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A NOTE ON THE AXIOM OF CHOICE AND THE CONTINUUM HYPOTHESIS

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In response to a request from Roy Davies for details on the proof of Lemma 3 of [4], the author worked out his proof sketch of (5) of Lemma 1 and found it to be erroneous. In fact, if x is infinite and well-ordered, then $x^{=+} \sim x$ and so $x^{=+} \epsilon\, x^{\#}$ although $x^{=} \epsilon\, x^{=+}$ and not $x^{=} \epsilon\, x^{=}$; that is, $x^{\#} \subseteq x^{=}\mathfrak{P}$ only holds when x is finite or not well-ordered.

Although disappointing, this error is not disastrous. With the aid of an alternative to (5), an iteration of a part of the antecedent of Lemma 3 leads to Theorem 3 with an analogous iteration and all the remaining lemmas and theorems of [4]. Some additional reasoning is needed, but far less than usual in proofs of the implication of the axiom of choice by the generalized continuum hypothesis. Also, the arithmetic of transfinite numbers is not employed. It is the aim of the present note to provide the corrections and additional reasoning, but some new results are also established.[1]

In what follows, $\{\,\}$ is the empty set while $\{x\}$ and $\{xy\}$ are the sets whose only members are x and x and y respectively. Also, $x - y$ is $\{a: a \epsilon\, x$ and not $a \epsilon\, y\}$, x, y is $\{\{x\}\{xy\}\}$, and $x \times y$ is $\{a, b\colon a \epsilon\, x$ and $b \epsilon\, y\}$. Additional notation is as in [4]. In particular, x^{+} is $x \cup \{x\}$ and $\bigcup$ is von Neumann's operation. The set-theoretic framework employed is Zermelo-Fraenkel set theory without the axioms of choice or regularity. Since the theory of cardinal numbers cannot be developed within this framework, neither cardinal arithmetic nor cardinal notation can be employed.

In the place of (5) of Lemma 1, put

(5) $x^{\#} < x^{=}\mathfrak{P}\mathfrak{P}$.

There is no problem if x is finite. Assume instead that x is infinite.

1. A previous short correction notice was printed on p. 464 of the *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 17 (1971), but the author was not sent the galley proof to read. The formula "$x^{\#} \prec x^{=}\mathfrak{P}\mathfrak{P}$" was there misprinted as the erroneous "$x^{\#} \subseteq x^{=}\mathfrak{P}\mathfrak{P}$".

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If x is not well-ordered, then there is no ordinal n such that $n \sim x$ and so $x^{\#}$ is $x^{=}$. Consequently, $x^{\#} < x^{=}\mathfrak{P} < x^{=}\mathfrak{P}\mathfrak{P}$. Assume then that x is well-ordered and let w be the set of all well-orderings of subsets of x. Clearly, $w \subseteq (x \times x)\mathfrak{P}$. Let f be the function which assigns to any $n \in x^{\#}$ the set of all members of w isomorphic to ϵ within n. Since f is one-to-one and its range is included in $w\mathfrak{P}$, $x^{\#} \leqslant w\mathfrak{P}$. But $w\mathfrak{P} \leqslant (x \times x)\mathfrak{P}\mathfrak{P}$ while $x \times x \sim x \sim x^{=}$ since x is well-ordered. Consequently, $x^{\#} \leqslant x^{=}\mathfrak{P}\mathfrak{P}$. If $x^{=}\mathfrak{P}\mathfrak{P} \leqslant x^{\#}$, $x^{=}\mathfrak{P}$ is well-ordered by (2) of Lemma 1 and $x^{=}\mathfrak{P} \sim x^{=}\mathfrak{P}^{=}$ by Lemma 2. Hence, not $x^{=}\mathfrak{P}^{=} \in x^{\#}$ by (1) of Lemma 1 and so $x^{\#} \subseteq x^{=}\mathfrak{P}^{=}$ by (2) of Lemma 1. That is, $x^{\#} \leqslant x^{=}\mathfrak{P} < x^{=}\mathfrak{P}\mathfrak{P} \leqslant x^{\#}$. This is impossible and so $x^{\#} < x^{=}\mathfrak{P}\mathfrak{P}$.

This new version of (5) also follows from result 1.43 on cardinals of [6] in Zermelo-Fraenkel set theory with a theory of cardinals.

