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A NOTE ON METAPHYSICS AND THE FOUNDATIONS OF MATHEMATICS

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1 *Introduction* An outline of the class of metaphysics to be associated with the transcendental modal logics **F*F** [1] was given in [2]. Concerning the foundations of mathematics and the 'intuitionist-classicist' debate it was implied in [2] that these logics **F*F** may help to clarify some of the issues in that debate. However, since paper [2] tended to stress only the critical function of metaphysics in relation to the foundations, in this paper we will focus attention more on the constructive side and indicate, at the same time, a new approach to foundational studies and constructive metaphysics.

From [2] there are essentially two alternative logics from the group **F*F**, namely, **S*S** and **E*E**. Also, the C_2-indeterminate truth-value occurring in **E*E** is associated only with future contingent events (although these may be of different kinds). Hence, if we confine ourselves to mathematics, then the relevant modal logic is **S*S** (or the modal predicate logic **PS*S** [3]) and consequently the possible area of agreement between the metaphysics M1 and M2 (see [2]) should include the realm of C_2-beliefs in mathematics generally, and the foundations in particular.

2 *Metaphysics and the Errors* We recall the two principal semantical notions underlying the logics **F*F**. From [1], they are:

A1. C_1-truth, which means fidelity to the human testament.
A2. C_2-truth, which means fidelity to the Divine testament.

A1 and A2 are two old hats. It is hoped that this presentation here will help to make a new one. Many contemporary philosophers may well grimace at our point of departure in [2], section **2**, and especially at our 'primitive notions'. Many of these notions will seem outmoded, mere residue of a bygone age, unfamiliar, perhaps even monstrous. Monstrous or no, with or without the grimace, this is the starting point: for our approach there can be no other. It is not new: it is Augustine's starting point.

From our point of view the fault of 'intuitionism' is that it wants to

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concern itself only with A1 and the mistake in 'classical' mathematics is that it fails to distinguish clearly A1 and A2. Here 'intuitionism' and 'classical' are to be understood in the context of the issues between Hilbert and Brouwer. The formalists cannot enter this debate directly, and then only at a distance since, in the main, formalists do not claim to be in pursuit of mathematical 'truth'.

We can thus discern three main sources of errors:

E1. To be concerned only with C_1-truth. We can call this error the collapse of C_2.
E2. To be concerned only with C_2-truth. We can call this error the collapse of C_1.
E3. To fail to distinguish C_1-truth and C_2-truth. We can call this error the collapse of the categories.

Although in the medieval period stretching roughly from Augustine to William of Ockham, the distinction between A1 and A2 above is kept well to the foreground, though only in varying degrees, examples of errors E2 and E3 are not difficult to find. A useful book for our purpose here is R. McKeon's work [4], a selection of basic texts surrounding the problem of knowledge. One example of E2: McKeon,[1] in his introductory remarks on Anselm's 'step in the building of a Christian Philosophy', notes that:

> It makes, of course, important omissions; it encounters significant dangers. The most pertinent of its omissions arises from the program which committed it to an examination of the nature and logic of eternal things; this it does so expertly that the doctrines have continued to be echoed and re-echoed with changes, modifications, ostensible oppositions from the eleventh century to the present; but as a consequence there is no doctrine of time, of changing things, of contingency, for these recall immediately the timeless, the changeless, the necessary on which they depend and by which they must be explained.

One example of E3: Although Duns Scotus[2] too is aware of the difference between A1 and A2, this distinction is not rigorously maintained. Thus, in his remarks on 'the certitude of first principles' and 'the certitude of conclusions' derived therefrom, he argues for examples of 'necessary truths', *but only on one level*; consequently, there is a blurring of the categories and a loss of aspects A1 and A2 in his account of 'formal composition'.

