ true, A has value T iff $\sim C_{\mathfrak{M}}(\alpha)$ has value T. But A must have value T. So we have $\mathfrak{M} \Vdash_{\overline{\mathrm{L}}} A$, but it is false that $\mathfrak{M} \vdash A$. This result is not paradoxical. For we already know that the formal system $\mathfrak{F}$ is not complete with respect to Leblanc logical consequence. The truth of A depends essentially on the truth of the members of infinite set $\mathfrak{M}$, and not on the truth of the members of any finite subset of $\mathfrak{M}$.

We might try to replace L by L′, and construct a similar set $\mathfrak{M}$ for Henkin and Gödel valuations of L′. Such an $\mathfrak{M}$ must contain more than is required for Leblanc valuations. For we must put some sentences in $\mathfrak{M}$ to insure that in extensions of L′, the various predicative expressions continue to represent what they are supposed to. Failing this, quantified sentences could be true (or false) for the wrong reasons. If we had a list of names (with respect to Φ) of the members of $\mathfrak{M}$, we might try to put in $\mathfrak{M}$ a sentence which has this effect:

$$(\forall x_1)\,[M(x_1) \supset \text{there is one of those names such that it} = x_1].$$

But we cannot do this with identity statements alone, for we are limited to finite conjunctions and disjunctions of them. So we might think of putting arithmetic axioms in $\mathfrak{M}$, "identifying" the integers with the constants of L′, and giving a numerical characterization of the members of $\mathfrak{M}$. But this will not work. Nothing will work, for we can easily prove the following theorem.

Theorem 6 *Let Φ be an assignment of the expressions and sequences of expression of* L′ *to some of the constants of* L′. *Then there is no set $\mathfrak{M}$ such as that described above for Henkin or Gödel valuations of* L′.

Gödel's Incompleteness Theorem shows that we can form a set $\mathfrak{M}$ for which we have certain intentions, but that if these intentions were realized we would have a paradoxical result. (I.e., $\mathfrak{M} \Vdash_{\overline{\mathrm{H}}} A$ and $\mathfrak{M} \Vdash_{\overline{\mathrm{G}}} A$, but not $\mathfrak{M} \vdash A$.) There is no set $\mathfrak{M}$ that places enough restrictions on extensions of L′ to make them fully conform to our intentions. This is an incompleteness result for L′, but the incompleteness it shows is an incompleteness of "expressive power."

6 *Higher-order languages* Let us consider what happens to the substitution-instance interpretations of quantifiers when we add predicate variables and put them in quantifiers. To understand this, it is sufficient to consider second-order languages. Let L_2 be the language obtained from L by adding:

monadic predicate variables: $f_0^1, f_1^1, f_2^1, \ldots;$
binary predicate variables: $f_0^2, f_1^2, f_2^2, \ldots;$ etc.

And allowing quantification of predicate variables. As far as valuations go, we can just extend our earlier definitions of Leblanc valuations, Henkin valuations, and Gödel valuations in the obvious ways.

However, there are certain complications in the higher-order case. In the customary formal systems that contain predicate variables and a rule of substitution for them, it is not just predicates (or predicate variables) that can replace predicate variables. Instead these variables are replaced by predicative expressions or formulas having the right number of distinct free variables. (This is the same use of 'predicative' as in the preceding section. A predicative expression has free occurrences of individual variables, and so acts like a predicate. Predicative expressions have nothing to do with Russell's predicative functions; a predicative expression may contain quantified predicate variables.) So it would seem appropriate to regard predicate variables as symbolizing predicative expressions. But we cannot do this, for it would prevent us from giving recursive (inductive) definitions of the various kinds of valuations (because the sentences that can be obtained by substitution are frequently longer than the quantified formula). On the other hand, if we regard predicate variables as ranging over just predicates, customary formal systems for higher-order languages are not sound.

We could easily construct new formal systems to accommodate our new approach to quantified predicate variables. But this would not lead to very interesting results; for a language without a predicate to match every predicative expression will be weak in expressive power. To resolve this dilemma, we will consider predicate variables as ranging over just predicates, and *then restrict our attention to those evaluated languages which are complete with respect to predicates.* These are the languages which have a predicate φ for every predicative expression $A(\alpha_1, \ldots, \alpha_n)$.