For the proof of the modification of Lemma 3, an additional lemma is needed.

Lemma 2.1 *If* $n \in \omega \lesssim x$ *and* $x\mathfrak{P} \lesssim (n^{+} \times x) \cup y$, *then* $x\mathfrak{P} \lesssim y$.

Proof: Assume that $\omega \in x$. Let x', z, z', z'', and y' be sets such that $\omega \sim z' \sim z'' \sim z \subseteq x$, z' and z'' are disjoint, z is $z' \cup z''$, x' is $x - z'$, and y' is $y - x$. Then $x' \sim x$ and $x'\mathfrak{P} \sim x\mathfrak{P} \lesssim x \cup y' \sim x' \cup y'$. Let f be a one-to-one function from ω unto z', let g be a one-to-one function from $x'\mathfrak{P}$ into $x' \cup y'$, and let x'' be $\{s: s \in x'\mathfrak{P} \text{ and } g(s) \in x'\}$. Clearly, g restricted to $x'\mathfrak{P} - x''$ is a one-to-one function into y' and so $x'\mathfrak{P} - x'' \leqslant y' \subseteq y$.

Now let h be a function on $x'\mathfrak{P}$ such that, if $g(s)$ is $a \in x'$, then $h(s)$ is $\{f(\{\})a\}$. Otherwise, $h(s)$ is s. Since there are no $s \in x''$ and $n \in \omega$ such that $f(n) \in s$, h is a one-to-one function into $x\mathfrak{P} - x''$ and $x\mathfrak{P} \lesssim x\mathfrak{P} - x''$. To establish the lemma, it will first be shown that $x\mathfrak{P} \lesssim x'\mathfrak{P} - x''$.

Let d be a set disjoint from $x'\mathfrak{P}$ such that $d \sim x\mathfrak{P} - x''$. By the non-arithmetical theorem of Tarski on p. 158 of [5], there are mutually disjoint a_1 through b_2 such that $a_1 \sim a_2$, $x'\mathfrak{P} - x''$ is $a_1 \cup b_1$, d is $a_2 \cup b_2$, and $x'' \cup b_1 \sim x'' \sim x'' \cup b_2$. Also, by the function-theoretic argument for the lemma of Sierpiński on p. 169 of [5], $x\mathfrak{P} \sim x'\mathfrak{P}$ and $x'' \cup b_2 \sim x''$ imply $b_2 < x'\mathfrak{P} - x''$. Since $x\mathfrak{P} \sim x\mathfrak{P} - x''$, $x\mathfrak{P} \lesssim d$. But d is $a_2 \cup b_2$, $a_2 \lesssim x'\mathfrak{P} - x''$ because $a_1 \sim a_2$, and so $d \lesssim (x'\mathfrak{P} - x'') \cup b_2 \lesssim \{\}^{++} \times (x'\mathfrak{P} - x'') \leqslant \{\}^{++} \times x\mathfrak{P} \leqslant x\mathfrak{P}$ since $\omega \in x$. That is, $x\mathfrak{P} \lesssim x'\mathfrak{P} - x''$ and so $x\mathfrak{P} \leqslant y$. Consequently, $x\mathfrak{P} \leqslant (\{\}^{+} \times x) \cup y$ implies $x\mathfrak{P} \leqslant y$. Also, if $n \in \omega$ and $x\mathfrak{P} \leqslant (n^{+} \times x) \cup y$ implies $x\mathfrak{P} \leqslant y$ for any y, then, since $(n^{++} \times x) \cup y$ is $(n^{+} \times x) \cup ((\{n^{+}\} \times x) \cup y)$, $x\mathfrak{P} \leqslant (\{n^{+}\} \times x) \cup y$ and so $x\mathfrak{P} \leqslant y$. The statement then holds by mathematical induction.

This lemma seems to be a new result.

The modified version of Lemma 3 is

Lemma 3 *If there is no* y *such that either* $x < y < x\mathfrak{P}$ *or* $x\mathfrak{P} < y < x\mathfrak{P}\mathfrak{P}$ *and* x *is well-ordered, so is* $x\mathfrak{P}$.