As defined in [2] the object of metaphysics based on **F*F** is rational C_2-beliefs. Thus the danger we must be careful to avoid is that of falling into error E2. Failure to take account of *both* categories will not result in right relationship. Consequently we need to consider C_1-truth, C_1-knowledge

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1. R. McKeon's introduction to St. Anselm's "Dialogue on truth," see [4], vol. 1, p. 148.

2. Duns Scotus, "The Oxford Commentary . . .," see [4], vol. 2, pp. 324-326.

and C_1-belief. For C_1-belief, we can take our bearings from Hume's account of belief (see [5]). In retrospect we can see that the weakness in Hume's skepticism is that it is not skeptical enough. We need to advance Hume's skepticism to discuss the role of beliefs in the mathematical and logical spheres. But it hardly needs saying that Hume's principal error is E1 (as it is for empiricism, quite generally).

"There is, it seems to us, at best, only a limited value in the knowledge derived from experience" (Eliot[3]). But that 'limited value' is indispensable. The difference between C_2-knowledge and C_2-belief is an unqualified difference of kind. In contrast, the difference between C_1-knowledge and C_1-belief is one of degree rather than kind; roughly, we can think of C_1-knowledge as compelling C_1-beliefs. Pressed to give definitions for C_1-knowledge and C_2-beliefs, I think we have to fall back on a kind of conventionalism. Thus, C_1-knowledge is human knowledge, by common consensus; and C_2-beliefs are human beliefs about matters under C_2, by common consensus. But we need to distinguish two cases:

D1. A consensus of human opinion without prior appeal to transcendental ideas.
D2. A consensus of human opinion with prior appeal to transcendental ideas.

We can illustrate this as follows. Considering the logic **S*S** [1], we can say that the $\mathbf{S}_1$-provable modal wff or $\mathbf{S}_1$-tautologies delineate the C_1-knowledge; it arises from D2 and depends upon us fixing these C_1-conventions in the light of the transcendental idea of C_2-truth. Also, we can see that the C_2-conventions of C_2-beliefs in relation to **S*S**, i.e., the $\mathbf{S}_2$-provable sentences or $\mathbf{S}_2$-tautologies, likewise arise from D2. Similarly, C_2-beliefs in the sciences arise from D2. In contrast, for the $\mathbf{S}_1$-provable non-modal sentences, and the resulting C_1-knowledge we proceed by D1. Also, to establish the C_1-knowledge in the sciences and the 'knowledge' derived from the senses we again proceed by D1. Thus in the latter three cases we are guided by Aristotle, whereas in the former three cases we are guided by Augustine. It is important to note that our desire to preserve the distinction between knowledge and belief, under C_1, depends inherently upon the idea of C_2-knowledge. Without such a transcendental idea as C_2-knowledge there would be no grounds for maintaining such a distinction between C_1-knowledge and C_1-belief. (This seems to me also the reason why 'knowledge' in the hands of Hume inevitably tends to evaporate, and this provides a novel solution to the 'problem of knowledge': there is no 'problem of knowledge' because there is no 'knowledge'.)

We have already made note of Angelelli's important findings on 'the ontological square', 'traditional predication theory' and 'the two dimensions'

3. T. S. Eliot, *Four quartets*, "East Coker," section 11.

of classical ontology (see [3]). Angelelli [6] summarises the general conclusion of his first chapter entitled "Ontology" as follows:

> It is true to say that classical ontology has explicitly considered two dimensions whereas this distinction is not "officially" introduced in contemporary philosophy. But it is equally true to affirm that the distinction of the two dimensions is far from having been well preserved in the past, whereas it is still discernible (though not acknowledged) in Frege and other recent authors (belonging, in particular, to foundational philosophy).

From our point of view, the bi-categorical view of ontology, namely, the C_1-real and the C_2-real mentioned in [2] are the two ontological distinctions that we associate with the logical aspects A1 and A2 respectively.