More specifically, let $\mathcal{V}$ be a Leblanc interpreting function of L_2. The Leblanc valuation of L_2 for $\mathcal{V}$ is *complete with respect to predicates* iff for every n-adic predicative expression $A(\alpha_1, \ldots, \alpha_n)$ in L_2, there is an n-adic predicate φ of L_2 such that $(\forall x_1) \ldots (\forall x_n)[A(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n)]$ has value $\mathbf{T}$ for the Leblanc valuation of L_2 for $\mathcal{V}$.

A Henkin valuation $\mathcal{V}$ of L_2 is *complete with respect to predicates* iff there is an extension L_2^+ of L_2 and a Leblanc valuation $\mathcal{V}'$ of L_2^+ which is complete with respect to predicates, and which agrees with $\mathcal{V}$ on the sentences of L_2.

We can show that for every pair $\langle \mathsf{L}_2, \mathcal{V} \rangle$, where $\mathcal{V}$ is a Leblanc interpreting function of L_2, there is an extension pair $\langle \mathsf{L}_2^+, \mathcal{V}' \rangle$ of $\langle \mathsf{L}_2, \mathcal{V} \rangle$ such that the Leblanc valuation $\mathcal{V}'$ of L_2^+ is complete with respect to predicates. So for every Leblanc interpreting function $\mathcal{V}$ of L_2, there is a Henkin valuation $\mathcal{V}^*$ of L_2 that agrees with $\mathcal{V}$ on the atomic sentences of L_2 and that is complete with respect to predicates. This result suggests the use of a formal system with a restricted rule of substitution (only predicates or predicate variables for predicate variables) and a rule of definition for predicates. Leśniewski's formal systems were of this sort, and in [6] Henkin suggests using such systems.

Let $\mathcal{S}$ be a Gödel Structure headed by $\langle \mathsf{L}_2, \mathcal{V} \rangle$. $\mathcal{S}$ is *complete with respect to predicates* iff for every n-adic predicative expression $A(\alpha_1, \ldots, \alpha_n)$ in $\langle \mathsf{L}_2^{+\beta}, \mathcal{V}_\beta \rangle$ at ε in $\mathcal{S}$, there is an n-adic predicate φ in $\langle \mathsf{L}_2^{+\gamma}, \mathcal{V}_\gamma \rangle$ at ε' in $\mathcal{S}$, where ε' is either ε or an extension of ε, such that

$(\forall x_1) \ldots (\forall x_n)[A(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n)]$ has value $\mathbf{T}$ in $\langle \mathsf{L}_2^{+\gamma}, \mathcal{V}_\gamma \rangle$ at ε'

for the Gödel valuation of $\mathcal{S}$. We will also say that the Gödel valuation of $\mathcal{S}$ is complete with respect to predicates.

Let $\mathfrak{F}_2$ be a formal system for L_2 obtained by adapting $\mathfrak{F}$ (as in [1]). $\mathfrak{F}_2$ is clearly sound for Leblanc, Henkin, and Gödel valuations that are complete with respect to predicates. $\mathfrak{F}_2$ is not complete with respect to Leblanc logical consequence, because the set $\{F_0^1(a_0), F_0^1(a_1), \ldots\}$ still implies $(\forall x_0) F_0^1(x_0)$ without our being able to prove it. But $\mathfrak{F}_2$ is complete for Henkin logical consequence for Henkin valuations that are complete with respect to predicates. Since Henkin valuations of L_2 "amount" to Gödel valuations given by a Gödel Structure with a single first-level pair, $\mathfrak{F}_2$ must also be complete for Gödel valuations that are complete with respect to predicates. But what does this do to Gödel's Incompleteness Theorem? To get that result we must consider a restricted class of Gödel Structures; and we need the following definitions:

If δ, γ are individual constants of $\langle \mathsf{L}_2^{+\alpha}, \mathscr{V}_\alpha\rangle$ at ε in Gödel Structure $\mathcal{S}$, then located individual constant $\langle \delta, \varepsilon\rangle$ is *distinguishable from* located individual constant $\langle \gamma, \varepsilon\rangle$ if there is a sentence A of $\mathsf{L}_2^{+\alpha}$ which contains δ and which has value T in $\langle \mathsf{L}_2^{+\alpha}, \mathscr{V}_\alpha\rangle$, which yields a sentence with value F when the occurrences of δ are replaced by occurrences of γ. (The definition of 'located individual constant' is given in the proof of Theorem 3 of section **4**.)