Proof: Assume the antecedent and that $x\mathfrak{P}$ is not well-ordered. Consequently, $\omega \lesssim x$. By Lemma 2, $x \sim x^{=}$ and so $x\mathfrak{P} \sim x^{=}\mathfrak{P}$ and $x\mathfrak{P}\mathfrak{P} \sim x^{=}\mathfrak{P}\mathfrak{P}$.

By Lemma 2 and the new (5) of Lemma 1, $x^{=} < x^{\#} < x^{=}\mathfrak{P}\mathfrak{P}$. If not $x^{\#} \lesssim x^{=}\mathfrak{P}$, not $x^{\#} \in x^{=}\mathfrak{P}^{\#}$ by (1) of Lemma 1. But, if $x^{=}\mathfrak{P}^{\#} \in x^{\#}$, $x^{=}\mathfrak{P}^{\#} \lesssim x \sim x^{=} < x^{=}\mathfrak{P}$. This is impossible by (3) of Lemma 1 and so $x^{=}\mathfrak{P}^{\#}$ is $x^{\#}$ by (2) of Lemma 1. Since not $x^{=}\mathfrak{P}^{\#} \lesssim x^{=}\mathfrak{P}$ by (3) of Lemma 1, $x^{=}\mathfrak{P} < x^{=}\mathfrak{P} \cup x^{=}\mathfrak{P}^{\#} \lesssim \{\ \}^{++} \times x^{=}\mathfrak{P}\mathfrak{P} \lesssim x^{=}\mathfrak{P}\mathfrak{P}$ since $\omega \lesssim x^{=}\mathfrak{P}$ and so $x^{=}\mathfrak{P}\mathfrak{P} \lesssim x^{=}\mathfrak{P} \cup x^{=}\mathfrak{P}^{\#}$ by the first assumption. Consequently, $x^{=}\mathfrak{P}\mathfrak{P} \lesssim x^{=}\mathfrak{P}^{\#}$ by Lemma 2.1 since $\omega \leqslant x^{=}\mathfrak{P}$. Hence, $x\mathfrak{P}$ is well-ordered by (2) of Lemma 1. This is contrary to assumption and so $x^{\#} \lesssim x^{=}\mathfrak{P}$. Consequently, $x\mathfrak{P} \lesssim x^{\#}$ by the first assumption and $x\mathfrak{P}$ is well-ordered after all.

The corresponding modification of Theorem 3 is

Theorem 3 *If* **CH** *and there is no y such that the continuum $< y <$ the continuum $\mathfrak{P}$, then the continuum is well-ordered.*

In Zermelo-Fraenkel set theory with a theory of cardinals, this theorem also follows from result 3.4 on cardinals of [6]. Indeed, Specker's result implies Lemma 3 without the assumption that x is well-ordered in such a framework, but at the cost of relying on Specker's result that not $2^{x} \leqslant x^{2}$ for cardinal $x \geqslant 5$. The proof depends not only on an involved series of transfinite calculations with cardinals, but also on a transfinite ordinal normal form theorem.

In spite of the fact that a few set-algebraic inequalities are employed in the present proof of Lemma 3, it can perhaps be claimed that, together with Theorem 1 or Lemma 4 of [4], the proof provides an approximation to what the author was after: a relatively short and set-theoretically transparent proof for the fact that **GCH** implies **AC** which does not rely on transfinite arithmetic.[2] Such proofs can perhaps also be applied in other set theories in which cardinal arithmetic cannot be developed.

Lemma 2.1 has some additional consequences which are of interest.

Lemma 2.2 *If $n \in \omega$ and x is infinite, then $n \times x < x\mathfrak{P}$.*

Proof: Assume the antecedent. If y has four members, let m be $(y \times n)^{=}$ and let f be a one-to-one function from m into x. Let g be a function on $n \times x$ such that, for any $a \in x$, $g(\{\ \}, a)$ is $\{f(\{\ \})a\}$. If $i^{+} \in n$, then $g(i^{+}, a)$ is $\{f(j^{++})a\}$ where j is the i'th empty or even ordinal if there is no $h \in j^{++}$ such that a is $f(h)$. Otherwise, $g(i^{+}, a)$ is $\{f(j^{+})\, f(b)\}$ where b is the h'th ordinal greater than j^{+}. The latter clause is proper if there are j^{++} many h left in m after those smaller than j^{++} are excluded; that is, the sum of j^{++} and j^{++} must be smaller than m. Since j is twice i, the sum is exactly four times i^{+}. But $i^{+} \in n$ and so four times $i^{+} \in m$. Hence, g is defined for i^{+}, a and so a function from $n \times x$ into $x\mathfrak{P}$. Also, since g restricted to $\{i\} \times x$ for an $i \in n$

