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MODAL TREE CONSTRUCTIONS

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1 The utility of truth tree constructions to determine the validity of truth-functional and/or quantificational arguments is well-known. In what follows, I have extended the procedure of Jeffrey[1] for the purpose of also handling arguments whose sentences contain the standard modal operators, $\Box$ and $\Diamond$. The basic program has been designed to accommodate modal system T (Hughes and Cresswell),[2] including first-order logic with the Barcan formula, $(x)\Box(\ldots x \ldots) \supset \Box(x)(\ldots x \ldots)$. The two modal notions which are essential to the modal tree constructions for T are: (a) if $\Diamond p$ is a sentence (see section **2.1** below) in tree A, then p is a sentence in some alternative-world tree to A; and (b) if $\Box p$ is a sentence in A, then p is a sentence in every alternative-world tree which has access to A (see section **2.4** below).

2 *Definitions and Notes*

2.1 "Sentence" in these contexts is elliptical for "either a sentence or a sentence-form" and it refers to a point in a tree, not to components of sentences which make up the point.

2.2 "Constructed configuration" means "All the sentences, trees, paths, and alternative-world trees which have been written down as a result of a particular application of the Program for modal tree constructions".

2.3 A *path* is a sequence of points in a tree such that the origin of the tree is in every path which is in the tree and such that every point below the origin is a successor of some previous point.

2.4 A path, B, *has access to* an alternative-world tree, V, just in case both

1. Richard C. Jeffrey, *Formal Logic: Its Scope and Limits*, McGraw-Hill, New York (1967).

2. G. E. Hughes and M. J. Cresswell, *An Introduction to Modal Logic*, M. J. Cresswell, London (1968), pp. 22-42. System T was originally propounded by Robert Feys in "Les logiques nouvelles des modalités," *Revue Néoscolastique de Philosophie*, vol. 40 (1937), pp. 517-553 and vol. 41 (1938), pp. 217-252.

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(1) the origin of V is a sentence of the form p only if B contains a sentence of the form $\Diamond p$ and (2) for every sentence of the form $\Box r$ which is contained in B, there is a sentence of the form r which is contained in every path of V.

2.5 That a path, B, has access to an alternative-world tree, V, is indicated in the constructed configuration by a line of dashes drawn from the bottom of B to the origin of V.

2.6 A path is *closed* just in case it contains both a sentence of the form p and a sentence of the form $-p$.

2.7 It is customary to indicate that a path is closed by placing an "X" at the bottom of that path.

2.8 In constructed configurations, paths do not extend into any alternative-world trees to which the paths have access, i.e., if B is a path in a tree, V, then B is not a path in any tree to which B has access.

2.9 A tree is *closed* just in case either every path in that tree is closed or every open path in that tree has access to at least one closed tree.

2.10 "$(\ldots n \ldots)$" in these contexts is the sentence which results from replacing all occurrences of x which are free in $(\ldots x \ldots)$ with an occurrence of n which is free in $(\ldots n \ldots)$.

2.11 It is usually a better procedure to pick for n constants rather than variables.

3 *Tree Rules*

3.1 Always take the highest (closest to the origin) applicable point when applying the rules.

3.2 *Denials:* (a) Erase "$--$" wherever it appears in unchecked sentences in open paths.
(b) Check sentences of the form $-(x)(\ldots x \ldots)$ in open paths and rewrite them as $(\exists x) - (\ldots x \ldots)$ at the bottom of each open path in which the checked sentence occurs.
(c) Check sentences of the form $-(\exists x)(\ldots x \ldots)$ in open paths and rewrite them as $(x) - (\ldots x \ldots)$ at the bottom of each open path in which the checked sentence occurs.
(d) Check sentences of the form $-\Box p$ in open paths and rewrite them as $\Diamond - p$ at the bottom of each open path in which the checked sentence occurs.
(e) Check sentences of the form $-\Diamond p$ in open paths and rewrite them as $\Box - p$ at the bottom of each open path in which the checked sentence occurs.

