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ŁUKASIEWICZ, LEIBNIZ AND THE ARITHMETIZATION OF THE SYLLOGISM

DAVID MARSHALL, Jr.

Jan Łukasiewicz wrote three separate treatises on the subject of Aristotle's assertoric syllogism. One appears as the final chapter of *Elements of Mathematical Logic* (1929) and is very brief. It shows that all the laws of conversion, of the square of opposition and all twenty-four valid moods of the syllogism (there are, in all, forty-eight theorems) can be demonstrated with the help of twelve theses from the sentential calculus (which he calls the theory of deduction), four axioms, two definitions, two undefined terms and three rules of inference. In 1939 he gave a more detailed paper on the syllogism at Cracow, a summary of which was published after the war in Polish. His definitive study, *Aristotle's Syllogistic*, appeared in 1950. In this last treatise, he uses the same axioms, rules of inference, definitions, and undefined terms as the earlier one, but makes important additions. The assertoric system is now shown to be capable of a nontrivial, infinite extension, so that the number of theorems is increased from forty-eight to aleph_0, and there are proofs of the consistency, independence, and completeness of the axioms. The proof of completeness introduces a rejection procedure, suggestions of which Łukasiewicz found in the text of Aristotle,[1] but which he himself was the first to formalize. It had been the wont of Aristotle to exhibit the invalidity of the invalid moods by adducing for the variables a set of values taken from the real world which seemed, empirically, to verify the premises while falsifying the conclusion. The tentative character of this method is obvious. Łukasiewicz' procedure is axiomatic. There is one syllogistic form which is rejected axiomatically, and one rule of rejection which he owes to his pupil, J. Słupecki.

Assuming, with Łukasiewicz, four figures, there are, in the unextended syllogistic, 256 formal assertoric moods, twenty-four of which are valid.

1. *Cf.* An. Pr. 26^{a} 2-9. *Cf.* also the comments of W. D. Ross in *Aristotle's Prior and Posterior Analytics*, Clarendon Press, Oxford (1949), pp. 28-29.

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The impression one gets in reading his different treatises, is that he supposed his four axioms of assertion capable of a decision procedure for the unextended system, but not for the infinitely extended one. This would be the case if none of the 232 rejected moods were formally consistent with the axioms. We should then say that it is the infinite extension of the system that makes the axiom of rejection necessary. It will be shown, however, in the course of this paper, that such is not the case. There are rejected syllogistic moods in the unextended system formally consistent with Łukasiewicz' axioms. Without the axiom of rejection, they are incapable of a decision procedure even for the unextended system. In showing this, I shall also demonstrate that the axiom of rejection is independent of the other four. I shall then describe an arithmetic model, due to Leibniz, proving all five consistent.

Łukasiewicz' four axioms are:

$$Aaa$$
$$Iaa$$
$$CKAbcAabAac$$
$$CKAbcIbaIac$$

The first two of these are ordinarily interpreted to mean: 'All a is a' and 'Some a is a' respectively; the last two being then the moods Barbara and Datisi. I shall interpret them as follows: The primitive forms Aab and Iab are propositions in simple arithmetic:

$$Aab \equiv (a > b > 0) \vee (a = b > 0) \vee (a = b = 0)$$
$$Iab \equiv (a > b > 0) \vee (a = b > 0) \vee (a = b = 0) \vee (a < b > 0)$$

in which a and b are positive integers greater than or equal to 0.

This translation has a vague intuitive correspondence to Aristotle's 'All a is b' and 'Some a is b', provided the latter be taken in a purely intensional way. 'All Greeks are men' can be taken to mean, among other things, that the intension of 'Greek' is greater than that of 'man' and that that of both is greater than 0; hence: $a > b > 0$. Sentences like 'All men are capable of laughter' can be taken to mean that the intension of both terms coincide; hence: $a = b > 0$. The third member of the disjunction corresponds to sentences of a non-aristotelian type like: 'All square circles are quadrilateral triangles'. Taking a contradictory intension as no intension we get: $a = b = 0$. There are, of course, other possibilities, but these three suffice and I shall take Aab to read as a disjunction with three members. The particular, 'Some a is b' can be taken to be the same as the universal except in this one respect that it can be true even if the intension of b is greater than that of a. 'Some men are Greeks' is an example. Hence a fourth disjunct: $a < b > 0$.

