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FREE S5 ALGEBRAS

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To Leon Mirsky on
his sixtieth birthday

A closure algebra is a system $\langle A, \wedge, \vee, -, 0, 1, C\rangle$ such that $\langle A, \vee, \wedge, -, 0, 1\rangle$ is a Boolean algebra with smallest element 0 and largest element 1, and C is a unary operation satisfying the identities $x \leqslant Cx$, $CCx = x$, $C(x \vee y) = Cx \vee Cy$, and $C0 = 0$. A member x of A is called closed if $Cx = x$. It is well known that the theorems of the Lewis system S4 are those formulas which are valid in every closure algebra (when we interpret the possibility operator by C). In S5, the theorems are the formulas which are valid in every closure algebra such that the complement of each closed element is closed. Let us call such a closure algebra an S5 algebra. The free closure algebra with one generator is already so complicated that its structure is unknown. However, S5 algebras are much simpler. In this note we shall determine the free S5 algebras with finitely many generators. This result was stated without proof in [1], Theorem 19.

An S5 subalgebra of an S5 algebra A is a Boolean subalgebra of A which is closed under C. Similarly an S5 homomorphism $f: A_1 \rightarrow A_2$ of S5 algebras is a Boolean homomorphism such that $f(Cx) = Cf(x)$ for all $x \in A_1$. If A is any Boolean algebra, we let A^* be the S5 algebra obtained by introducing in A the closure operation C such that $Cx = 1$ for all $x \neq 0$ and $C0 = 0$. It is known that a formula is a theorem of S5 if and only if it is valid in every S5 algebra of the form A^*. We shall not make use of this fact. Indeed it will be a consequence of our theorem proved below.

Lemma 1 *If A is a closure algebra, then A is an* S5 *algebra, if and only if* $C(x \wedge y) = Cx \wedge y$ *for any $x \in A$ and any closed $y \in A$.*

Proof: If A is an S5 algebra, and y is closed, then $C(x \wedge y) \leqslant Cx \wedge Cy = Cx \wedge y$. Also $x \wedge y \leqslant C(x \wedge y)$ implies $x \leqslant -y \vee C(x \wedge y)$. Since $-y$ is closed, it follows that $Cx \leqslant -y \vee C(x \wedge y)$, and so $Cx \wedge y \leqslant C(x \wedge y)$. Thus $C(x \wedge y) = Cx \wedge y$.

Conversely suppose this identity holds and y is a closed element of A. Then $0 = C(-y \wedge y) = C(-y) \wedge y$. Hence $C(-y) \leqslant -y$ and therefore $-y$ is closed.

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Now suppose F is the free S5 algebra freely generated by $v_1, \ldots, v_n$. Let B be the Boolean subalgebra of F generated by $v_1, \ldots, v_n$. Let C be the set of closed elements of F. C is also a Boolean subalgebra of F.

Lemma 2 *B is a free Boolean algebra freely generated by $v_1, \ldots, v_n$.*

Proof: Let A be any Boolean algebra and $a_1, \ldots, a_n$ be arbitrary members of A. There exists an S5 homomorphism $f: F \to A^*$ such that $f(v_i) = a_i$ for all i. Then $f|B: B \to A$ is a Boolean homomorphism.

Lemma 3 *C is the Boolean subalgebra of F generated by $\{\mathsf{C}x: x \in B\}$, and F is Boolean generated by $B \cup C$.*

Proof: Let C_1 be the Boolean subalgebra of F generated by $\{\mathsf{C}x: x \in B\}$, and let F_1 be Boolean subalgebra of F generated by $B \cup C_1$. If $x \in F$, then x is of the form $\bigvee_i (b_i \wedge c_i)$, where $b_i \in B$ and $c_i \in C_1$. By Lemma 1,

$$\mathsf{C}x = \bigvee_i \mathsf{C}(b_i \wedge c_i) = \bigvee_i \mathsf{C}b_i \wedge c_i \in C_1 \subseteq F_1.$$

Thus F_1 is an S5 subalgebra of F containing $v_1, \ldots, v_n$. Hence $F = F_1$ and $C \subseteq C_1$. Since $C_1 \subseteq C$ obviously, we have $C_1 = C$.

From Lemma 3, it immediately follows that every formula of S5 can be reduced to the modal conjunctive normal form described in [2], p. 55.