3 *Metaphysics, the Foundations and Methodology* When we turn to the current problem of the foundations from the above, we have this immediate consequence for metaphysics which can be stated easily in a few words but which is a very tall order: we are required to reconstruct the foundations of mathematics. Regarding such a reconstruction we will be guided by the two semantical aspects A1 and A2 and hence we will have to operate with a bi-categorical notion of mathematical theoremhood. *In some respects*, this approach will enable us to accommodate *both* the 'intuitionist' view of mathematical truth and the 'classicist' view. Under C_2, the mathematical theorems will have the logical status of rational beliefs under the absolute category. Thus, from the mathematical side this approach to the foundations will carry with it, what we might call—a certificate of metaphysics—that is, it is an approach whereby the philosophical presuppositions are laid bare from the start, i.e., it means doing mathematics on a metaphysical ticket. At the same time, from the metaphysical side, we will be involved with a significant branch of constructive metaphysics. Hence it is an activity that brings with it a new kind of sophistication, and a new kind of precision, and hopefully with this synthesis, "thereby render useful service to the public" (Kant [7]). A full metaphysics in our sense will not be forthcoming from any one man though, as in other sciences, individual authors will make determinative contributions.

This noble conception of metaphysics is not new. A finely balanced example is furnished by Robert Grosseteste's work "On truth."[4] Grosseteste's idea on the role of mathematics was not merely that it provided an important subject matter for metaphysics (as it does to-day); rather, it was (or at least could be made) a fitting instrument to be used towards a constructive end for metaphysics.

On the methodology of our new approach there is also this consequence: we cannot proceed, in the first instance, by means of axiomatics, because axiomatics, as usually conceived, depends upon a uni-categorical notion of

4. Robert Grosseteste, "On truth," see [4], vol. 1, pp. 263-281, and McKeon's introduction, pp. 259-262.

theoremhood. Other genetic methods,[5] like Smullyan's 'analytic tableaux', Beth's 'semantic tableaux', Hintikka's 'tableaux' and Gentzen's 'natural deduction', will have to be developed. The axiomatic method has been the dominating method so far this century, yet there have been signs that it is not always entirely without limitations. Axioms may be self-evident truths, but in some recent modal logic this has often meant that the axioms are "evident to one's self, but to nobody else" (Bierce[6]). On Gödel's paper of 1931,[7] Emil Post[8] made the following observations in 1944:

> The conclusion is inescapable that even for such a fixed, well-defined body of mathematical propositions, mathematical thinking is, and must remain, essentially creative. To the writer's mind, this conclusion must inevitably result in at least a partial reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth as being of the essence of mathematics.

Post's 'anticipations' would seem here to be fully corroborated. Weyl's final paragraph from his "Comments"[9] runs as follows:

> What "truth" or objectivity can be ascribed to this theoretic construction of the world, which presses far beyond the given, is a profound philosophical question. It is closely connected with the further question: what impels us to take as a basis precisely the particular axiom system developed by Hilbert? Consistency is indeed a necessary but not a sufficient condition for this. For the time being we probably cannot answer this question except by asserting our belief in the reasonableness of history, which brought these structures forth in a living process of intellectual development—although, to be sure, the bearers of this development, dazzled as they were by what they took for self-evidence, did not realize how arbitrary and bold their construction was. Hilbert's appeal to the practical success of the method, too, seems to me to rest upon such a belief. Or is it his opinion that, the nearer we bring the construction of the axiomatic system to its completion, the more we shall eliminate arbitrariness and bring to the fore that which is unambiguously compelling? . . . .

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5. Genetic Methods: R. M. Smullyan, *First-Order Logic*, Springer-Verlag, New York (1968); R. M. Smullyan, "Abstract quantification theory," in Kino, Myhill, Vesley (editors), *Intuitionism and Proof Theory*, North-Holland, Amsterdam (1970); E. W. Beth, *The Foundations of Mathematics*, North-Holland, Amsterdam (1959); K. J. J. Hintikka, "Form and content in quantification theory," *Acta Philosophia Fennica*, vol. 8 (1955), pp. 7-55; G. Gentzen, "Investigations into logical deduction" (1935), reproduced in M. E. Szabo, *The Collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam (1969).