On this interpretation the verification of Łukasiewicz' first two axioms is intuitively easy.

$$Aaa \equiv (a < a < 0) \vee (a = a > 0) \vee (a = a = 0)$$

The first disjunct is necessarily false and of the remaining two one must be true. a is necessarily equal to itself and is either greater than or equal to 0. Similar considerations will apply to the second axiom.

$$Iaa \equiv (a > a > 0) \vee (a = a > 0) \vee (a = a = 0) \vee (a < a > 0)$$

Either the second or the third disjunct must be true.

The verification of the last two axioms is more complicated and must be shown in detail. The translation of axiom 3 is as follows:

$$\begin{array}{rl} CKAbcAabAac \equiv & (b > c > 0) \vee (b = c > 0) \vee (b = c = 0) \\ & \underline{(a > b > 0) \vee (a = b > 0) \vee (a = b = 0)} \\ \text{ergo} & (a > c > 0) \vee (a = c > 0) \vee (a = c = 0) \end{array}$$

The premises state that one of nine pairs of disjuncts is true. The argument as a whole states that for each pair one disjunct in the conclusion is true. In some cases the pair of premised disjuncts will contradict each other making the argument trivial. If it is not trivial, I shall indicate which disjunct in the conclusion verifies it.

$$\begin{array}{l} (b > c > 0) \cdot (a > b > 0) \rightarrow (a > c > 0) \\ (b > c > 0) \cdot (a = b > 0) \rightarrow (a > c > 0) \\ (b > c > 0) \cdot (a = b = 0) \rightarrow \text{contradiction} \\ (b = c > 0) \cdot (a > b > 0) \rightarrow (a > c > 0) \\ (b = c > 0) \cdot (a = b > 0) \rightarrow (a = c > 0) \\ (b = c > 0) \cdot (a = b = 0) \rightarrow \text{contradiction} \\ (b = c = 0) \cdot (a > b > 0) \rightarrow \text{contradiction} \\ (b = c = 0) \cdot (a = b > 0) \rightarrow \text{contradiction} \\ (b = c = 0) \cdot (a = b = 0) \rightarrow (a = c = 0) \end{array}$$

The translation of axiom 4 is as follows:

$$\begin{array}{rl} CKAbcIbaIab \equiv & (b > c > 0) \vee (b = c > 0) \vee (b = c = 0) \\ & \underline{(b > a > 0) \vee (b = a > 0) \vee (b < a > 0) \vee (b = a = 0)} \\ \text{ergo} & (a > c > 0) \vee (a = c > 0) \vee (a < c > 0) \vee (a = c = 0) \end{array}$$

Here twelve cases must be examined.

$$\begin{array}{l} (b > c > 0) \cdot (b > a > 0) \rightarrow (a > c > 0) \vee (a = c > 0) \vee (a < c > 0) \\ (b > c > 0) \cdot (b = a > 0) \rightarrow (a > c > 0) \\ (b > c > 0) \cdot (b < a > 0) \rightarrow (a > c > 0) \\ (b > c > 0) \cdot (b = a = 0) \rightarrow \text{contradiction} \\ (b = c > 0) \cdot (b > a > 0) \rightarrow (a < c > 0) \\ (b = c > 0) \cdot (b = a > 0) \rightarrow (a = c > 0) \\ (b = c > 0) \cdot (b < a > 0) \rightarrow (a > c > 0) \\ (b = c > 0) \cdot (b = a = 0) \rightarrow \text{contradiction} \\ (b = c = 0) \cdot (b > a > 0) \rightarrow \text{contradiction} \\ (b = c = 0) \cdot (b = a > 0) \rightarrow \text{contradiction} \\ (b = c = 0) \cdot (b < a > 0) \rightarrow \text{contradiction} \\ (b = c = 0) \cdot (b = a = 0) \rightarrow (a = c = 0) \end{array}$$