If δ is an individual constant of $\langle \mathsf{L}_2^{+\alpha}, \mathscr{V}_\alpha\rangle$ at ε in $\mathcal{S}$, and γ is an individual constant of $\langle \mathsf{L}_2^{+\beta}, \mathscr{V}_\beta\rangle$ at ε', then $\langle \delta, \varepsilon\rangle$ is *distinguishable from* $\langle \gamma, \varepsilon'\rangle$ if there are constants ρ, σ of $\langle \mathsf{L}_2^{+\mu}, \mathscr{V}_\mu\rangle$ at ε'' which are such that ρ at ε'' expresses the same concept as δ at ε, constant σ at ε'' expresses the same concept as γ at ε', and $\langle \rho, \varepsilon''\rangle$ is distinguishable from $\langle \sigma, \varepsilon''\rangle$.

Let φ be a predicate in $\langle \mathsf{L}_2^{+\alpha}, \mathscr{V}_\alpha\rangle$ at location ε of Gödel Structure $\mathcal{S}$. Then $\langle \varphi, \varepsilon\rangle$ is a *located predicate* of $\mathcal{S}$.

Let φ, ψ be n-adic predicates in $\langle \mathsf{L}_2^{+\alpha}, \mathscr{V}_\alpha\rangle_\varepsilon$ in Gödel Structure $\mathcal{S}$. And let $(\forall x_1) \ldots (\forall x_n) [\varphi(x_1, \ldots, x_n) \equiv \psi(x_1, \ldots, x_n)]$ have value T in $\langle \mathsf{L}_2^{+\alpha}, \mathscr{V}_\alpha\rangle_\varepsilon$. Then $\langle \varphi, \varepsilon\rangle$ is *extensionally equivalent* to $\langle \psi, \varepsilon\rangle$.

Let φ be an n-adic predicate in $\langle \mathsf{L}_2^{+\alpha}, \mathscr{V}_\alpha\rangle_\varepsilon$, and let ψ be an n-adic predicate in $\langle \mathsf{L}_2^{+\beta}, \mathscr{V}_\beta\rangle_{\varepsilon'}$ of $\mathcal{S}$. Let φ in $\langle \mathsf{L}_2^{+\alpha}, \mathscr{V}_\alpha\rangle_\varepsilon$ express the same concept as n-adic predicate θ in $\langle \mathsf{L}_2^{+\gamma}, \mathscr{V}_\gamma\rangle_{\varepsilon''}$, and let ψ in $\langle \mathsf{L}_2^{+\beta}, \mathscr{V}_\beta\rangle_{\varepsilon'}$ express the same concept as n-adic predicate Ξ in $\langle \mathsf{L}_2^{+\gamma}, \mathscr{V}_\gamma\rangle_{\varepsilon''}$. Let $\langle \theta, \varepsilon''\rangle$ be extensionally equivalent to $\langle \Xi, \varepsilon''\rangle$. Then $\langle \varphi , \varepsilon\rangle$ is *extensionally equivalent* to $\langle \psi, \varepsilon'\rangle$.

Let $\boldsymbol{\theta}$ be the set of equivalence classes of nondistinguishable located individual constants of $\mathcal{S}$. And let P^n be the set of located (n-adic) predicates that are extensionally equivalent to some located n-adic predicate $\langle \varphi, \varepsilon\rangle$. We can talk about P^n as if it were a predicate, and consider its value for a given n-tuple of elements of $\boldsymbol{\theta}$. So we will say, informally, that P^n is an n-adic predicate of (n-tuples of) elements of $\boldsymbol{\theta}$.

A *Strong Gödel Structure* is one which contains every n-adic predicate P^n of elements of $\boldsymbol{\theta}$ for every finite $n > 0$. (I.e., it is a Gödel Structure for which every such P^n is nonempty.)

$\mathfrak{F}_2$ is not complete for Strong Gödel Structures and their valuations. For consider the set X with these sentences in it (the identity symbol is defined by Leibniz's definition)[7]:

$$\sim(\exists x_0)\, F_0^2(x_0, a_0)$$
$$F_0^2(a_0, a_1),\ F_0^2(a_1, a_2),\ \ldots$$
$$(\forall x_0)(\forall x_1)(\forall x_2)\,[F_0^2(x_0, x_1)\ \&\ F_0^2(x_0, x_2) \supset x_1 = x_2]$$
$$(\forall x_0)(\forall x_1)(\forall x_2)\,[F_0^2(x_0, x_2)\ \&\ F_0^2(x_1, x_2) \supset x_0 = x_1]$$
$$F_0^1(a_0),\ (\forall x_0)(\forall x_1)\,[F_0^1(x_0)\ \&\ F_0^2(x_0, x_1) \supset F_0^1(x_1)]$$
$$(\forall f_0^1)\,[f_0^1(a_0)\ \&\ (\forall x_0)(\forall x_1)\,[f_0^1(x_0)\ \&\ F_0^2(x_0, x_1) \supset f_0^1(x_1)] \supset (\forall x_0)\,[F_0^1(x_0) \supset f_0^1(x_0)]]$$