3.3 *Truth-functional connectives:* Apply the following rules to any sentence, S, having the form $p\ R\ q$ or $-(p\ R\ q)$, where R is either "$\supset$", "v", ".", or "$\equiv$", in open paths; check S and write the result of applying the rule at the bottom of each open path in which S occurs:

(a)√ $p \supset q$

$-p$	q

(b)√ $p \vee q$

p	q

(c)√ $p \,.\, q$

p
q

(d)√ $p \equiv q$

p	$-p$
q	$-q$

(e)√ $-(p \supset q)$

p
$-q$

(f)√ $-(p \vee q)$

$-p$
$-q$

(g)√ $-(p \,.\, q)$

$-p$	$-q$

(h)√ $-(p \equiv q)$

$-p$	p
q	$-q$

3.4 *Necessities:* For each sentence of the form $\Box p$ in an open path, check it and write p at the bottom of each open path in which $\Box p$ occurs.

3.5 *Alternative-world necessity rule:* For every alternative-world tree which is being formed at the bottom of an open path in which a sentence of the form $\Box p$ occurs, write p at the bottom of every open path in those alternative-world trees, unless it is already in the path, and erase the check which is beside $\Box p$. (For modal system T plus the predicate calculus, no alternative-world tree, V, is considered to be at the bottom of an open path, P, just in case there is an intervening alternative-world tree between the bottom of P and the origin of V.)

3.6 *Possibilities:* For each sentence of the form $\Diamond p$ in an open path, begin to form an alternative-world tree at the bottom of every open path in which $\Diamond p$ occurs by writing p at the origin of each of those alternative-world trees. Check $\Diamond p$. (See section **2.5** above.)

3.7 *Universal quantifiers:* Given an open path in which a sentence of the form $(x)(\ldots x \ldots)$ occurs: for each name n that appears free anywhere in the path or in a path in an alternative-world tree to which the path has access, write the sentence $(\ldots n \ldots)$ at the bottom of the path in which $(x)(\ldots x \ldots)$ occurs unless that sentence already occurs in the path. (If no name occurs free in the path or in some open path in an alternative-world tree to which the path has access, choose some name n, and write $(\ldots n \ldots)$ at the bottom of the path.) When you are done, do *not* check $(x)(\ldots x \ldots)$. (See sections **2.10** and **2.11** above.)

3.8 *Existential quantifiers:* Given an open path in which a sentence of the form $(\exists x)(\ldots x \ldots)$ occurs, inspect the path to see if it contains a sentence of the form $(\ldots n \ldots)$, where n is some name. If it does not contain such a sentence, choose a name n that is not free anywhere in the path and write the sentence $(\ldots n \ldots)$ at the bottom of the path. When this has been done for every open path in which $(\exists x)(\ldots x \ldots)$ occurs, check that sentence.

4 *Program for Modal Tree Constructions*

1. To determine the validity of arguments whose sentences contain the modal operators, $\Box$ and/or $\Diamond$; or to determine the validity of arguments whose sentences contain only the standard quantifiers and truth-functional

connectives as operators; *list at the origin of the tree the premises and the negation of the conclusion, go to* 2.

2. Is there a sentence unchecked in an open path in the constructed configuration to which one of the rules for *denial* applies?

YES: *Apply it, go to* 2.
NO: *Close all paths containing both a sentence and its denial, go to* 3.

3. Are all the paths closed?

YES: *Stop! The argument is valid.*
NO: *Go to* 4.

4. Is there a sentence unchecked in an open path to which the rule for *necessities* applies?

YES: *Apply it, go to* 2.
NO: *Go to* 5.

5. Is there a sentence unchecked in an open path to which one of the rules for *truth-functional connectives* applies?

YES: *Apply it, go to* 2.
NO: *Go to* 6.