This interpretation exhibits as valid all the asserted theorems of the system, i.e., all the laws of conversion and of the traditional square of opposition, as well as the twenty-four valid moods of the syllogism. It also exhibits as valid several rejected moods. The following is an example:

$$\begin{array}{rl} CKAcbAabIac \equiv & (c > b > 0) \vee (c = b > 0) \vee (c = b = 0) \\ & \underline{(a > b > 0) \vee (a = b > 0) \vee (a = b = 0) \qquad\qquad\qquad\quad} \\ \text{ergo} & (a > c > 0) \vee (a = c > 0) \vee (a < c > 0) \vee (a = c = 0) \end{array}$$

Nine cases must be examined:

$$\begin{array}{l} (c > b > 0) \cdot (a > b > 0) \rightarrow (a > c > 0) \vee (a = c > 0) \vee (a < c > 0) \\ (c > b > 0) \cdot (a = b > 0) \rightarrow (a < c > 0) \\ (c > b > 0) \cdot (a = b = 0) \rightarrow \text{contradiction} \\ (c = b > 0) \cdot (a > b > 0) \rightarrow (a > c > 0) \\ (c = b > 0) \cdot (a = b > 0) \rightarrow (a = c > 0) \\ (c = b > 0) \cdot (a = b = 0) \rightarrow \text{contradiction} \\ (c = b = 0) \cdot (a > b > 0) \rightarrow \text{contradiction} \\ (c = b = 0) \cdot (a = b > 0) \rightarrow \text{contradiction} \\ (c = b = 0) \cdot (a = b = 0) \rightarrow (a = b = 0) \end{array}$$

This invalid mood is, therefore, formally consistent with Łukasiewicz' four axioms. That it is indeed invalid is easily seen if we interpret *Aab* and *Iab* as $a = b$ and $a > b$ respectively. We would then have

$$(c = b) \cdot (a = b) \rightarrow (a > c)$$

It is the form that Łukasiewicz rejects axiomatically. Since it is formally consistent with his axioms of assertion, its rejection cannot be deduced from them and is, therefore, independent. Łukasiewicz himself has shown that it is consistent with them, and the way he did appears to involve an interesting historical misapprehension.

Leibniz had made several attempts to arithmetize the syllogism, i.e., to find arithmetic translations of the four propositional types of the square of opposition that would make all the valid assertoric moods into truths of arithmetic and all the invalid ones into arithmetic falsities. The last attempt that he made was the only successful one. It was never published until 1903[2] and in this edition Łukasiewicz came across it.[3] The rules of translation are as follows: Every proposition has a subject and a predicate term. Each term is assigned an ordered sequence of two relatively prime numbers, the first positive, the second negative. If each number assigned the predicate term divides the corresponding number of the subject term, the proposition is of the form: 'All *a* is *b*'. The negation of this form, i.e.,

2. L. Couturat, *Opuscules et Fragments inédits de Leibniz*, F. Alcan, Paris (1903), pp. 77-82. In the account which I shall give of this interpretation, the numbers assigned to terms will be written as products of primes to facilitate checking.

3. *Cf.* Jan Łukasiewicz, *Aristotle's Syllogistic*, Clarendon Press, Oxford (1950), p. 126, footnote.

