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CLASSICAL LOGICAL RELATIONS

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The logical relations of classical logic—i.e., the five relations based on the square of opposition together with equivalence and independence—are usually assumed, in logic textbooks and elsewhere, to be familiar and easily defined, but in fact standard discussions of these relations are always imprecise on vital points. I want to illustrate this and then go on to discuss the precise nature of the relations.

David H. Sanford has recently drawn attention to one source of difficulty.[1] Many textbooks, he argues, are inconsistent in their treatment of contraries and subcontraries in that they fail to allow for the distinction between contingent and noncontingent propositions. For example, two propositions are said to be contraries if and only if they cannot both be true but can both be false. If, however, we happen to have a necessarily true proposition of the form "All a are b", it appears that this proposition and its contrary cannot *both* be false, which goes against the stated conditions for the relation.

But textbook formulations of the other relations also create problems. As an example I will refer to M. R. Cohen and E. Nagel's well-known work, *An Introduction to Logic and Scientific Method*, which contains (Chapter III) the fullest account of logical relations I have been able to locate. Later books, so far as I can discover, have not cleared up the problems that arise. Cohen and Nagel list nine possible relations and in each case specify *two* conditions for the relation. For example, the conditions for contradictory relation are given as "If p is true q is false. If p is false q is true", and the conditions for contrary relation as "If p is true q is false. If p is false q is undetermined". Cohen and Nagel's list contains two more relations than the standard seven and in explanation of this they claim that three of their relations are of the same type, independence. The sets of conditions they give for these three relations are as follows (pp. 55-56):

1. "Contraries and subcontraries," *Noûs*, vol. II (1968), pp. 95-96.

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(a) If p is true q is true. If p is false q is true.
(b) If p is true q is false. If p is false q is false.
(c) If p is true q is undetermined. If p is false q is undetermined.

It is, however, a mistake to assimilate these three relations. Cohen and Nagel's account is complicated by their references to p's and q's being *undetermined* and needs further explanation. When in (c) it is said that "If p is true q is undetermined" what is meant is that "If p is true then q may be true" and *also* "If p is true q may be false". Similarly, what is meant by "If p is false q is undetermined" is that "If p is false q may be true" and "If p is false q may be false". In other words, (c) incorporates all the conditions for independence (or indifference), there really being *four* conditions for this relation. At the same time, Cohen and Nagel's formulation is inaccurate in that it fails to distinguish clearly between *specific* propositions and *forms* of propositions. When we do make this distinction we can express the four conditions for independence as follows:

Two specific propositions A_1 and B_1 are independent when they are of forms A and B such that with propositions of this form there are the four possibilities: (i) both propositions are true; (ii) both propositions are false; (iii) the first proposition is true and the second false; (iv) the first proposition is false and the second true.

This means that Cohen and Nagel's cases (a) and (b) are irrelevant and as I shall point out later are not cases of independence at all. The ambiguity of Cohen and Nagel's position is further revealed when they suggest (in a footnote, p. 56) that if questions about reversible relations are introduced then it will be possible to express logical relations by *tetrads* of conditions, for example that we could express the relation of superaltern to subaltern by the tetrad: "If p is true q is true; if p is false q is undetermined; if q is true p is undetermined; if q is false p is false". But if this case counts as a tetrad, we can in fact produce tetrads in all cases. For example, we could turn the two conditions for contradictory relation given above into a tetrad by adding the conditions: "If q is true p is false; if q is false p is true", and so on in the other cases. But if we do this in every case we find that some of the extra conditions are superfluous—for instance the two extra conditions for contradictory relation are unnecessary and can be inferred (by transposition) from the first two conditions. Consequently, the reference to tetrads of conditions is misleading; tetrads with superfluous conditions have to be carefully distinguished from the genuine tetrad already noted, namely, the four conditions for independence.

Let us, then, in stating the conditions for traditional relations, omit superfluous conditions. When we do so we find that there are three types of cases, dyadic, triadic, and tetradic sets of conditions. These conditions can be conveniently expressed by specifying what true-false possibilities each relation leaves open. Let us take, for example, the case where a proposition of the form A is in subaltern relation to a proposition of the form B. We can express the conditions for this relation briefly at **TT**, **TF**, **FF**, understanding by this that when two actual propositions are in this relation, the first proposition of form *A* being in subaltern relation to the second proposition of form *B*, three possibilities are left open, (i) that both propositions are true, (ii) that the first proposition is true and the second proposition is false, and (iii) that both propositions are false, while the absence of **FT** shows that the relation does not permit the combination: the first proposition false and the second proposition true. Using this method we can set out the conditions for the seven relations as follows:

TABLE 1

Relation	Truth-Combinations Left Open
(1) equivalence	**TT, FF**
(2) contradictory	**TF, FT**
(3) subcontrary	**TT, TF, FT**
(4) subaltern	**TT, TF, FF**
(5) superaltern	**TT, FT, FF**
(6) contrary	**TF, FT, FF**
(7) independence	**TT, TF, FT, FF**

From this table it can be seen that equivalence and contradictory relation each has two conditions, independence has four conditions, and the remaining four relations each has three conditions. So far as the ordinary system of *contingent* A, E, I, O propositions is concerned these seven cases provide us with a comprehensive and precisely differentiated set of relations.

