Notre Dame Journal of Formal Logic
Volume XVIII, Number 2, April 1977
NDJFAM

THE ONE-ONE EQUIVALENCE OF SOME GENERAL COMBINATORIAL DECISION PROBLEMS

CHARLES E. HUGHES and W. E. SINGLETARY

1 *Introduction* A *general combinatorial decision problem* may be defined quite simply to be a family of related decision problems concerned with some class of combinatorial systems. E.g., the general halting problem for Turing machines is the family of halting problems ranging over all Turing machines. Let G_1 and G_2 be two general combinatorial decision problems. G_1 is said to be *one-one* (*many-one*) *reducible* to G_2 if there exists an effective mapping ψ from the problems p in G_1 into the problems $\psi(p)$ in G_2 such that p is of the same one-one (many-one) degree as $\psi(p)$. (Actually if p is solvable we only require that $\psi(p)$ be also solvable.) G_1 and G_2 are said to be *one-one* (*many-one*) *equivalent* if each is one-one (many-one) reducible to the other. Recent research by the authors and Overbeek [2, 3, 4, 5, 6, 7, and 10] has demonstrated the many-one equivalence of a large number of general combinatorial decision problems. In this paper* we will show that some of these general decision problems are in fact one-one equivalent. Our method of proof, which has been used by Cleave [1] to study "system functions", is to show that each non-recursive instance of the general decision problems under consideration is a cylinder. Since many-one equivalence of cylinders implies one-one equivalence, the desired results are achieved.

2 *Cylinders and their properties* Let R be an arbitrary recursively enumerable (r.e.) set. R is called a *cylinder* if the decision problem for membership in R is of the same one-one degree as that for the set of pairs $\{\langle x, n\rangle \mid x \in R$ and n is a natural number$\}$. That is to say, R is a cylinder if it may be placed in an effective one-one correspondence with the cartesian

---

*The research of the first author was partially supported under a faculty research grant from The University of Tennessee. The research of the second author was partially supported under NSF grants GP-23779 and GJ-35721.

Received June 14, 1975

product of itself cross the natural numbers. Cylinders may also be defined by the following characterization due to Young [11]. R is a cylinder if there exists an effective method f which, when applied to an arbitrary natural number x, produces an infinite r.e. set S_x such that S_x is wholly contained in R, or in the complement of R, depending upon whether $x \in R$, or $x \notin R$, respectively. It is this latter characterization that we will use. Our reason for discussing cylinders is due to the following property possessed by them.

Property 1 *Let D_1 and D_2 be an arbitrary pair of cylinders. If the decision problem for membership in D_1 and that for D_2 are of the same many-one degree, then they are of the same one-one degree.* (See, for example, Rogers [9], p. 89.)

From Property 1 we attain Property 2 below which forms the basis for the results presented here.

Property 2 *Let G_1 and G_2 be an arbitrary pair of general combinatorial decision problems such that each non-recursive instance of G_1 and G_2 is a cylinder. If G_1 and G_2 are many-one equivalent, then they are one-one equivalent.*

3 *Background* In order to simplify the statement of results in this and the following section we introduce some useful abbreviations. M_D shall denote the general derivability problem for Turing machines, A_W, P_W, and T_W the general word problems for Markov algorithms, tag systems and Thue systems, respectively, M_H, A_H, and P_H the general halting problems for Turing machines, Markov algorithms and tag systems, respectively, M_C and A_C the general confluence problems for Turing machines, and Markov algorithms, respectively, T_A the general decision problem for Thue systems with axiom, and finally PC and IC the general decision problems for partial propositional calculi and partial implicational propositional calculi, respectively. The following summarizes the results on many-one equivalences which we require.

Theorem 1 *The general decision problems M_D, M_H, M_C, A_W, A_H, A_C, P_W, P_H, T_W, T_A, PC, and IC are many-one equivalent.*

Proof: For most of these general decision problems a sketch of the proof of their equivalence was given in [4]. A formal proof is obtained by combining together, in the manner of [4], the Turing machine results of [6], Markov algorithm results of [2], tag systems results of [3], Thue systems results of [7] and [5], and the partial calculi results of [10].

4 *One-one equivalences* The following series of lemmas leads us to the desired results on the one-one equivalence of the general decision problems cited in Theorem 1.

Lemma 1 *Let D be a non-recursive instance of any of the general decision problems considered here. Then there is an infinite* r.e. *set S in the complement of the set associated with D.*

That is, for example, if M is a Turing machine whose halting problem is unsolvable, then there is an infinite r.e. set S of immoral configurations of M.

Proof: This theorem in effect says that no instance D may be of the same one-one degree as a simple set. (See Post [8] for a description and proof of the existence of such r.e. sets which contain no infinite r.e. sets in their complements.) This theorem was proved for A_H in [2], T_A in [5], and PC and IC in [10]. The proofs for M_H and P_H are exactly as for A_H. Those for M_D, M_C, A_W, A_C, P_W, and T_W are analogous to that for T_A.

Lemma 2 *Let D be a non-recursive instance of M_D, A_W, or P_W. Then D is a cylinder.*

Proof: Let C_1 and C_2 be an arbitrary pair of configurations (words). The following procedure will generate an infinite r.e. list of pairs R such that if $\langle C_3, C_4\rangle \in R$ then C_3 derives C_4 iff C_1 derives C_2. This clearly shows that D is a cylinder.