6. Is there a sentence unchecked in an open path to which the rule for *existential quantifiers* applies?

YES: *Apply it, go to* 6.
NO: *Go to* 7.

7. Is there a sentence in an open path to which the rule for *universal quantifiers* applies?

YES: *Apply it, go to* 8.
NO: *Go to* 9.

8. Has there been any change in the constructed configuration since last entering stage 7?

YES: *Go to* 7.
NO: *Go to* 9.

9. Has there been any change in the constructed configuration since last entering stage 2?

YES: *Go to* 2.
NO: *Go to* 10.

10. Is there a sentence unchecked in an open path to which the rule for *possibilities* applies?

YES: *Apply it, go to* 11.
NO: *Go to* 11.

11. Is there a checked sentence in an open path to which the *alternative-world necessity rule* applies?

YES: *Apply it, go to* 11.
NO: *Go to* 12.

12. Have all sentences of the form $\Box p$ been checked in every open path at the bottom of which an alternative-world tree is being formed?

YES: *Go to* 13.
NO: *Check them, go to* 12.

13. Has there been any change in the constructed configuration since last entering stage 2?

YES: *Go to* 2.
NO: *Go to* 14.

14. Does every open path have access to at least one alternative-world tree which is closed?

YES: STOP! *The argument is valid.*
NO: STOP! *The argument is invalid.*

5 *Examples*

1. Show that $\Box(p \supset q)$ /∴ $\Diamond p \supset \Diamond q$ is a valid argument.

1 √ $\Box(p \supset q)$
2 √ $-(\Diamond p \supset \Diamond q)$
3 √ $p \supset q$
4 √ $\Diamond p$
5 √ $-\Diamond q$
6 √ $\Box -q$
7 $-q$
8 $-p$ 9 q
X
10 p
11 √ $p \supset q$
12 $-q$
13 $-p$ 14 q
X X

Explanation: Lines 1 and 2 are the listing of the premise and the negation of the conclusion. Line 3 comes from line 1 by rule **3.4**. Lines 4 and 5 come from line 2 by rule **3.3**. Line 6 comes from line 5 by rule **3.2**. Line 7 comes from line 6 by rule **3.4**. Lines 8 and 9 come from line 3 by rule **3.3**. The path consisting of lines 1, 2, 3, 4, 5, 6, 7, and 9 is closed since it contains both q and $-q$. Line 10 comes from line 4 by rule **3.6**. Line 11 comes from line 1 by rule **3.5**. Line 12 comes from line 6 also by rule **3.5**. Lines 13 and 14 come from line 11 by rule **3.3**. The path consisting of lines

10, 11, 12, and 13 is closed since it contains both p and $-p$. The path consisting of lines 10, 11, 12, and 14 is also closed since it contains both q and $-q$. The alternative-world tree which has p as its origin and consisting of lines 10, 11, 12, 13, and 14 is closed since all its paths are closed. The open path in the original tree, the path consisting of lines 1, 2, 3, 4, 5, 6, 7, and 8, has access to at least one closed tree. Hence, since every open path has access to at least one alternative-world tree which is closed, the argument is valid.

2. Show that $\{[\Diamond(\exists x)Fx \vee \Diamond(\exists x)Gx] \equiv \Diamond(\exists x)(Fx \vee Gx)\}$ is valid.

$$1\surd\ -\{[\Diamond(\exists x)Fx \vee \Diamond(\exists x)Gx] \equiv \Diamond(\exists x)(Fx \vee Gx)\}$$