This set has $(\forall x_0)\,[F_0^1(x_0) \supset (\exists x_1)\, F_0^2(x_0, x_1)]$ as a *Strong Gödel consequence* ($\Vdash_{SG}$), but the sentence cannot be deduced from the set by means of $\mathfrak{F}_2$. (The induction sentence is required, for without it the set would not keep other constants—in extensions of L_2—from having 'F_0^1' truly applied to them.) This incompleteness of $\mathfrak{F}_2$ is very much akin to the incompleteness of $\mathfrak{F}$ for Leblanc valuations of L.

Now consider the set Y with these sentences in it:

$$\sim(\exists x_0)\, F_0^2(x_0, a_0)$$
$$F_0^1(a_0)$$
$$(\forall x_0)\,[F_0^1(x_0) \supset (\exists x_1)\,[F_0^2(x_0, x_1)\ \&\ F_0^1(x_1)]]$$
$$(\forall x_0)(\forall x_1)(\forall x_2)\,[F_0^2(x_0, x_1)\ \&\ F_0^2(x_0, x_2) \supset x_1 = x_2]$$
$$(\forall x_0)(\forall x_1)(\forall x_2)\,[F_0^2(x_0, x_2)\ \&\ F_0^2(x_1, x_2) \supset x_0 = x_1]$$
$$(\forall f_0^1)\,[f_0^1(a_0)\ \&\ (\forall x_0)(\forall x_1)\,[f_0^1(x_0)\ \&\ F_0^2(x_0, x_1) \supset f_0^1(x_1)] \supset (\forall x_0)\,[F_0^1(x_0) \supset f_0^1(x_0)]]$$

This set guarantees the existence of a numerical structure. We could use the procedure of Gödel's Incompleteness Theorem to show that there is a sentence A which is a Strong Gödel consequence of Y, but which cannot be deduced from Y by $\mathfrak{F}_2$. (Doing this would be much more complicated than Gödel's own proof, because the set Y does not identify any numerals except 'a_0.' And all the numerals may not be found in L_2, or even in any one first-level extension pair of $\langle \mathsf{L}_2, \mathscr{V} \rangle$. A more manageable set with Y's effect would be possible if L_2 were enriched with functional expressions. Then we could use a finite set to identify the numerals with a_0, a_0', a_0'', etc.) Since Y is finite, there is a sentence which is a Strong Gödel logical truth, but which is not deducible by $\mathfrak{F}_2$. So $\mathfrak{F}_2$ turns out to be doubly incomplete for Strong Gödel Structures and their valuations.

If we try to describe carefully what is going on, we can say that a finite set (or a single sentence) is used to "generate" an infinite set (the numerical structure). We can then construct a predicative expression which forms a true sentence when its free variable is replaced by any member of the set. This predicative expression signifies that the substituted (numerical) constant is not the name of a proof of the unprovable sentence. So there is a predicative expression true for all members of the infinite set (if $\mathfrak{F}_2$ is consistent of course), but this truth cannot be proved by induction. The truth of the unprovable quantified consequence depends essentially on the truth of its infinitely many instances. By restricting Gödel Structures to Strong Gödel Structures we have increased the expressive power of the language L_2. This expressive power now outstrips deducibility, for there are logical truths and logical consequences which cannot be established with our formal system. We cannot similarly strengthen the formal system, so long as we stick to rules (procedures) that people might actually be able to use.

7 *Existence* For every nonempty domain θ and referential valuation of L or L_2 for θ, there is a substitution-instance valuation which awards the same values to sentences of L (or L_2). And there are substitution-instance valuations whose quantifiers "cover as much territory" as the quantifiers of referential valuations. This raises questions about the existential significance of the existential quantifier. For substitution-instance quantifiers do not have existential import, but they "coincide" with referential quantifiers which are alleged to have such import. If we proceed semantically with referential valuations, we really settle questions of existence by our choice of a domain—not by using a quantifier. (Consider the difference between choosing a domain whose members are astronomical objects, and choosing the domain of real numbers.) If we proceed logistically (in Church's sense, as explained in [1]), we cannot distinguish a referential from a substitution-instance interpretation. For with both interpretations we will have the same axioms and rules of inference. And in both cases we will find the same sentences coming out true. Existential considerations fall outside the scope of the formalisms of our formal languages.

