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DEGREES OF PARTIAL FUNCTIONS

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In this paper we consider a new notion of relative recursion on partial functions, which allows for an easy definition of the recursive infimum of two functions.

1 Let U be the set of partial mappings: $\omega \rightarrow \{0, 1\}$. We write $f(x) = *$ if $f(x)$ is undefined. U will be the universum for our recursion theory. Computations are introduced using Kleene brackets [1]. So we have a relation

$$\{e\}^{\alpha}(\overrightarrow{m}) \cong n$$

where e, m_i, $n \in \omega$, $\alpha \in U$. A computation $\{e\}^{\alpha}(\overrightarrow{m})$ is undefined if either it never stops or it uses $\alpha(n)$ for an n s.t. $\alpha(n) = *$.

1.1 Definition α is recursive in $\beta(\alpha \leqslant \beta)$ if for some $e \in \omega$

$$\alpha \subseteq \lambda x \cdot \{e\}^{\beta}(x).$$

It is easy to see that $\leqslant$ is a transitive relation on U. We write $\alpha \equiv \beta$ if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$. Of course $\equiv$ is an equivalence relation. The equivalence classes are called degrees. The lowest degree, 0, is the degree of the partial recursive functions.

Motivation We see $\alpha \in U$ as an object containing information (concerning its arguments). If $\alpha \subseteq \beta$ then β contains at least as much information as α does. Hence we insist to have $\alpha \leqslant \beta$ in this case. A similar argument holds if $\alpha = \lambda x\{e\}^{\beta}(x)$ for some e. These two requirements generate $\leqslant$.

As the total functions are included in U, $U/\equiv$ has cardinality $2^{\aleph_0}$, *cf.* [2]. On the other hand some equivalence classes of $\equiv$ do have cardinality $2^{\aleph_0}$ themselves. It is not difficult to find α which is not equivalent to any total function. Furthermore a straightforward spoiling construction shows that there are no minimal degrees in U. Some motivation for considering U lies in the following theorem.

1.2 Definition $1 - \mathrm{sc}(\alpha)$ is the set of *total* functions recursive in α.

1.3 Theorem *Let $V \subseteq \omega^{\omega}$ be countable and closed under recursion. Then for some $\alpha \in U$ $V = 1 - \mathrm{sc}(\alpha)$.*

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Proof: Straightforward construction.

Example Let ${}^nF$ be a type $-n$ functional. Let $\alpha = \lambda p \cdot \{p\}({}^nF)$. Then $1 - \mathrm{sc}(\alpha) = 1 - \mathrm{sc}({}^nF)$.

(The following problem is open: $\forall\alpha\exists^4F[1 - \mathrm{sc}(\alpha) = 1 - \mathrm{sc}(F)]$?)

2 In $U/\equiv$ we can define easily the supermum and infimum of two degrees.

2.1 Definition (i) for α, $\beta \in U$, $\alpha \vee \beta$ is defined by

$$\alpha \vee \beta(n) = \begin{cases} \text{if } n = 2k \text{ then } \alpha(k) \\ \text{if } n = 2k + 1 \text{ then } \beta(k) \end{cases}$$

(ii) for α, $\beta \in U$, $\alpha \wedge \beta$ is defined by

$$\alpha \wedge \beta(n) = \alpha \wedge \beta(\langle x, y, z\rangle) = \begin{cases} \text{if } \{x\}^\alpha(z) = p = \{y\}^\beta(z) \text{ then } p (p \in (0, 1)) \\ * \text{ otherwise.} \end{cases}$$

2.2 Theorem $\vee$ *and* $\wedge$ *give* sup *and* inf *in* $U/\equiv$.

Proof: $\vee$: Suppose $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$ then $\alpha \vee \beta \leqslant \gamma$ immediately.
$\wedge$: Suppose $\gamma \leqslant \alpha$ and $\gamma \leqslant \beta$. Choose x, y such that

$$\gamma \subseteq \lambda n\{x\}^\alpha(n) \text{ and } \gamma \subseteq \lambda n \cdot \{y\}^\beta(n).$$

Clearly $\gamma \subseteq \lambda n \cdot \{x\}^\alpha(n) \cap \lambda n \cdot \{y\}^\beta(n)$. Now $\gamma \subseteq \lambda n \cdot \alpha \wedge \beta(\langle x, y, n\rangle)$. Hence $\gamma \leqslant \alpha \wedge \beta$.

An interesting problem is whether or not there exist α and β of non-zero degree such that $\alpha \wedge \beta$ is of degree zero. We were unable to solve this question. However we can solve it for an operation Δ which assigns, in a natural way, a function $\alpha \Delta \beta$ to α and β which is below $\alpha \wedge \beta$.

2.3 Definition

$$\alpha \Delta \beta(\langle x, y\rangle) = \begin{cases} \text{if } \alpha(x) = \beta(y) = p \text{ then } p \\ \text{else } *. \end{cases}$$

$\alpha \Delta \beta$ allows one to compare all values of α with all values of β.