2 $\surd$ $\Diamond(\exists x)Fx \vee \Diamond(\exists x)Gx$		3 $\surd$ $-[\Diamond(\exists x)Fx \vee \Diamond(\exists x)Gx]$
4 $\surd$ $-\Diamond(\exists x)(Fx \vee Gx)$		5 $\surd$ $\Diamond(\exists x)(Fx \vee Gx)$
6 $\surd$ $\Box-(\exists x)(Fx \vee Gx)$		9 $\surd$ $-\Diamond(\exists x)Fx$
7 $\surd$ $-(\exists x)(Fx \vee Gx)$		10 $\surd$ $-\Diamond(\exists x)Gx$
8 $(x)-(Fx \vee Gx)$		11 $\surd$ $\Box-(\exists x)Fx$
		12 $\surd$ $\Box-(\exists x)Gx$
17 $\surd$ $\Diamond(\exists x)Fx$	18 $\surd$ $\Diamond(\exists x)Gx$	13 $\surd$ $-(\exists x)Fx$
		14 $\surd$ $-(\exists x)Gx$
19 $\surd$ $-(Fa \vee Ga)$	20 $\surd$ $-(Fa \vee Ga)$	15 $(x)-Fx$
23 $-Fa$	25 $-Fa$	
		16 $(x)-Gx$
24 $-Ga$	26 $-Ga$	
		21 $-Fa$
		22 $-Ga$
37 $\surd$ $(\exists x)Fx$	44 $\surd$ $(\exists x)Gx$	27 $\surd$ $(\exists x)(Fx \vee Gx)$
38 $\surd$ $-(\exists x)(Fx \vee Gx)$	45 $\surd$ $-(\exists x)(Fx \vee Gx)$	28 $\surd$ $-(\exists x)Fx$
39 $(x)-(Fx \vee Gx)$	46 $(x)-(Fx \vee Gx)$	29 $\surd$ $-(\exists x)Gx$
40 Fb	47 Gb	30 $(x)-Fx$
41 $\surd$ $-(Fb \vee Gb)$	48 $\surd$ $-(Fb \vee Gb)$	31 $(x)-Gx$
42 $-Fb$	49 $-Fb$	32 $\surd$ $Fb \vee Gb$
43 $-Gb$	50 $-Gb$	33 $-Fb$
X	X	34 $-Gb$
		35 Fb 36 Gb
		X X

Explanation: Line 1 is the listing at the origin of the negation of the conclusion. Lines 2, 3, 4, and 5 come from line 1 by rule **3.3**. Line 6 comes from line 4 by rule **3.2**. Line 7 comes from line 6 by rule **3.4**. Line 8 comes from line 7 by rule **3.2**. Lines 9 and 10 come from line 3 by rule **3.3**. Line 11 comes from line 9 and line 12 comes from line 10 by rule **3.2**. Line 13 comes from line 11 and line 14 comes from line 12 by rule **3.4**. Line 15 comes from line 13 and line 16 comes from line 14 by rule **3.2**.

Lines 17 and 18 come from line 2 by rule **3.3**. Line 19 comes from line 6 and line 20 comes from line 6 by rule **3.5**. Line 21 comes from line 15 and line 22 comes from line 16 by rule **3.7**. Lines 23 and 24 come from line 19 and lines 25 and 26 come from line 20 by rule **3.3**. Line 27 comes from line 5 by rule **3.6**. Lines 28 and 29 come from lines 11 and 12 respectively by rule **3.5**. Lines 30 and 31 come from lines 28 and 29 respectively by rule **3.2**. Line 32 comes from line 27 by rule **3.8**. Lines 33 and 34 come from lines 30 and 31 respectively by rule **3.7**. Lines 35 and 36 come from line 32 by rule **3.3**. Both paths in the alternative-world tree which has line 27 as its origin are closed. Line 37 comes from line 17 by rule **3.6**. Line 38 comes from line 6 by rule **3.5**. Line 39 comes from line 38 by rule **3.2**. Line 40 comes from line 37 by rule **3.8**. Line 41 comes from line 39 by rule **3.7**. Lines 42 and 43 come from line 41 by rule **3.3**. The sole path in the alternative-world tree which has line 37 as its origin is closed. Line 44 comes from line 18 by rule **3.6**. Line 45 comes from line 6 by rule **3.5**. Line 46 comes from line 45 by rule **3.2**. Line 47 comes from line 44 by rule **3.8**. Line 48 comes from line 46 by rule **3.7**. Lines 49 and 50 come from line 48 by rule **3.3**. The sole path in the alternative-world tree which has line 44 as its origin is closed. Since every open path has access to at least one alternative-world tree which is closed, the sentence is valid. It should be noted that lines 15, 16, 21, 22, 19, 20, 23, 24, 25, and 26 need not have been written down, i.e., sometimes a bit of intelligence, rather than blind adherence to the rather mechanical program, will produce more efficient results.