6. A. Bierce, *The Devil's Dictionary*, Dover, New York (1958), p. 123.

7. K. Gödel, "On formally undecidable propositions . . .," reproduced in [8].

8. E. Post, "Recursively enumerable sets of positive integers and their decision problems," *Bulletin of the American Mathematical Society*, vol. 50 (1944), p. 295.

9. H. Weyl, "Comments on Hilbert's second lecture on the foundations of mathematics," reproduced in [8]. The reader should bear in mind that quotation in this paper is meant to function as 'ideogram'. For the definitions of 'ideogram' see *The Cantos of Ezra Pound.*

The usual requirements (as laid down by formalists and Hilbert's Formalism[10] in particular) for a formal system to be useful or even meaningful are too stringent. For example, in the modal sub-logics $\mathsf{F}^*_4\mathsf{F}_4$ [1], $\mathsf{PF}^*_4\mathsf{F}_4$ [3], the formal sub-systems F_4, PF_4 are formally inconsistent, yet the sub-logics $\mathsf{F}^*_4\mathsf{F}_4$ and $\mathsf{PF}^*_4\mathsf{F}_4$ are semantically consistent. Although it is correct when we view these sub-logics $\mathsf{F}^*_4\mathsf{F}_4$ and $\mathsf{PF}^*_4\mathsf{F}_4$ in isolation (i.e., incorrectly), the F_4-tautology underlying them is not, on its own, a reasonable semantical notion of 'truth' to correlate with a kind of provability in a formal system, yet, when they are viewed (i.e., correctly) as sub-logics of the full logics $\mathsf{F}^*\mathsf{F}$ and $\mathsf{PF}^*\mathsf{F}$ respectively, then an F_4-tautology is such a reasonable and natural semantical notion of 'truth'. If we examine Ł*Ł, the derived $Ł_3$-tautology was defined in [1] as:

X is an $Ł_3$-tautology iff X is an $Ł_1$-tautology and X is an $Ł_2$-tautology (i)

If we grant that an $Ł_3$-tautology is reasonable (and the Ł-modal system, under our interpretation, depends upon it) then we must grant that an $Ł_4$-tautology (replace 'and' by 'or' in (i)) is also reasonable. Hence we must enlarge the kinds of formal systems allowable; formal consistency—which was such a decisive consideration for Hilbert and the formalist programme—is no longer a necessary prerequisite for a mathematical system to be meaningful and perhaps useful. Weyl[11] almost said as much in 1946:

> In defending his program against Brouwer, Hilbert pointed emphatically to the situation in theoretical physics. The individual physical statements and laws have no "meaning" verifiable in immediate observation; only the system as a whole can be confronted with experience. Here consistency is absorbed into the farther-reaching requirement of *"concordance."*

What importance can we ascribe to Logicism?

> Logicism is the thesis that mathematics is reducible to logic, hence nothing but a part of logic. Frege was the first to espouse this view (1884). In their great work *Principia Mathematica*, the English mathematicians A. N. Whitehead and B. Russell produce a systematization of logic from which they construed mathematics. (R. Carnap[12])

We must distinguish Logicism proper—'the thesis that mathematics is reducible to logic'—from narrow methodological associations (especially since the logics $\mathsf{S}^*\mathsf{S}$ and $\mathsf{PS}^*\mathsf{S}$ are at our disposal). Logicism, as developed and as usually presented, has been connected with the axiomatic method, and hence is no longer tenable exclusively in that form. Against the charge that the appearance of many-valued systems of logic renders obsolete the

10. D. Hilbert, "On the infinite," reproduced in [9].

11. H. Weyl, "Review: The philosophy of Bertrand Russell," reproduced in Weyl's *Gesammelte Abhandlungen*, Springer-Verlag, New York (1968), vol. IV, p. 603.

