Notre Dame Journal of Formal Logic
Volume XVIII, Number 3, July 1977
NDJFAM

# THE NUMERAL AXIOMS

RICHARD BUTRICK

The introduction of the numerals as individual constants of formal number theory is generally done by appeal to a pretheoretic or intuitively given concept of the succession of numbers. A typical account might run as follows:

The terms '0', '$0^1$', '$0^{11}$', . . . we shall call numerals abbreviated by (accounts frequently and mistakenly say 'denoted by') '0', '$\bar{1}$', '$\bar{2}$', . . . . In general, if $n$ is a non-negative integer, we shall let '$\bar{n}$' stand in place of (mistake: 'stand for') the corresponding numeral $\ulcorner 0^{11 \cdots 1} \urcorner$, with $n$ strokes.

What is pedagogically prior is not necessarily epistemologically prior, but certainly one is taught the numerals before one's "intuition" of the succession of numbers is "awakened." Regardless, it is possible to introduce the numerals without appeal to some intuitively given concept of the natural number sequence. The following axioms and axiom schemata may, for convenience, be given the title of 'the theory of numeral succession,' (**NS**).

The following axiom and definition schemata provide for the usual correlation of "numbers" with numerals and a characterization of the successor function. Definition of 'numeral': '0' . . . '9' are simple numerals. If $n_1$ and $n_2$ are simple numerals and $n_1 \neq$ '0' then $\ulcorner n_1\, n_2 \urcorner$ is a compound numeral. If $n_1$ and $n_2$ are compound numerals, $\ulcorner n_1\, n_2 \urcorner$ is a compound numeral. (In the following $n_1$ and $n_3 \neq$ '0'.)

Axiom schema (1): $\ulcorner \mathrm{z}(n_1, n_2) = n_1 n_2 \urcorner$.

Let $n_1 =$ '12', $n_2 =$ '3'. Then $\ulcorner \mathrm{z}(n_1, n_2) = n_1 n_2 \urcorner =$ 'z'⁀'('⁀'12'⁀','⁀'3'⁀')'⁀'='⁀'12'⁀'3'. The use of quasi-quotes has the effect of quoting the constant contextual background for $n_1$ and $n_2$. 'z' is intended as a function constant on a par with '(', '=', and ')' as far as quasi-quotes are concerned, i.e., part of the constant context for $n_1$ and $n_2$. Thus 'z(12, 3) = 123' is an instance of axiom schema (1), and **NS** asserts that z(12, 3) = 123.

*Received September 14, 1972*

Axioms (0) . . . (9): '$s(0) = 1$', . . . '$s(8) = 9$', '$s(9) = 10$'.
Axiom schema (2): *If* $n_1$ *is a numeral and* $n_2$ *is a simple numeral* $\neq$ '9', *then* $\ulcorner s(n_1\, n_2) = z(n_1,\, s(n_2))\urcorner$ *is an axiom schema of* **NS**.
Axiom schema (3): *If* $n_2$ = '9' *and* $n_1$ *is simple and* $\neq$ '9', *then* $\ulcorner s(n_1 n_2) = z(s(n_1), \text{'0'})\urcorner$ *is an axiom schema of* **NS**.
Axiom schema (4): *If* $n_1$ = '9' *and* $n_2$ = '9', *then* $\ulcorner s(n_1 n_2) = \text{'100'}\urcorner$ *is an axiom schema of* **NS**.
Axiom schema (5): *If* $n_2$ = '9' *and if* $n_1$ *is compound,* i.e., $\ulcorner n_1 = z(n_3, n_4)\urcorner$ *is the form of a theorem, with* $n_4$ *simple, then* $\ulcorner s(n_1 n_2) = z(s(n_3 n_4), \text{'0'})\urcorner$ *is an axiom schema of* **NS**.

Consider '$s(1999)$'. Axiom schemata (1) . . . (4) do not apply, i.e., '1999' is not an instance of $\ulcorner n_1 n_2\urcorner$ where $n_2$ is simple $\neq$ '9', nor can $n_1$ and $n_2$ be simple. However, $\ulcorner\text{'1999'} = z(\text{'199'}, \text{'9'})\urcorner$ is an instance of axiom schema (1) as is $\ulcorner n_1 = z(n_3, n_4)\urcorner$ with $n_1$ = '199' and $n_3$ = '19' and $n_4$ = '9'. Therefore axiom (5) applies and **NS** asserts that $s(1999) = z(s(199), 0) = z(z(s(19), 0), 0) = z(z(z(s(1), 0), 0), 0) = 2000$. It would seem that formal number theory does not characterize the successor function except by appeal to an intuitive notion of the successor function. In order to prove that $s(0^{111}) = 0^{1111}$ (for '$s(0^{111}) = 0^{1111}$' to be a theorem) one must rely on a pre-theoretic, pre-formal ability to count up. If formal number theory is to be formal, axioms along the foregoing lines must be added. The axioms rely on a syntatic matching rather than an explicit knowledge of the "numbers".

*Ohio University*
*Athens, Ohio*