3. Examine the argument $\Box(p \supset q) / \therefore \Box(\Diamond p \supset \Diamond q)$ for validity.

1 $\surd$ $\Box(p \supset q)$
2 $\surd$ $-\Box(\Diamond p \supset \Diamond q)$
3 $\surd$ $\Diamond-(\Diamond p \supset \Diamond q)$
4 $\surd$ $p \supset q$

5 $-p$	6 q
7 $\surd$ $-(\Diamond p \supset \Diamond q)$	8 $\surd$ $-(\Diamond p \supset \Diamond q)$
9 $\surd$ $p \supset q$	10 $\surd$ $p \supset q$
11 $\surd$ $\Diamond p$	15 $\surd$ $\Diamond p$
12 $\surd$ $-\Diamond q$	16 $\surd$ $-\Diamond q$
13 $\surd$ $\Box -q$	17 $\surd$ $\Box -q$
14 $-q$	18 $-q$
19 $-p$ 20 q X	21 $-p$ 22 q X
23 p	25 p
24 $-q$	26 $-q$

Explanation: A detailed line-by-line account will not be given here, since by this time the procedures should be clear. However, let me point out a few items of interest. First of all, the argument is not valid since the open paths in the constructed configuration do not have access to any closed alternative-world trees. A model demonstrating the invalidity of the argument can be constructed by simply reading up any open path, assigning 'true' to the unnegated sentences and 'false' to the negated ones. The second item of interest is that in this constructed configuration we constructed alternative-world trees to alternative-world trees. In system T the relation of alternativeness is not transitive, i.e., if V is an alternative-world tree to W and W is an alternative-world tree to Y, it is not the case that V is an alternative-world tree to Y. This means, of course, that if a sentence of the form $\Box p$ is a sentence in an open branch of Y which has access to W and there is an open branch of W which has access to V, although p will be a member of every open branch of W, it will not be written down as a sentence in the open branches of V by the Alternative-World Necessity Rule. However, there are modal systems in which the relation of alternativeness is transitive, e.g., S.4 and S.5. In both S.4 and S.5 the argument which we have just found to be invalid in T is valid. In the next section of this paper modifications to the *program for modal tree constructions* will be given which will permit the examination of arguments for validity using other modal systems.

Finally, it should be pointed out that although we reached stage 14 of the Program in this example, there will be some arguments for which stage 14 will not be reached. Since there is no decision procedure for quantificational logic and since this Program has been written for quantificational logic plus, it should come as no surprise that this Program is not a decision procedure for modal quantificational logic.

6 *Modifications to Accommodate Other Modal Systems* In order to accommodate Lewis' S.4 plus quantificational logic including the Barcan Formula, one simply replaces the parenthetical remark in the alternative-world necessity rule (rule **3.5**) with the following: (If V is an alternative-world tree being formed at the bottom of an open path of tree W and if W is an alternative-world tree which has been formed at the bottom of an open path of tree T, then V is an alternative-world tree which is being formed at the bottom of an open path of T.) This is the only change that is required for S.4.

The accommodation of Lewis' S.5, however, requires a bit more work. First of all, the modification of the *alternative-world necessity rule* which was made for S.4 must also be made for S.5. Secondly, stage 3 of the *Program* must be changed to read: "Are all the paths in the original tree closed?