Let M be the Turing machine, Markov algorithm or tag system associated with D. Using the rules of M, list pairs $\langle C, C_2\rangle$ where C_1 derives C until (i) (C_2, C_2) is listed or (ii) no new pairs may be found and case (i) has not been fulfilled. If neither case (i) nor (ii) ever occurs then C_1 does not derive C_2 and the specified procedure lists an infinite set of pairs $\langle C, C_2\rangle$ where C does not derive C_2. If case (i) occurs then C_1 derives C_2. Continue the above procedure by listing all pairs $\langle C_3, C_4\rangle$ such that C_3 derives C_4. Clearly this is an r.e. set and therefore satisfies our needs. Finally if case (ii) occurs then C_1 does not derive C_2. Continue the above procedure by listing the infinite r.e. set S whose existence has been proven in Lemma 1. This set satisfies our requirements since each member is a pair $\langle C_3, C_4\rangle$ where C_3 does not derive C_4.

Lemma 3 *Let D be a non-recursive instance of M_H, A_H, or P_H. Then D is a cylinder.*

Proof: Let C be an arbitrary configuration (word). The following procedure will generate an infinite r.e. list R such that if $C_1 \in R$ then C_1 is mortal iff C is mortal. This clearly shows that D is a cylinder.

Let M be the Turing machine, Markov algorithm or tag system associated with D. Using the rules of M list configurations (words) C_1 where C derives C_1 until (i) C_1 is terminal or (ii) no new C_1's can be found due to the system M looping when started on C. If neither (i) nor (ii) is ever satisfied then C is immortal and the above will list an infinite number of other immortal elements. If (i) holds then continue the process by listing all mortal words. If (ii) holds list the set S whose existence was shown in Lemma 1.

Lemma 4 *Let D be a non-recursive instance of M_C or A_C. Then D is a cylinder.*

Proof: Let C_1 and C_2 be an arbitrary pair of configurations (words). The following procedure will generate an infinite r.e. list of pairs R such that if $\langle C_3, C_4\rangle \in R$ then C_3 and C_4 conflue iff C_1 and C_2 conflue. This clearly shows that D is a cylinder.

Let M be the Turing machine or Markov algorithm associated with D. Using the rules of M list pairs $\langle C_3, C_4\rangle$ where C_1 derives C_3 and C_2 derives C_4 until (i) a pair is listed such that $C_3 = C_4$ or (ii) no new pairs may be found and case (i) has not been satisfied. The rest of the proof is exactly the same as the last part of the proof of Lemma 2 except that everywhere derivability is discussed we replace such a phrase by its confluence analogue.

Lemma 5 *Let D be a non-recursive instance of* T_W. *Then D is a cylinder.*

Proof: Since C_1 derives C_2 in some Thue system iff C_2 derives C_1, the word problem for Thue systems can be analyzed as was the confluence problem for other systems in Lemma 4.

Lemma 6 *Let D be a non-recursive instance of* T_A. *Then D is a cylinder.*

Proof: This can be shown in the same manner as Lemma 5.

Lemma 7 *Let D be a non-recursive instance of PC or IC. Then D is a cylinder.*

Proof: Let M be the calculus under consideration and let W be an arbitrary wff of the (implicational) propositional calculus. Let $p_1, p_2, \ldots$ be the set of all propositional variables. Let W contain some n distinct variables $p_{j_1}, p_{j_2}, \ldots, p_{j_n}$. W_i, $i \geq 1$, is defined to be the substitution instance of W obtained by simultaneously rewriting variable p_{j_1} as p_i, p_{j_2} as $p_{i+1}, \ldots, p_{j_n}$ as p_{i+n-1}. Then W_i is deducible in M iff W is deducible. Hence $\{W_i \mid i \in (1, 2, \ldots)\}$ is an infinite r.e. set each member of which is deducible or not depending upon whether or not W is deducible.

Theorem 2 *The general decision problems* M_D, M_H, M_C, A_W, A_H, A_C, P_W, P_H, T_W, T_A, *PC, and IC are one-one equivalent.*

Proof: This theorem is a direct consequence of Theorem 1, Lemmas 2 through 7 and Property 2.

REFERENCES

[1] Cleave, J. P., "Combinatorial systems I, cylindrical problems," *Journal of Computer and System Sciences*, vol. 6 (1972), pp. 254-266.

[2] Hughes, C. E., "Degrees of unsolvability associated with Markov algorithms," *International Journal of Computer and Information Sciences*, vol. 1 (1972), pp. 355-365.

[3] Hughes, C. E., "Degrees of unsolvability associated with problems of tag," *The Journal of Symbolic Logic*, vol. 38 (1973), pp. 1-17.

[4] Hughes, C. E., R. Overbeek, and W. E. Singletary, "The many-one equivalence of some general combinatorial decision problems," *Bulletin of the American Mathematical Society*, vol. 77 (1971), pp. 467-472.

[5] Hughes, C. E., and W. E. Singletary, "Combinatorial systems with axiom," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 354-360.

[6] Overbeek, R., "The representation of many-one degrees by decision problems of Turing machines," *Proceedings of the London Mathematical Society*, vol. 26 (1973), pp. 167-183.

[7] Overbeek, R., "The representation of many-one degrees by the word problem for Thue systems," *Proceedings of the London Mathematical Society*, vol. 26 (1973), pp. 184-192.

[8] Post, E. L., "Recursively enumerable sets of positive integers and their decision problems," *Bulletin of the American Mathematical Society*, vol. 50 (1944), pp. 284-316.

[9] Rogers, H., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York (1967).

[10] Singletary, W. E., "Many-one degrees associated with partial propositional calculi," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 335-343.

[11] Young, P. R., "Linear orderings under one-one reducibility," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 70-85.

The University of Tennessee
Knoxville, Tennessee

and

Northern Illinois University
DeKalb, Illinois