2. About a year after this study had been written, the author was informed that G. Takeuti and W. Zaring also proved that **GCH** implies **AC** without the use of transfinite arithmetic in their textbook *Introduction to Axiomatic Set Theory* (New York, Springer-Verlag, 1971). However, the present proof was initiated at least as early in [4] and differs considerably from that of Takeuti and Zaring.

is obviously one-to-one and its range is disjoint from the range of g restricted to $\{h\} \times x$ for all $h \in i$, g must be one-to-one. Finally, the range of g consists entirely of doubletons and so is a proper subset of $x\mathfrak{P}$. Thus, g is a one-to-one function from $n \times x$ into a proper subset q of $x\mathfrak{P}$. If $x\mathfrak{P} \sim n \times x \sim q$, $x\mathfrak{P}$ is one-to-one with a proper subset of itself and so Dedekind infinite; that is, $\omega \lesssim x\mathfrak{P} \sim q$. But then $\omega \sim p \subseteq q$ for some p. Let f' be a function which assigns to any $i \in n$ the intersection of p with the range of g restricted to $\{i\} \times x$. Since p is well-ordered, a subset of p is infinite just when Dedekind infinite. But p is finite if $f'(i)$ is for any $i \in n$. Hence, there is an $i \in n$ such that $\{i\} \times x$ is Dedekind infinite and so $\omega \lesssim \{i\} \times x \sim x$. Also, $x\mathfrak{P} \lesssim n \times x \lesssim (n^+ \times x) \cup \{\,\}$. Hence, $x\mathfrak{P} \lesssim \{\,\}$ by Lemma 2.1. This is impossible and so not $x\mathfrak{P} \sim n \times x$.

Proven results in the literature which imply Lemma 2.2 without the assumption that x can be well-ordered in Zermelo-Fraenkel set theory with a theory of cardinals (such as 2.52 of [6] and 2 on p. 115 of [1]) rely on both transfinite arithmetic and Specker's difficult result that not $2^x \leqslant x^2$ for cardinal $x \geqslant 5$. Also, the assumption that $n \cdot x \leqslant 2^x$ for finite cardinal n and infinite cardinal x is usually made without proof. The author has only been able to find a single proof through a large amount of transfinite arithmetic of this inequality (on p. 147 of [5]).

Lemma 2.3 *If there is no y such that $x < y < x\mathfrak{P}$ and $\{\,\}^+ < x$, then $\omega \lesssim x \sim \{\,\}^{++} \times x$.*

Proof: Assume the antecedent. Then x is infinite and so $x \lesssim x^+ \lesssim \{\,\}^{++} \times x < x\mathfrak{P}$ by Lemma 2.2. Consequently, $x \sim x^+ \sim \{\,\}^{++} \times x$ by reapplying the assumption and $\omega \lesssim x \sim \{\,\}^{++} \times x$.

3.1 and 3.2 of [6] are analogues on cardinals obtained by means of not $2^x \leqslant x^2$ for cardinal $x \geqslant 5$ to this lemma. Notice that Lemma 2.3 explains the application of Lemma 2.1 in the proof of Lemma 4 of [4]. It also provides a deeper justification for the application of Lemma 2.1 in the proof of the present version of Lemma 3.

In the alternative proof of Theorem 1 of [4], it was assumed without proof that the union of a well-ordered set of well-ordered sets was well-ordered. The axiom of regularity and the assumption that $x\mathfrak{P}$ is well-ordered when x is for any x were presupposed. For the sake of completeness, a lemma by which the union principle follows from these presuppositions will be established.