By Lemma 1, B has $N = 2^n$ atoms $a_1, \ldots, a_N$, each of the form $(\pm v_1) \wedge \ldots \wedge (\pm v_n)$. Let $S_1, \ldots, S_M$, where $M = 2^N - 1$, be the set of all proper subsets of $\{1, \ldots, N\}$. For $1 \leq j \leq M$, let

$$d_j = \bigwedge_{i \notin S_j} \mathsf{C}a_i \wedge \bigwedge_{i \in S_j} (-\mathsf{C}a_i)$$

and let I_j be the ideal in B generated by $\bigvee_{i \in S_j} a_j$. (If S_j is the empty set, $\bigwedge_{i \in S_j} (-\mathsf{C}a_i) = 1$ and $\bigvee_{i \in S_j} a_j = 0$.) Note that every proper ideal in B is of the form I_j, and that $j \neq k$ implies $I_j \neq I_k$. Now let $f_j: F \to (B/I_j)^*$ be the S5 homomorphism such that $f_j(v_i) = v_i/I_j$ for all i.

Lemma 4 *We have $f_j(d_j) = 1/I_j$ and $f_j(d_k) = 0/I_j$ for $k \neq j$.*

Proof: If $i \notin S_j$, then $a_i \notin I_j$, so $f_j(a_i) = a_i/I_j \neq 0/I_j$, hence $f_j(\mathsf{C}a_i) = 1/I_j$. If $i \in S_j$ then $a_i \in I_j$ and $f_j(a_i) = a_i/I_j = 0/I_j$, hence $f_j(\mathsf{C}a_i) = 0/I_j$. The lemma now follows immediately.

Lemma 5 (i) *The atoms of C are $d_1, \ldots, d_M$.* (ii) *Every element of F is of the form $\bigvee_{j=1}^{M} x_j \wedge d_j$, where $x_j \in B$.* (iii) *The atoms of F are the elements of the form $a_i \wedge d_j$, where $i \notin S_j$.*

Proof: By Lemma 2, C is Boolean generated by $\mathsf{C}a_1, \ldots, \mathsf{C}a_N$, since every element of B is a join of the atoms $a_1, \ldots, a_N$. Therefore the atoms of C are of the form

$$c_S = \bigwedge_{i \notin S} \mathsf{C}a_i \wedge \bigwedge_{i \in S} - \mathsf{C}a_i,$$

where $S \subseteq \{1, \ldots, M\}$. If $S = \{1, \ldots, M\}$, then $c_S = 0$. If $S = S_i$, then $c_S = d_i$. Also $d_i \neq 0$ since $f_i(d_i) = 1/I_i \neq 0/I_i$ by Lemma 4. This proves (i). (ii) follows immediately from (i) and Lemma 2. To prove (iii), we need only show $a_i \wedge d_j \neq 0$ for $i \notin S_j$, and $a_i \wedge d_j = 0$ for $i \in S_j$. If $i \notin S_j$, then $f_j(a_i \wedge d_j) = a_i/I_j \neq 0/I_j$, hence $a_i \wedge d_j \neq 0$. If $i \in S_j$, then $a_i \wedge d_j = 0$ since $a_i \leqslant \mathrm{C}a_i$ and $d_j \leqslant -\mathrm{C}a_i$.

Note that the atoms of F are the elements of the form

$$(\pm\mathrm{C}a_1) \wedge \ldots \wedge (\pm\mathrm{C}a_{i-1}) \wedge a_i \wedge (\pm\mathrm{C}a_{i+1}) \wedge \ldots \wedge (\pm\mathrm{C}a_N)$$

so that F has $N \cdot 2^{N-1}$ atoms.

Theorem *Let F be the free* S5 *algebra with n free generators and B be the free Boolean algebra with n free generators. Let $\mathcal{J}$ be the set of all ideals of B. Then F is isomorphic to the direct product* $\prod_{I \in \mathcal{J}} (B/I)^*$.

Proof: By Lemma 1, we may assume B is the Boolean subalgebra of F generated by the free generators $v_1, \ldots, v_n$ of F. As remarked above, $\mathcal{J} = \{I_1, \ldots, I_M, B\}$, and therefore

$$\prod_{I \in \mathcal{J}} (B/I)^* \approx \prod_{j=1}^{M} (B/I_j)^*.$$

Let $f: F \to \prod_{j=1}^{M} (B/I_j)^*$ be defined by $f(x) = (f_1(x), \ldots, f_M(x))$. If x_j/I_j is an arbitrary element of $(B/I_j)^*$, let $x = \bigvee_{j=1}^{M} x_j \wedge d_j$. Then $f_j(x) = f_j(x_j) = x_j/I_j$ by Lemma 4. Therefore f is onto. To show f is one to one, suppose $x \in F$ and $f(x) = (0/I_1, \ldots, 0/I_M)$. By Lemma 5(ii), $x = \bigvee_j x_j \wedge d_j$, where $x_j \in B$. By Lemma 4, $0/I_j = f_j(x) = x_j/I_j$ and so $x_j \in I_j$. Hence $\mathrm{C}x_j \leqslant \bigvee_{i \in S_j} \mathrm{C}a_i$, and therefore $\mathrm{C}x_j \wedge d_j = 0$. This implies $x_j \wedge d_j = 0$ for all j, so $x = 0$.

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