174
Notre Dame Journal of Formal Logic
Volume XIX, Number 1, January 1978
NDJFAM

A NOTE ON THE LAW OF IDENTITY AND THE CONVERSE PARRY PROPERTY

KOSTA DOŠEN

On a few occasions Anderson and Belnap in [2] are eager to stress the importance of the law of identity for E. In this note we shall give some results bearing upon the role of the law of identity in the implicational and implication-negation fragment of E. Our notation and basic conceptual apparatus will be the same as in [2]. Moreover we define the following.

An entailment subformula (**ef**) of a wff A of $\mathbf{E}_{\rightarrow}$ or $\mathbf{E}_{\overset{\sim}{\rightarrow}}$ is any subformula of A of the form $B \rightarrow C$. An elementary **ef** (**eef**) of A is any **ef** which has only a propositional variable on at least one side of the arrow (e.g., $D \rightarrow p$, $p \rightarrow D$, and $p \rightarrow q$ are all **eef**s). A minimal **ef** (**mef**) of A is any **eff** of the form $p \rightarrow q$.

We can now state:

Lemma 1 *If every* **ef** B *of* $\vdash_{\mathbf{E}_{\rightarrow}} A$ *is* $\vdash_{\mathbf{E}_{\rightarrow}} B$, *then*

1.1. *every* **eef** *of* A *is of the form* $p \rightarrow p$,
1.2. *A contains only one propositional variable.*

Proof: 1.1. Every **mef** will be of the form $p \rightarrow p$ in virtue of variable-sharing. **Eef**s of the form $p \rightarrow C$ and $C \rightarrow p$, where C is an **ef**, are ruled out, the first because of the Ackermann property, the second because by *modus ponens* p would be a theorem. So every **eef** is a **mef**, and Lemma 1.1. follows.

1.2. Let A contain two or more propositional variables. In virtue of variable-sharing every **ef** of A containing two or more propositional variables will have on at least one side of its arrow a subformula containing at least two propositional variables (so this subformula will be an **ef**). (E.g., imagine an **ef** B containing two propositional variables, p and q; then it will contain p either on both sides of the arrow, in which case q is on at least one side, or on only one side, in which case q must be on both.)*

*Occasionally, in virtue of the converse Parry property (v. infra) we know that the **ef** on the right-hand side of the arrow will always contain two or more propositional variables. Of course this fact is not essential for the proof.

Received July 20, 1977

Analysing A, and then repeating this analysis for every **ef** containing two or more propositional variables upon which we come, in a finite number of steps of such an analysis we must reach the **mefs**. But as on every step we shall get at least one **ef** containing at least two propositional variables, we shall get also at least one **mef** containing two propositional variables; and this is impossible in virtue of variable-sharing (if A is a **mef** this result is reached in zero steps). So we conclude that A cannot contain two or more propositional variables, and Lemma 1.2. follows.

It is obvious that 1.1. and 1.2. are a sufficient condition for every **ef** B of A to be $\vdash_{E_{\rightarrow}} B$. It can also be easily shown that under the assumption of Lemma 1. A co-entails $p \rightarrow p$. We get nothing but identity if, so to say, we make $E_{\rightarrow}$ speak about itself only. If we have any diversity in theorems of $E_{\rightarrow}$, some of the nested entailments in them cannot be true entailments of $E_{\rightarrow}$.

An appropriate form of Lemma 1.2. could be proved also for $RM_{\rightarrow}$, i.e., Lemma 1.2. is provable for the implicational fragment of a logical system whenever variable-sharing holds. (Lemma 1.1. is not provable for $RM_{\rightarrow}$.) Lemma 1.2. is a distinguishing mark of implicational fragments of relevant logical systems. We can prove also:

Lemma 2 *If a wff A of $E_{\overset{\sim}{\rightarrow}}$ has some* **efs** *and every* **ef** *B of A is $\vdash_{E_{\overset{\sim}{\rightarrow}}} B$, then*

2.1. *every* **eef** *of A co-entails a wff of the form $p \rightarrow p$,*
2.2. *A contains only one propositional variable.*

Proof: **Efs** co-entailing wffs of the form $\overline{D \rightarrow F} \rightarrow. G \rightarrow H$ and $D \rightarrow F \rightarrow \overline{G \rightarrow H}$ are ruled out, the first in virtue of a Theorem in [2], p. 120, the second because by *modus ponens* we should have as a theorem the negation of an **ef**. **Efs** co-entailing wffs of the form $\overline{D \rightarrow F} \rightarrow p$ and $p \rightarrow \overline{D \rightarrow F}$ are also ruled out, as well as **efs** co-entailing wffs of the form $\overline{p} \rightarrow p$ and $p \rightarrow \overline{p}$. So only those **efs** are left which co-entail wffs of $E_{\rightarrow}$, and the proof can be carried as in $E_{\rightarrow}$.