We could impose restrictions on substitution-instance valuations to make them more existential. If we suppose that we have some independent criteria for distinguishing the names of existents from names of non-existents, we could limit the names in L (or L_2) and its extensions to the names of existents. Then, even in substitution-instance valuations, the "existential" quantifier would indicate existence when individual variables were quantified. And existentially quantified predicate variables would correspond to sets of n-tuples of existing things. However, this seems pointless. The equivalence of the two sorts of valuations shows that quantification is not well-suited for distinguishing between what exists and what does not. And we do not increase our understanding of this difference by imposing external restrictions on the languages. Instead, with substitution-instance valuations in mind, we can consider formal languages that, like natural languages, contain both empty and nonempty names. Perhaps this can serve as a first step towards getting clear about reference, and all that it involves. For the banishing of empty names from standard logical languages, or giving them a treatment which makes the difference between empty and nonempty names depend on their relation to referential quantifiers (as in [7]), has blocked a satisfactory account of reference. (Frege and Russell have so strongly influenced subsequent philosophers that it is very difficult to recognize that speakers of natural languages use empty and nonempty names in the same way.)

8 *Speculations* Russell once thought that there is a difference between the meanings of 'all' and 'any' that is important for logic and mathematics. He claimed that this difference corresponds to the difference between bound and free occurrences of variables (between apparent and real variables). If this difference is overlooked, contradictions will result. In the Preface to the second edition of *Principia Mathematica*, Russell gave up his claims about 'all' and 'any.' (Of course, there is a difference between them, but it is not what he had thought.) This was because he recognized that free occurrences of variables could be construed as bound by initially placed universal quantifiers.

Even though Russell did not succeed in expressing, or understanding, the difference he was after, I think there is a genuine difference at stake. It is primarily a difference between two kinds of generality—between two kinds of general statements. The first kind I will call *schematic generality*. This generality is expressed by quantifiers when these are given a substitution-instance interpretation; for such valuations construe variables as schematic letters (in Quine's sense). The second kind of generality is *total generality*. Totally general statements involve a totality (a set perhaps) that is something over and above its elements. Total generality is part of the subject matter of set theory; a totally general statement indicates set inclusion or some other relation between sets.

The two kinds of general statements are distinct, and confusing them might lead to contradiction (since not every statable general condition determines a set). There is an important connection between schematic generality and the open character of a natural language. Such a language is never closed or complete; there are always new expressions that might be added. (Even a Gödel Structure may distort the essential openness of a natural language, since we regard such a Structure as completed.) Henkin and Gödel substitution-instance quantifiers, and the customary formal systems for quantification, are well suited to this openness. Note that from the perspective of a user of $\mathbf{L}$ or $\mathbf{L}_2$ (with its atomic sentences evaluated), the extensions of the language are not yet available. He knows that extensions are possible, though none has yet been realized. And he may even be justified in making some universally quantified statements. Customary formal systems codify the inferences it is legitimate for him to make. Quantification is a logical (or linguistic) device suited to express schematic generality. Such generality is pretty well understood, since quantification theory is well developed. When schematic generality is involved, so-called impredicative definitions are permissible. For schematic generality does not (necessarily) involve a completed totality. Although total generality is studied in set theory, it is not so well understood. For there is no complete account of the conditions that determine a totality.

If one grants that there are these two kinds of generality, and accepts my claim that quantification is best suited to express schematic generality, then it can seem inappropriate to use set-theoretic structures to evaluate formal languages. For this amounts to "cutting down" schematic generality to those cases where it is equivalent to total generality. But there must be true schematically general sentences for which there are no equivalent totally general sentences. (Even when we talk about referential valuations, we make general statements which "outstrip" totally general statements. For we talk about sentences true for all valuations of all nonempty domains. But there is no set which contains all these domains as its elements.)

But set-theoretic structures are not so inadequate as they may seem. Recall the aim of treating formal languages like natural ones. Substitution-instance quantifiers are used in a language regarded as being already a going concern. So that its nonlogical expressions are (supposed to be) already meaningful; its individual constants already either denote or fail to denote real objects. But with respect to a natural language, there are always things without names. If all the things waiting to be named cannot be contained in a single set, then the possible extensions to our language will not all fit in a set-theoretic structure. But no matter how our language is extended, it seems plausible to claim that it will never contain more than denumerably many expressions. It also seems plausible that there are at most denumerably many candidates (shapes or sounds) for new expressions of our language. If these are really limitations on our language and its extensions, and if from the viewpoint of logic we are only concerned with the truth values of atomic sentences in the various extensions (and not with the different meanings these sentences have in the different extensions that happen to agree about the truth values of those sentences), then set-theoretic structures will be adequate for logical purposes. There is a sense in which a set-theoretic structure furnishes only a kind of "scale model" of an interpreted language whose quantifiers are understood in the substitution-instance way. But this is sufficient to convince us that substitution-instance interpretations are consistent.