a proposition of the form 'Some a is not b' will be given if one of the two conditions is not met. For example, let us assign to the term 'sapiens' the numbers: 2^2x5, $-3x7$, and to 'pius' $2x5$, -3. The proposition 'Omnis sapiens est pius' will be translated: 2^2x5, $-3x7$; $2x5$, -3. The reason is that $2x5$ divides 2^2x5 and -3 divides $-3x7$. On the other hand, 'Quidam pius non est sapiens' is rendered: $2x5$, -3; 2^2x5, $-3x7$, because 2^2x5 does not divide $2x5$ and because $-3x7$ does not divide -3; either condition would have sufficed. If two of the non-corresponding numbers (i.e., the positive assigned one term and the negative assigned the other) have a common divisor, the proposition is of the type 'No a is b'. If this is not the case, i.e., if neither pair has a common divisor, the proposition is of the form 'Some a is b'. For example, let 'miser' be assigned 5, $-2x7$ and 'fortunatus' 11, -3^2. Then $2x5$, -3; 5, $-2x7$ will mean 'Nullus pius est miser' because $2x5$ and $-2x7$ have a common divisor; and 11, -3^2; 5, $-2x7$ will mean 'Quidam fortunatus est miser' because neither do 5 and -3^2 nor $-2x7$ and 11 have a common divisor. The syllogism

Omnis sapiens est pius.
Nullus pius est miser.
<u> </u>
Nullus miser est sapiens.

can now be verified. Its translation is:

$$\begin{array}{l} 2^2x5,\ -3x7;\ 2x5,\ -3 \\ \underline{2x5,\ -3;\ 5,\ -2x7} \\ 5,\ -2x7;\ 2^2x5,\ -3x7 \end{array}$$

In the major premise, each number in the predicate position divides the corresponding number in the subject position. In the minor premise, the first number in the subject position and the last number in the predicate position have a common divisor, which is 2. The premises are, therefore, verified. The last number in the subject position in the conclusion, and the first number in the predicate position also have the common divisor 2; hence the conclusion is verified as well. It must be admitted that this is a difficult way to verify the validity of syllogisms. But it exhibits as valid all the laws of conversion, of the square of opposition, and all the valid moods of the assertoric syllogism. It does the same for Łukasiewicz' four axioms of assertion and also, as he himself takes pains to point out,[4] for his axiom of rejection. This proves all five consistent.

Presumably for these reasons, and also because there is nothing in the printed text indicating dissatisfaction of the part of the author, Łukasiewicz seems to have thought that Leibniz not only wrote it, but also suscribed to it. L. Couturat, who saw the manuscript, seems to have found some handwritten evidence that Leibniz himself found his system unsatisfactory.

4. *Ibid.*, pp. 126-129. The interpretation that Łukasiewicz gives is not Leibniz' but a slightly modified variant. He drops the requirement that one of the numbers assigned a term be negative.

Having translated, we are told, the two premises of the invalid mood *AOO* of the third figure, he found that instead of falsifying the conclusion, he had verified it. 'And so', writes Couturat, 'he crossed it out, noticing that his method did not succeed in showing this syllogism to be invalid.' 'Hence, this system of notation is invalid.' 'Leibniz seems to have abandoned it, presumably because of its defects and its complexity.'[5]

When Couturat edited the fragment, he did not deem the crossed-out mood worthy of inclusion in the printed text. In his own book, *La Logique de Leibniz* (Paris, 1901), he gives an example which may be the original. It is the following:

Omnis pius est felix.
Quidam pius non est fortunatus.
Quidam fortunatus non est felix.

The numerical translations are:

$$\begin{aligned}&\text{pius} = 2x5,\ -3\\ &\text{fortunatus} = 2^3,\ -11\\ &\text{felix} = 5,\ -1\end{aligned}$$

which gives:

$$\frac{\begin{array}{l}2x5,\ -3;\ 5,\ -1\\ 2x5,\ -3;\ 2^3,\ -11\end{array}}{2^3,\ -11;\ 5,\ -1}$$