2.4 Theorem *There are* α, $\beta \in U$ *such that* $\alpha \not\equiv 0$, $\beta \not\equiv 0$ *and* $\alpha \Delta \beta \equiv 0$.

Proof: We find α and β such that

(i) $\alpha \Delta \beta(\langle x, y\rangle)$ is defined implies $\alpha \Delta \beta(\langle x, y\rangle) = \{x\}(y)$ (hence $\alpha \Delta \beta \equiv 0$).
(ii) for all n, $\alpha \not\subseteq \lambda x \cdot \{n\}(x)$ and $\beta \not\subseteq \lambda x\{n\}(k)$.

α and β are defined in an infinite construction. We, in fact, define strictly increasing infinite sequences l_n and $l_n'(n \in \omega)$ and initial segments α_n, β_n of α and β with lengths l_n, $l_n' \cdot l_0 = l_0' = 0$, α_0, β_0 are the empty sequence. At step $2n$ we extend α_{2n-1} to ensure $\alpha_{2n} \not\subseteq \lambda x \cdot \{n\}(x)$. And at step $2n + 1$ we extend β_{2n} to ensure that $\beta_{2n+1} \not\subseteq \lambda x \cdot \{n\}(x)$ (and hence $\beta \not\subseteq \lambda x \cdot \{n\}(x)$). During the construction we ensure that the following two conditions remain satisfied.

a. Whenever $\alpha_n(i) = q = \beta_n(j)$ then $q = \{i\}(j)$ $(q \in \{0, 1\})$
b. If $\alpha_n(i) \neq *$ then for some $t \in \omega$ we have: $\forall y > t\{i\}(y) \cong \{y\}(i)$.

The motivation of condition a. is clear. Condition b. helps us to do the odd steps (and complicates the even ones). We will now describe the steps:

Step $2n$. Suppose β_{2n-1} is defined on $y_1 \ldots y_k$ with values $z_1 \ldots z_k$. Find, using the recursion theorem, an index $x \in \omega$ such that:

(i) $x > 1_{2n-1}$

(ii) $\{x\}(y) \cong \begin{cases} \text{if } y = y_j \text{ for some } j \leq k \text{ then } z_j \\ \text{else if } y \leq l'_{2n-1} \text{ then } * \\ \text{else } \{y\}(x). \end{cases}$

We take $l_{2n} = x + 1$, $l'_{2n} = l'_{2n-1} \cdot \alpha_{2n}$ is found as follows:

$$\alpha_{2n}(i) = \begin{cases} \text{if } i < l_{2n-1} \text{ then } \alpha_{2n-1}(i) \\ \text{else if } i = x \text{ then } p \text{ else } *. \end{cases}$$

Here $p \in \{0, 1\}$ is a value such that $\{n\}(x) \not\simeq p$ (this ensures $\alpha \not\subset \lambda x\{n\}(x)$). We must verify conditions a. and b.

a. Suppose $\alpha_{2n}(i) = q = \beta_{2n}(j)(i \leq l_{2n}, j \leq l'_{2n})$ then $q = \beta_{2n-1}(j)$. Clearly $j = y_l$ for some $l \leq k$. Further we may assume that $i = x$. Thus: $q = \beta_{2n-1}(y_l) = z_l = \{x\}(y_l) = \{i\}(j)$.

b. Immediate by the construction of x.

Step $2n + 1$. Let α_{2n} be defined on $x_1 \ldots x_l$ with values $z_1 \ldots z_l$. Choose $t > l'_{2n}$ such that for $i \leq l$ and $t' > t$:

$$\{x_i\}(t') \cong \{t'\}(x_i).$$

Using the recursion theorem we find an $y > t$ such that: $\{y\}(x_i) = z_i$ for $i \leq l$. We extend β_{2n} to β_{2n+1} by giving it a value q on y which ensures that $\beta_{2n} \not\subset \lambda x\{n\}(x)$. Again we must check conditions a. and b. (We have $l_{2n+1} = l_{2n}$, $l'_{2n+1} = y + 1$).

b. Immediate (no change).
a. Suppose $\alpha_{2n+1}(i) = q = \beta_{2n+1}(j)$, $(q \in \{0, 1\})$. We may assume that $j = y$ and $i = x_m$ for some $m \leq l$. $q = \alpha_{2n+1}(x_m) = z_m = \{y\}(x_m) = \{x_m\}(y) = \{i\}(y)$. This completes the proof of 2.4.

REFERENCES

[1] Kleene, S. C., "Recursive functionals and quantifiers of finite type," *Transactions of the American Mathematical Society*, vol. 91, pp. 1-52.

[2] Sacks, G. E., *Degrees of Unsolvability*, Princeton University Press, 1963.

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