Meyer in [3], p. 183, notes that it follows from [1] that we can prove that $\vdash_{E} A$ iff $\vdash_{E} (p_1 \rightarrow p_1) \& \ldots \& (p_n \rightarrow p_n) \rightarrow A$, where $p_1, \ldots, p_n$ are *all* the propositional variables of which A is built. Using the strategy of this proof we can have a stronger result of the same sort for $E_{\overset{\sim}{\rightarrow}}$:

Lemma 3 *If A is any subformula of B, then $\vdash_{E_{\overset{\sim}{\rightarrow}}} B$ iff $\vdash_{E_{\overset{\sim}{\rightarrow}}} A \rightarrow A \rightarrow B$.*

Proof: We have: $\vdash_{E_{\overset{\sim}{\rightarrow}}} A \rightarrow A \rightarrow. A \rightarrow A$, $\vdash_{E_{\overset{\sim}{\rightarrow}}} A \rightarrow A \rightarrow. \overline{A} \rightarrow \overline{A}$, $\vdash_{E_{\overset{\sim}{\rightarrow}}} A \rightarrow A \rightarrow. A \rightarrow C \rightarrow. A \rightarrow C$, $\vdash_{E_{\overset{\sim}{\rightarrow}}} A \rightarrow A \rightarrow. C \rightarrow A \rightarrow. C \rightarrow A$. An induction on the length of B then suffices to prove that $\vdash_{E_{\overset{\sim}{\rightarrow}}} A \rightarrow A \rightarrow. B \rightarrow B$. Since if $\vdash_{E_{\overset{\sim}{\rightarrow}}} B$, then $\vdash_{E_{\overset{\sim}{\rightarrow}}} B \rightarrow B \rightarrow B$, it follows that if $\vdash_{E_{\overset{\sim}{\rightarrow}}} B$, then $\vdash_{E_{\overset{\sim}{\rightarrow}}} A \rightarrow A \rightarrow B$. The converse being trivial, this proves Lemma 3.

Lemma 3 can serve to prove and explicate a property for which Meyer discovered by a matrix method that it holds for $RM_{\overset{\sim}{\rightarrow}}$ and which puzzled Anderson and Belnap in [2], p. 149.

Lemma 4 (the converse Parry property) *If* $\vdash_{E_\rightarrow} A \rightarrow B$ *and* $\vdash_{E_\rightarrow} A$, *then* A *cannot have propositional variables foreign to* B.

Proof: Suppose that A has a propositional variable p foreign to B. We know from Lemma 3 that if $\vdash_{E_\rightarrow} A$, then $\vdash_{E_\rightarrow} p \rightarrow p \rightarrow A$, and if $\vdash_{E_\rightarrow} A \rightarrow B$, by transitivity we get $\vdash_{E_\rightarrow} p \rightarrow p \rightarrow B$, which is impossible in virtue of variable-sharing. From this Lemma 4 follows.

With the help of Lemma 4 we can prove:

Lemma 5 *If* $\vdash_{E_\rightarrow} A \rightarrow B$ *and* $\vdash_{E_\rightarrow} B \rightarrow A$, *then* A *and* B *have all their propositional variables in common.*

Proof: From $\vdash_{E_\rightarrow} A \rightarrow B \rightarrow . \, B \rightarrow A \rightarrow . \, A \rightarrow A$, $\vdash_{E_\rightarrow} A \rightarrow B \rightarrow . \, B \rightarrow A \rightarrow . \, B \rightarrow B$ and $\vdash_{E_\rightarrow} A \rightarrow B$ it follows that (1) $\vdash_{E_\rightarrow} B \rightarrow A \rightarrow . \, A \rightarrow A$ and (2) $\vdash_{E_\rightarrow} B \rightarrow A \rightarrow . \, B \rightarrow B$. Since $\vdash_{E_\rightarrow} B \rightarrow A$, both (1) and (2) must satisfy the converse Parry property, and Lemma 5 follows.

Appropriate forms of Lemma 3 can be given for $R_\rightarrow$ and $RM_\rightarrow$, with analogous proofs. In $RM_\rightarrow$ we could moreover have that if A is any subformula of B, then $\vdash_{RM_\rightarrow} B$ only if $\vdash_{RM_\rightarrow} A \rightarrow B$. Appropriate forms of Lemmas 4 and 5 hold for $T_\rightarrow$, $R_\rightarrow$, and $RM_\rightarrow$. None of Lemmas 1-5 holds for the whole system E.

REFERENCES

[1] Anderson, A. R. and N. D. Belnap, Jr., "Modalities in Ackermann's 'rigorous implication'," *The Journal of Symbolic Logic*, vol. 24 (1959), pp. 107-111.

[2] Anderson, A. R. and N. D. Belnap, Jr., *Entailment, the Logic of Relevance and Necessity*, vol. 1, Princeton University Press (1975).

[3] Meyer, R. K., "E and S4," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 181-199.

St. John's College
University of Oxford
Oxford, England