The middle term occupies the subject position in both premises so that the syllogism is clearly in the third figure. The premises verify the conditions for the *A* and *O* propositions respectively, and the conclusions for the *O*. These values, therefore, exhibit the syllogism as valid. Does this mean that Leibniz' system of notation is deficient? Couturat is no doubt right in thinking Leibniz was dissatisfied with it. But we may wonder whether he was right in saying that this is because of its 'defects'. Łukasiewicz has shown that the system is not invalid but valid. It is true that there are sets of values that can cause it to exhibit as valid the invalid mood shown above. There are others that exhibit it as invalid. What is required for rejection? If we translate the three terms of the example given above as follows:

$$\begin{aligned}&\text{pius} = 2^4x7,\ -3^3x5\\ &\text{fortunatus} = 2^2,\ -3^9\\ &\text{felix} = 2,\ -3^3\end{aligned}$$

we get:

$$\frac{\begin{array}{l}2^4x7,\ -3^3x5;\ 2,\ -3\\ 2^4x7,\ -3^3x5;\ 2^2,\ -3^9\end{array}}{2^2,\ -3^9;\ 2,\ -3^3}$$

5. L. Couturat, *La Logique de Leibniz*, F. Alcan, Paris (1901), p. 334.

In this translation, the premises are of the type *A* and *O* respectively, and the position of the middle term identifies the third figure. But the numbers in the conclusion do not fulfill the conditions required by an *O* proposition so that the mood *AOO* of the third figure is shown to be invalid. The argument for rejection should be as follows: if invalidating instantiations can be produced, the mood is invalid; it is valid if they cannot. It would not be difficult to show that no invalidating instantiation can be produced for the valid mood given above.

Leibniz' system is, therefore, not a failure but a success. Is it possible that he failed to notice it? If, as Łukasiewicz claims, no rejection procedure was ever formalized until his own in 1950, earlier logicians who used the procedure were following their good instincts. Formally, one invalidating instance is enough. On this point, the instincts of Aristotle did not fail him. What are we to say of Leibniz? Were his instincts inferior? Does it not seem better to assume that the reason he abandoned his attempts at arithmetization is the one he gave—that it made practical application too difficult? Leibniz, one of the earliest to attempt a reconciliation of the Christian churches, was preoccupied to find an unimpeachable method for resolving issues at debate. He dreamed of arithmetic solutions to all questions. This was one, at least, of the driving forces behind the *Ars Combinatoria* and behind his numerous attempts at arithmetizing the syllogism. He was hardly the man to fail at such an attempt: but technical success was accompanied by the realization of impracticability. And so he turned his thoughts to other matters.

This view has the advantage of some textual support, but more profound considerations, it seems, must incline us to believe that while Couturat was certainly wrong in thinking the system of notation a technical failure, he may well have been right in thinking that Leibniz conceived of it as such. It is one of the striking characteristics of Leibniz' philosophy that he held all true propositions to be analytically true and all false ones to be analytically false. If we assume that this is false (as it seems we must),[6] we may say that in this matter his conscious views inhibited the play of his instincts, so that they failed him indeed where Aristotle's did not. The very attempt to arithmetize the syllogism is as foreign to the philosophy of Aristotle as it is natural to that of Leibniz. Sentences in simple arithmetic do not obviously contain variables so that their truth or falsity seems immediately given. Prima facie, at least, the science seems to be without propositional functions. This is presumably the characteristic that caused Leibniz to think an arithmetization of the syllogism desirable. If so, the experience must have been something of a disappointment, for the arithmetization that he finally came up with is not, as L. Couturat points out,[7] an example of arithmetic, but of algebra. Instead of numbers, it contains

6. *Cf.* B. Russell, *Critical Exposition of the Philosophy of Leibniz*, George Allen and Unwin, London (1900), p. 18.

7. Couturat, *La Logique de Leibniz*, p. 337.

variables whose values are numbers. Even this would have been acceptable if the algebra in question had been such as to yield none but isomorphic instantiations; i.e., none but confirming ones for the valid moods and for the invalid ones, none but counter-examples. Finding for the variables of his system a single set of values that made the premises and conclusion of an invalid mood simultanteously true must have caused Leibniz to think that he had failed to accomplish his task. In this respect, Couturat may well be right. If he is, Łukasiewicz, in showing that the 'arithmetization' is a complete success, has in effect rescued it from its author's rejection.

St. Joseph's College
Philadelphia, Pennsylvania