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INDEPENDENT NECESSARY CONDITIONS FOR FUNCTIONAL COMPLETENESS IN m-VALUED LOGIC

YALE N. PATT

A function f is *functionally complete* in m-valued logic if the set of functions which can be defined explicitly from f is exactly the set of all functions of m-valued logic. A *Sheffer function* is a two-place functionally complete function. Post [1] and Webb [2], among others, have identified some Sheffer functions in m-valued logic. Martin [3] identified four properties (i.e., proper substitution, co-substitution, proper closing, and $\dagger$-closing), the absence of which are necessary conditions for functional completeness. In this paper, we will prove that co-substitution implies proper substitution; or with respect to our necessary conditions, if f does not have the proper substitution property, then f does not have the co-substitution property. Consequently, the co-substitution property can be discarded from our set of necessary conditions for functional completeness. Finally, we show that the remaining three necessary conditions are independent.

Theorem *If $f(p, q)$ is a two-place function satisfying the co-substitution property, then $f(p, q)$ satisfies the proper substitution property as well.*

Proof: Let $K = \{1, 2, 3, \ldots m\}$ be the set of m truth values, and D be a decomposition of K into $\hat{m}$ disjoint non-empty classes, $2 \leq \hat{m} < m$. We will say $i \sim j$ (D) if i and j are elements of the same class, $i, j \in K$. Further, let $\hat{\mathsf{D}}$ be the decomposition of the two-dimensional space K^2 such that $(p, q) \sim (r, s)(\hat{\mathsf{D}})$ if and only if $p \sim r$ (D) and $q \sim s$ (D). Let f satisfy the co-substitution law of Martin; that is, for any $h, i, j, k \in K$, whenever $f(h, i) \sim f(j, k)$ (D), then $h \sim j$ (D) or $i \sim k$ (D).

Assume there exist $(a, b) \sim (c, d)$ ($\hat{\mathsf{D}}$) such that $f(a, b) \not\sim f(c, d)$ (D). There are $\hat{m}$ - 1 classes of $\hat{\mathsf{D}}$, we will call them $C_1, C_2, \ldots C_{\hat{m}-1}$, such that if $(w_i, x_i) \in C_i$ then $a \not\sim w_i$ (D), $b \not\sim x_i$ (D), $c \not\sim w_i$ (D), and $d \not\sim x_i$ (D). Further, if $(w_i, x_i) \in C_i$ and $(w_j, x_j) \in C_j$, $i \neq j$, then $w_i \not\sim w_j$ (D) and $x_i \not\sim x_j$ (D). Since f satisfies co-substitution, $f(a, b) \not\sim f(w_1, x_1)$ (D) and $f(c, d) \not\sim f(w_1, x_1)$ (D). Further, $f(a, b) \not\sim f(w_2, x_2)$ (D), $f(c, d) \not\sim f(w_2, x_2)$ (D) and $f(w_1, x_1) \sim f(w_2, x_2)$ (D). Continuing, we reach the case that $f(w_{\hat{m}-1}, x_{\hat{m}-1})$

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cannot be specified without violating the co-substitution property. Therefore, our original assumption is contradicted; that is, that $(a, b) \sim (c, d)$ $(\hat{D})$ and $f(a, b) \not\sim f(c, d)$ (D). But $(a, b) \sim (c, d)$ $(\hat{D})$ implies $f(a, b) \sim f(c, d)$ (D) is exactly the proper substitution property. Q.E.D.

Remark: Obviously, if D is the trivial decomposition consisting of m classes, then every one-place function trivially satisfies co-substitution. However, as is well known, one-place functions cannot be functionally complete, so this case is not considered.

It remains to show that the three remaining properties, i.e., proper substitution, proper closing, and t-closing are indeed independent. This, we easily do by means of examples. First, we review the definitions of t-closing and proper closing from [3]. Let $t(p)$ be a one-place function which satisfies the following:

$$t^m(j) = j,\ 1 \leq j \leq m$$
$$t^i(j) \neq j,\ 1 \leq i \leq m - 1,\ 1 \leq j \leq m.$$

Then $f(p, q)$ is t-closing if there exists a $t(p)$ such that, for all i, j, there exists a k such that $f(t^i(p), t^j(p)) = t^k(p)$. A function $f(p, q)$ is proper closing if some non-empty proper subset of the m truth values is closed under $f(p, q)$.

Example 1: $f_1(p, q)$ has the proper substitution property, but is not proper closing and is not t-closing.

f_1: (p rows, q columns)

$p \backslash q$	1	2	3	4
1	3	4	4	4
2	3	4	3	3
3	4	3	2	1
4	4	3	2	1

Example 2: $f_2(p, q)$ is proper closing, but not t-closing and does not have the proper substitution property.

f_2: (p rows, q columns)

$p \backslash q$	1	2	3	4
1	3	2	1	1
2	4	3	2	4
3	1	2	4	3
4	2	4	4	3

Example 3: $f_3(p, q)$ is t-closing, but not proper closing and does not have the proper substitution property.

f_3:

		q		
p	1	2	3	4
1	2	4	3	1
2	2	3	1	4
3	1	3	4	2
4	3	2	4	1

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North Carolina State University
Raleigh, North Carolina