YES: *Stop! The argument is valid.*
NO: *Go to* 3.5.

Finally, an additional stage must be added to the *Program*: stage 3.5: Is there a sentence in an open path to which the S.5 RULE applies?

YES: *Apply it, go to* 3.5.
NO: *Go to* 4.

S.5 RULE: For every sentence of the form $\Box p$ which is in V, where V is an alternative-world tree to tree T, if $\Box p$ is not a member of that path of T which has access to V, write $\Box p$ at the bottom of that path and write p at the bottom of every open path in V.

To eliminate the Barcan Formula from the quantificational portion of the *Program* all that is needed is to delete the phrase 'or in an alternative-world tree to which the path has access' which occurs twice in the rule for *Universal quantifiers* (rule **3.7**).

To accommodate Hintikka's system M,[3] three changes must be made. First the *Alternative-world necessity rule* must be changed to the following:

For every alternative-world tree which is being formed at the bottom of an open path in which a sentence of the form $\Box p$ occurs and for each free individual variable or constant which occurs in p, if that alternative-world tree contains other sentences in which those free individual variables or constants occur, then write p at the bottom of every open path in those alternative-world trees, unless it's already in the path, and erase the check which is beside $\Box p$. (The parenthetical remark that is found in the original *Alternative-world necessity rule* also applies here.)

Secondly, the rule for *Universal quantifiers* must also be rewritten as follows:

Given an open path in which a sentence of the form $(x)(\ldots x \ldots)$ occurs: for each name n that appears free anywhere in the path, write the sentence $(\ldots n \ldots)$ at the bottom of the path in which $(x)(\ldots x \ldots)$ occurs unless that sentence already occurs in the path. When you are done, do *not* check the sentence $(x)(\ldots x \ldots)$.

The final modification that is necessary for the *Program* to accommodate system M is a rewriting of note **2.4** of section **2**:

A path, B, *has access to* an alternative-world tree, V, just in case the origin of V is a sentence of the form p only if B contains a sentence of the form $\Diamond p$ and for every sentence of the form $\Box r$ which is a member of B there is a sentence of the form r which is a member of every path of V unless $\Box r$ contains free individual variables or constants which occur in no sentences of V.

To accommodate Hintikka's system M*,[4] only one change is required in the original *Program*, viz., a rewriting of the rule for *Universal quantifiers*:

3. K. J. Hintikka, *Models for Modalities*, D. Reidel, Dordrecht-Holland (1969), pp. 60-64.

4. *Ibid*., pp. 63-64.

Given an open path in which a sentence of the form $(x)(\ldots x \ldots)$ occurs: for each name n that appears free anywhere in the path or in any path that has access to the alternative-world tree (if it is one) in which the path occurs, write the sentence $(\ldots n \ldots)$ at the bottom of the path in which $(x)(\ldots x \ldots)$ occurs unless that sentence already occurs in the path. When you are done, do not check the sentence $(x)(\ldots x \ldots)$.

To accommodate Hintikka's system M**,[5] the only change that is required in the original *Program* is the following rewriting of the rule for *Universal quantifiers*:

Given an open path in which a sentence of the form $(x)(\ldots x \ldots)$ occurs: for each name n that appears free anywhere in the path or in a path in an alternative-world tree to which the path has access or in any path that has access to the alternative-world tree (if it is one) in which the path occurs, write the sentence $(\ldots n \ldots)$ at the bottom of the path in which $(x)(\ldots x \ldots)$ occurs unless that sentence already occurs in the path. When you are done, do not check $(x)(\ldots x \ldots)$.

Additional modal systems may be accommodated by adding to the above three modifications of the rule for *Universal quantifiers* the following phrase: If no name occurs free in the path, choose some name n which does not occur in the constructed configuration and write $(\ldots n \ldots)$ at the bottom of the path.

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5. *Ibid.*, pp. 65-66.