NOTES

1. Hugues Leblanc deals with such valuations in [9], among other places.

2. In [9], Leblanc was concerned to show that a substitution-instance interpretation can give the same results as a referential interpretation. He succeeded, and his success has some philosophical significance, since a substitution-instance interpretation is ontologically less committing than the referential variety. But he was clearly not developing the substitution-instance interpretation "on its own terms."

3. In [1], Alonzo Church stipulates that he uses the term 'language' in such a way that, if a new expression is introduced into a given language, the result is a new language (p. 48, note 111). And this usage seems to be the most common in logic. However, it must not be thought that his use of 'language' is *the* (only) correct use. For the ordinary way of talking about languages has it that a language can be the same language even though it changes in various ways—we might say, for

example, that residents of the United States spoke English in 1850, and that residents of the U.S. continue to speak English today, though the language has changed considerably during that time. This manner of using the term 'language' is just as legitimate as Church's. The difference between the two uses really amounts to the difference between two systems of classification. But alternative systems of classification are not right or wrong, they are only more or less useful for a given purpose.

4. The terms 'Henkin' and 'Gödel' have not been chosen because either Henkin or Gödel investigated the valuations named for them. I have chosen these names because Henkin valuations lead naturally to Henkin's Completeness Theorem, and Gödel valuations lead to Gödel's Incompleteness Theorem.

5. For Henkin valuations, it is sufficient to respect true identities '$a_i = a_j$' by making sure that a_i and a_j are everywhere interchangeable. We do not have to require that for false identities there must be a schema $A(\alpha_1)$ which is true for a_i and false for a_j. But for Gödel valuations, it would be reasonable to impose this requirement, i.e., that such a schema $A(\alpha_1)$ be located someplace in the Gödel Structure. However, I will not bother to give a precise formulation of this requirement.

6. Theorems 5 and 5a are simpler to state (and prove) in a language which has functional expressions, so that an expression with places for names is well-formed when these places are occupied by functional expressions. In such a language, sentence A can turn out to be $B(\alpha)$; but α will be a functional expression, not a simple constant.

7. The sentences in the set can obviously be regarded as numerical axioms. But I would not say that they are axioms for *the* natural numbers; there are no such things as *the* natural numbers. Instead there is a characteristic *natural number structure*, which has many exemplifications. In the set in question, it is exemplified by the constants $a_0, a_1, a_2, \ldots$ as ordered by F_0^2.

REFERENCES

[1] Church, A., *Introduction to Mathematical Logic*, vol. 1, Princeton University Press, Princeton (1956).

[2] Dunn, J. M., and N. D. Belnap, "The substitution interpretation of the quantifiers," *Noûs*, vol. 2 (1968), pp. 177-185.

[3] Henkin, L., "The completeness of the first-order functional calculus," *The Journal of Symbolic Logic*, vol. 14 (1949), pp. 159-166.

[4] Henkin, L., "Completeness in the theory of types," *The Journal of Symbolic Logic*, vol. 15 (1950), pp. 81-91.

[5] Henkin, L., "Some notes on nominalism," *The Journal of Symbolic Logic*, vol. 18 (1953), pp. 19-29.

[6] Henkin, L., "Banishing the rule of substitution for functional variables," *The Journal of Symbolic Logic*, vol. 18 (1953), pp. 201-208.

[7] Hintikka, J., "Studies in the logic of existence and necessity," *The Monist*, vol. 50 (1966), pp. 55-76.

[8] Kearns, J. T., "Two views of variables," *Notre Dame Journal of Formal Logic*, vol. 10 (1969), pp. 163-180.

[9] Leblanc, H., "Truth-value semantics for a logic of existence," *Notre Dame Journal of Formal Logic*, vol. 12 (1971), pp. 153-168.

[10] Luschei, E. C., *The Logical Systems of Leśniewski*, North-Holland Publishing Co., Amsterdam (1962).

[11] Tarski, A., "The concept of truth in formalized languages," in *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford (1956), pp. 152-278.

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