Lemma 2.4 *If m and n are ordinals and $m^+ \bigcup^{\#} \mathfrak{P} \lesssim n$, then $m^+ \bigcup \lesssim n$.*

Proof: Assume the antecedent and let f be a one-to-one function from $m^+ \bigcup^{\#} \mathfrak{P}$ into n. Also for $j \in m^+$, let g be a function on m^+ such that $g(j)$ is $g_j \cup \{a, b\colon a \in j^+ \bigcup - j \bigcup$ and b is $f_j(\{g_j(c)\colon c \in a\})\}$ where g_j is $\{a, b\colon a\, g(i)\, b$ for some $i \in j\}$ and f_j is the one-to-one function into $B_j^{+=} - B_j^{=}$ induced by f from $A_j^{+} - A_j$ with B_j the range of f restricted to A_j and A_j $\{\{g_j(c)\colon c \in a\}\colon a \in j \bigcup\}$. Such a g exists by transfinite recursion on m^+ since $g(j)$ is function of j and the range of g restricted to j for $j \in m^+$.

Assume now that $j \in m^+$ and, for any $i \in j$, $g(i)$ is a one-to-one function from $i^+\bigcup$ into $i^+\bigcup^= \subseteq n$. If g_j is defined for c and c' and $g_j(c)$ is $g_j(c')$, then there are $i \in j$ and $i' \in j$ such that $c \in i^+\bigcup$ and $c' \in i'^+\bigcup$. Let u be $i \cup i'$. Since $h \in u$ implies $h^+\bigcup \subseteq u^+\bigcup$, $g(u)$ is defined for c and c'. Also, $g(u) \subseteq g_j$ and is one-to-one by the inductive assumption. Hence, $g(u)(c)$ is $g(u)(c')$, c is c', and g_j is one-to-one. Since $c \in a \in j^+\bigcup \subseteq m^+\bigcup$ implies that $c \subseteq i^+\bigcup$ for some $j \in j$ and $g_j(c) < j^+\bigcup$, $g_j(c) < m^+\bigcup$ and so $g_j(c) \in m^+\bigcup^\#$ for $c \in a \in j^+\bigcup$ by (1) of Lemma 1 of [4]. Hence, if $a \in j^+\bigcup$, $\{g_j(c): c \in a\}$ is in the domain of f and $g(j)$ is a function from $j^+\bigcup$ into $j^+\bigcup^= \subseteq n$.

Assume finally that $g(j)$ is defined for a and a' and that $g(j)(a)$ is $g(j)(a')$. If $\{aa'\} \subseteq j\bigcup$, a is a' since g_j is one-to-one. Assume then that $\{aa'\} \subseteq j^+\bigcup - j\bigcup$. Since f_j is one-to-one and both $\{g_j(c): c \in a\}$ and $\{g_j(c): c \in a'\}$ are in the domain of f_j, these sets are identical. Hence, a is a' because g_j is one-to-one and $g(j)$ is as well since not both $a \in j\bigcup$ and $a' \in j^+\bigcup - j\bigcup$ or vice versa. By applying transfinite induction up to m^+, it follows that $g(m)$ is a one-to-one function from $m^+\bigcup$ into n.

Of course, this lemma also provides a direct proof of Theorem 1. The original proofs are in [2] and [3].

In conclusion, some new results which supplement Lemma 2.1 will be proved.

Lemma 2.5 *If* $\omega \lesssim x$ *and* $y \lesssim \omega \times x$, *then* $x\mathfrak{P} \cup y \lesssim x\mathfrak{P}$.

Proof: Assume the antecedent. Let y' be $y - x\mathfrak{P}$, let z', x', and f be as in the proof of Lemma 2.1, and let g be a one-to-one function from y' into $\omega \times x'$. Also, let h be a function on $x'\mathfrak{P} \cup y'$ such that $h(s)$ is $\{f(n)a\}$ if $s \in y'$ and $g(s)$ is n, a, for some n and a. Otherwise, $h(s)$ is s. Then h is a one-to-one function from $x'\mathfrak{P} \cup y'$ into $x\mathfrak{P}$ and so $x\mathfrak{P} \cup y \lesssim x'\mathfrak{P} \cup y' \lesssim x\mathfrak{P}$.

Theorem 4 *If* $\omega \lesssim x$, *then* $\omega \times x < x\mathfrak{P}$.

This is an immediate consequence of Lemma 2.5 and Specker's result.

By replacing ω with non-empty w such that $\{\ \}^{++} \times \omega \lesssim w$ and imitating the proof of the above theorem, a more general statement can be established.

Theorem 5 *If* $\{\ \}^{++} \times w \lesssim w \lesssim x$, *then* $w \times x < x\mathfrak{P}$.

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