12. R. Carnap, "The Logicist foundations of mathematics," reproduced in [9].

idea of 'absolute truths', Łukasiewicz[13] affirms:

> Absolute truths of thought did not collapse in 1930. Whatever discredit anyone may try to cast upon many-valued logics, he cannot deny that their existence has not invalidated the principle of exclusive contradiction. This is an absolute truth which holds in all logical systems under the penalty that should this principle be violated then all logic and all scientific research would lose their purpose. Also valid remain the rules of inference, namely the rule of substitution, which corresponds to the Aristotelian *dictum de omni*, and the rule of detachment, analogous to the Stoic syllogism called *modus ponens*. Owing precisely to these rules we are building today not one but many logical systems, each of which is consistent and free of contradiction. It may be that other absolute principles with which all logical systems must comply, also exist.

In the sub-systems $\mathbf{F}_4$ [1] and $\mathbf{PF}_4$ [3], the rule modus ponens does not hold. Thus, concerning the kinds of axiom systems and rules of inference permissible for Logicism here, we cannot give to modus ponens the kind of 'absolute' status afforded it by Łukasiewicz (and a distinguished line of logicians). The suggestion that formal consistency and the rule modus ponens may not in fact be universally applicable in mathematics has already been suggested recently from another approach, in Yessenin-Volpin's "The ultra-intuitionistic criticism and the antitraditional program for foundations of mathematics" (*cf.* Kino in footnote 5).

> In connection with [the postulates of the intuitionistic predicate calculus] I must say that conjunctions $A \& B$ may differ according to the possibilities that parts of A and B are or are not collated to one another. If one has a proof of A and another one of $\neg A$ then the collations of different parts of A within the first A belong to the first proof, and those written $\neg A$ to the second; but if one merely combines both proofs one makes by this action no identification of both A's in the theorems A and $\neg A$. On similar grounds it is possible that there are two theorems A and $A \supset B$ such that B is not a theorem.

Thus, if we permit new methodological approaches, I think we can still speak of a programme for Logicism. I do not envisage that the logics **S*S** and **PS*S** will provide, on their own, the 'calculi' that could realise such a Logicism. No, I see the main service of the transcendental modal logic **S*S** as that of a fundamental logic of appeal: **S*S** is one logic we may wear in our lapels while holding a mathematical conversation. In some respects Logicism can still be the most challenging approach to the foundations. Even if only applicable to parts of mathematics applying Ockham's *razor*, it is the one approach we should be very reluctant to give up too easily.

But, recalling the reception given to Frege's 'ideography',[14] it is to be expected:

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13. J. Łukasiewicz, "In defence of logistic," reproduced in L. Borkowski (editor), *Jan Łukasiewicz, Selected Works*, North-Holland, Amsterdam (1970), p. 248.

14. G. Frege, "Begriffsschrift," (1879), reproduced in [8], p. 31.

> [We] must give up hope of securing as readers all those mathematicians who, when they come across logical expressions like 'concept', 'relation', 'judgment', think: *Metaphysica sunt, non leguntur*; and those philosophers who at sight of a formula call out:
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> *Mathematica sunt, non leguntur.*
>
> The number of these people cannot be very small. (Frege[15])

4 *Conclusion* 'These are only hints and guesses.' It remains to see if this new approach can yield useful results in foundational studies, beginning perhaps with Arithmetic, Set Theory and 'elucidating the infinite'. Some contemporary mathematicians will no doubt be offended by this intrusion of a foreign body into their human mathematics. The intuitionists have been objecting for quite some time. Among these great mathematicians of that era, many now gone—against reductions, against divisions, against multiloquio—at least Hilbert would have wished to keep his eye on such an enterprise.

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15. G. Frege, "Preface to foundations," see [10], reproduced in *Translations from the Philosophical Writings of Gottlob Frege*, edited by P. Geach and M. Black, Blackwell, Oxford (1966), p. 143.