But there remains the question, raised by Sanford, of relations in the case of non-contingent A, E, I, O propositions—with which is connected Cohen and Nagel's reference to two extra cases of independence. Thus, suppose *p* is related to *q*, but *p* is a proposition of a necessarily true form, then no matter what the relation is between *p* and *q*, none of the seven sets of conditions will apply correctly since each of them specifies the possibility that propositions of *p*'s form can be *false*.

Now, as I have pointed out in a different context,[2] it is possible to expand the list of classical relations. This expansion enables us to solve the present problem. Thus given the four types of truth-combinations, **TT**, **TF**, **FT**, **FF**, referred to in Table 1 above, if we were to specify all the mathematically possible sets of four conditions we should have $4^4 = 256$ cases. But most of these cases involve redundant or superfluous conditions—citing them would be like citing **TT**, **FF**, **TT**, **FF**, as four conditions for equivalence. So confining attention, as before, to non-superfluous conditions we arrive at fifteen relevant cases, consisting of the relations listed in Table 1 and the following additional cases:[3]

2. "Non-empty complex terms," *Notre Dame Journal of Formal Logic*, vol. VII (1966), pp. 55-56.

3. There is a further case (16) where *no* truth-combinations are left open, and which could be applied to paradoxical propositions.

TABLE 2

	Relation	Truth-Combinations Left Open
(8)	subcontrary type (ii)	TT, TF
(9)	subcontrary type (iii)	TT, FT
(10)	contrary type (ii)	TF, FF
(11)	contrary type (iii)	FT, FF
(12)	non-contingent equivalence type (i)	TT
(13)	non-contingent inconsistency type (i)	TF
(14)	non-contingent inconsistency type (ii)	FT
(15)	non-contingent equivalence type (ii)	FF

This table introduces four additional relations with dyadic sets of conditions and four new relations each with a single condition. Of these, (8) to (11) will enable us to deal with relationships between contingent and non-contingent propositions, for in each of these cases one of the related propositions can be either true or false but the other proposition is either restricted to being true—as with (8) and (9)—or else is restricted to being false—as with (10) and (11).

Referring back now to Cohen and Nagel's two supposed extra cases of independence, it can be seen that these cases, stated above as (a) and (b) are in fact relations (9) and (10) in the table, and can be more accurately described as special cases of subcontrary relation and contrary relation respectively. On the other hand, the remaining relations (12) to (15) can be applied to relations between propositions both of which are non-contingent.

These eight new cases have an obvious application to relations between truth-functions, but can also be applied to syllogistic logic. Suppose we have the relation between a necessarily true proposition of the form "All *a* are *a*", and its contrary according to the square of opposition "No *a* are *a*". The conditions for contrary relation stated in (6) in Table 1 are then quite inadequate as a description of the relation between these two propositions. But so are the conditions for (10) and (11) in the second table, for they also contain the inapplicable condition FF. In fact, it is relation (13) that precisely fits the relation that "All *a* are *a*" has to "No *a* are *a*", just as (12) fits the relation between "All *a* are *a*" and "Some *a* are *a*", and (15) fits the relation between "No *a* are *a*" and "Some *a* are not *a*".

It might be thought that these relations (12) to (15) will also apply in the case of standard examples of necessarily true propositions. Sandford, for example, (*op. cit.*, p. 95) says that some A form propositions, for example, "All squares are rectangles", are necessarily true, and so will not have an ordinary contrary. Now if we consider these cases in the context of Table 2, it might be thought that the relation "All squares are rectangles" has to "No squares are rectangles" is that defined by (13), and so on. But this can be said to be so only in an extra-syllogistic, truth-functional sense. For within classical logic, the conditions for the relations refer to *forms* of propositions. So while, e.g., "All *a* are *a*" and "No *a* are *a*'' are genuine forms which can be instanced by specific propositions like ''All squares are squares'', ''No squares are squares'', we cannot say in a parallel way that ''All *a* are *b*'' is a necessarily true *form* of which ''All squares are rectangles'' is a specific instance. Hence if we do go on to say ''All squares are rectangles'' is in relation (13) to ''No squares are rectangles'', this will be for truth-functional type reasons—the relation will hold because the one *specific* proposition has the truth-value **T** and the other *specific* proposition has the truth-value **F**.

Relations (8) to (11) will, as noted above, apply to relations between contingent and non-contingent propositions, but examples of these relations are harder to find within traditional logic than might be imagined. However, suppose we have a proposition of the form, (i) ''Some *a* are *ab*'' (e.g., ''Some trees are deciduous trees''), then since we can regard this as implying that there are *ab*'s it is necessarily true. But the related form, (ii) ''All *a* are *ab*'' is contingent, and the relation a proposition of form (i) has to a proposition of form (ii) exemplifies relation (8). Similarly, the relation a proposition of the form ''No *a* are *ab*'' has to a proposition of the form ''All *a* are *ab*'' will exemplify relation (11), and so on. Or, as a different type of example, we might perhaps say that relation (8) is the relation which holds between a proposition of the form ''Some *a* are *a*'' and a proposition of the form ''All *a* are *b*'', and so on.

The conclusion I wish to draw is thus, that while the classical relations are often dealt with inadequately, this does not preclude a systematic and coherent account being given of these relations. Error arises when the conditions for the relations are not specified precisely, or when allowance is not made for the distinction between contingent and non-contingent propositions. But when the conditions for the seven ordinary relations are set out carefully we can (a) deal in a straightforward way with relations between ordinary contingent A, E, I, O, propositions, and when the list of relations is expanded we can also deal in a straightforward way (b) with relations between non-contingent propositions, and (c) with any relations between contingent and non-contingent propositions that arise.

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