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Notre Dame Journal of Formal Logic
Volume XIX, Number 1, January 1978
NDJFAM

FORBIDDEN SUBGRAPHS IN TERMS OF FORBIDDEN QUANTIFIERS

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In 1930, Kuratowski characterized planar graphs as those graphs which fail to contain either of two special subgraphs; see Theorem 11.13 of [4]. Since then, such "forbidden subgraph" characterizations have been sought and prized by graph theorists. The nature of such characterizations is considered in [2] and [3]. In particular, [3] is based on the simple observation that a class of graphs has a forbidden subgraph characterization if and only if the class contains each subgraph of each of its members.

We will show that the properties characterizable using forbidden subgraphs are precisely those which are expressible in a natural symbolic language from which existential quantifiers have been forbidden. Of course, this is exactly what is expected from combining the observation of [3] with the well known result of Tarski and Łoś on properties preserved under subsystems (Theorem U of [5]). But unlike either of these approaches, ours uses only very simple symbolic logic and is actually able to produce the set of forbidden subgraphs.

While following the graph-theoretic terminology of [4], one important distinction must be stressed. We call H an *induced subgraph* of G if H results by removing points from G (along with each line incident with a removed point). On the other hand, H is a *subgraph* of G if H results by removing lines or points from G (along with each line incident with a removed point). (The notion of containment in Kuratowski's theorem is slightly different from each of these.)

Consider the language $\mathcal{L}$ involving variables $x, y, \ldots$ (interpreted as points) and the binary relations $=$, $\neq$, $\sim$, and $\nsim$ (interpreted as equality, nonequality, adjacency, and nonadjacency). Also, $\mathcal{L}$ has the connectives $\wedge$ and $\vee$ (for conjunction and disjunction) and universal and existential quantifiers. The universal $\mathcal{L}$-sentences are defined in the expected manner. Note that the omission of a symbol for negation in no way limits the expressiveness of $\mathcal{L}$, since occurrences of negation can be reduced to uses of $\neq$ and $\nsim$.

Received June 22, 1977

Theorem 1 *A graph-theoretic property can be characterized in terms of finitely many forbidden induced subgraphs if and only if the property is expressible as a universal $\mathcal{L}$-sentence.*

To prove the "only if" direction of the theorem, suppose a graph has a particular property exactly when it contains none of $H_1, \ldots, H_n$ as induced subgraphs. For each H_i there is a universal $\mathcal{L}$-sentence $\sigma(H_i)$ which is true of precisely those graphs not containing H_i as an induced subgraph. (For instance, if C_4 denotes a cycle with four points and no diagonals, then $\sigma(C_4)$ could be written

$$\begin{aligned}\forall x_1 \forall x_2 \forall x_3 \forall x_4 (x_1 = x_2 \vee x_1 \nsim x_2 \vee x_2 = x_3 \vee x_2 \nsim x_3 \vee \\ x_3 = x_4 \vee x_3 \nsim x_4 \vee x_4 = x_1 \vee x_4 \nsim x_1 \vee \\ x_1 = x_3 \vee x_1 \sim x_3 \vee x_2 = x_4 \vee x_2 \sim x_4).\end{aligned}$$

Thus $\sigma(C_4)$ "says" that C_4 is not an induced subgraph.) And so the property in question can be expressed as the universal $\mathcal{L}$-sentence $\sigma(H_1) \wedge \ldots \wedge \sigma(H_n)$.

To prove the "if" direction of the theorem, suppose a graph G has a particular property exactly when the universal $\mathcal{L}$-sentence σ is true of G. As specified in the next sentence, σ can be put into the form $\sigma_1 \wedge \ldots \wedge \sigma_n$ where each σ_i consists of a string of universal quantifiers followed by a disjunction of one of the formulas

$$x_i = x_j \vee x_i \sim x_j, \; x_i = x_j \vee x_i \nsim x_j, \text{ or } x_i \neq x_j \vee x_i \sim x_j$$

for each pair x_i, x_j of quantified variables. Specifically, in the standard jargon of [6], σ can be put into prenex normal form with its matrix in full conjunctive normal form; conjuncts involving $x_i \neq x_j \vee x_i \nsim x_j$ can be dropped as tautologous; then the quantifiers can be pulled inside the conjunctions. (Happily, there are ways to shorten this procedure considerably.) For each σ_i there will be a graph $H(\sigma_i)$ (determined directly from the negation of σ_i) such that σ_i will be true of precisely those graphs not containing $H(\sigma_i)$ as an induced subgraph. (For instance, the sentence $\sigma(C_4)$ displayed above is in the form of one of the σ_i's. Its negation asserts the existence of four distinct points x_1, x_2, x_3, x_4 joined cyclicly but not diagonally. Thus $H(\sigma(C_4))$ would be C_4.) Therefore σ will be true of G if and only if each σ_i is true of G, and so if and only if none of the $H(\sigma_i)$'s are induced subgraphs of G.

Observe that this proof gives a method for producing a list of forbidden subgraphs from the characterizing $\mathcal{L}$-sentence. After weeding out duplications and inclusions, the minimal set of forbidden subgraphs results, each subgraph having no more points than the sentence has quantifiers. This method is admittedly involved, but it is feasible in at least one important case. Namely, Beineke's forbidden (induced) subgraph characterization ([1] or Theorem 8.4 of [4]) of "derived" graphs (or, synonymously, "interchange" or "line" graphs) can be thus deduced from van Rooij and Wilf's earlier characterization ([7] or [4]), the latter being easily written as a (lengthy) universal $\mathcal{L}$-sentence.

To look at subgraphs (rather than induced subgraphs), it is more natural to consider a language $\mathcal{L}'$ with the point variables of $\mathcal{L}$ plus new variables $X, Y, \ldots$ (interpreted as lines) and with binary relations $=, \neq, \in$, and $\notin$ (interpreted as expected), $\wedge$, $\vee$, and universal and existential quantifiers for both sorts of variables. Essentially the same proof as above shows the following.

Theorem 2 *A graph-theoretic property can be characterized in terms of finitely many forbidden subgraphs if and only if the property is expressible as a universal $\mathcal{L}'$-sentence.*

For instance, not having C_4 as a subgraph can be expressed by the universal quantification of the variables $x_1, x_2, x_3, x_4, X_1, X_2, X_3, X_4$ followed by the disjunction of all the formulas $x_i = x_j$ and $X_i = X_j$ (where $i \neq j$) and $x_i \notin X_i$, $x_{i+1} \notin X_i$, $x_{i+2} \in X_i$, and $x_{i+3} \in X_i$ (in the obvious modulo 4 sense). Theorem 2 could also be stated in terms of those $\mathcal{L}$-sentences without existential quantifiers and without the adjacency relation (but allowing nonadjacency), since saying two points are adjacent is tantamount to requiring the existence of a line. Then not containing C_4 as a subgraph could be expressed by the above-displayed $\mathcal{L}$-sentence $\sigma(C_4)$ without the disjuncts $x_1 \sim x_3$ and $x_2 \sim x_4$.

For infinite families of forbidden subgraphs or forbidden induced subgraphs, these theorems can be restated in terms of infinite families of universal sentences. In this way we could (awkwardly) forbid all subgraphs homeomorphic (see [4]) to particular graphs and thus treat Kuratowski's characterization of planar graphs. It would be preferable, however, to find a language better suited for discussion of homeomorphism.

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